THE ASTROPHYSICAL JOURNAL, 187:425–438, 1974 February 1

FORMATION OF GALAXIES AND CLUSTERS OF GALAXIES BY SELF-SIMILAR GRAVITATIONAL CONDENSATION*

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Received 1973 August 1

ABSTRACT

We consider an expanding Friedmann cosmology containing a "gas" of self-gravitating masses. The masses condense into aggregates which (when sufficiently bound) we identify as single particles of a larger mass. We propose that after this process has proceeded through several scales, the mass spectrum of condensations becomes "self-similar" and independent of the spectrum initially assumed. Some details of the self-similar distribution, and its evolution in time, can be calculated with the linear perturbation theory. Unlike other authors, we make no ad hoc assumptions about the spectrum of long-wavelength initial perturbations: the nonlinear N-body interactions of the mass points randomize their positions and generate a perturbation to all larger scales; this should fix the self-similar distribution almost uniquely. The results of numerical experiments on 1000 bodies are presented; these appear to show new nonlinear effects: condensations can "bootstrap" their way up in size faster than the linear theory predicts. Our self-similar model predicts relations between the masses and radii of galaxies and clusters of galaxies, as well as their mass spectra. We compare the predictions with available data, and find some rather striking agreements. If the model is to explain galaxies, then isothermal "seed" masses of $\sim 3 \times 10^7 M_{\odot}$ must have existed at recombination. To explain clusters of galaxies, the only necessary seeds are the galaxies themselves. The size of clusters determines, in principle, the deceleration parameter q_0; presently available data give only very broad limits, unfortunately.

Subject headings: cosmology — galaxies — galaxies, clusters of

I. INTRODUCTION

The observed matter content of the Universe is very clumpy over a range of mass exceeding 15 orders of magnitude, from stars ($\sim 1 M_{\odot}$) through clusters and galaxies, to clusters of galaxies of $10^{15} M_{\odot}$. However, on progressively larger scales the evidence for clumpiness is less striking. Considerable effort has been required to demonstrate the existence of superclusters (see Bogart and Wagoner 1973 for a recent treatment). On scales larger than ~ 50 Mpc (but smaller than the present horizon of $\sim 10^3$ Mpc) the Universe is probably isotropic and homogeneous, corresponding to an expanding Friedmann cosmology. Even if the earliest epochs were characterized by chaos and large-scale inhomogeneity (Misner 1968; Rees 1972; Peebles 1972), the isotropy of the cosmic microwave background argues for a Friedmann model at recombination and subsequently. After recombination (and the roughly coincident transition to matter dominance) the Universe probably evolves according to the pressureless dynamical equations (see, e.g., Peebles 1971 or Weinberg 1972).

In this context, how are the various scales of clumpiness to be explained? Star formation from a diffuse medium of sufficient density (and suitable other parameters) may be a purely astrophysical—as opposed to cosmological—problem: stars form at various epochs, and the process can be observed and studied in the present. In contrast, the condensation of substantially larger scales, especially galaxies and clusters of galaxies, seems to be a unique cosmological event. The accepted view, convincingly stated by Peebles (1965), Silk (1968) and others, puts the formation of these large-scale objects at some epoch between recombination and the present, because only in this period have the large condensing masses been substantially smaller than the cosmological horizon but bigger than their Jeans lengths.

A linear theory of inhomogeneous perturbations has been extensively developed for both isothermal and adiabatic disturbances (Lifshitz 1946; Zel'dovich 1967; for a review see Field 1974), and much recent work has been directed toward propagating an initially postulated spectrum of matter perturbations through the complicated era of recombination, into the era where the perturbations condense into (hopefully) observed objects. This program has yielded considerable cosmological understanding in many respects, but it has not thus far been completely successful in explaining the basic observational data: the mass distribution of galaxies and their linear sizes, the masses and sizes of clusters of galaxies, the fact that there is no strong clustering on larger scales.

In the usual framework of the linear perturbation analysis, physical processes at or before recombination

* Supported in part by the National Science Foundation [GP-36687X, GP-28027].

(such as photon viscosity) can modify a postulated spectrum of perturbations (Michie 1967; Silk 1968; Field and Shepley 1968; Peebles and Yu 1970). Nevertheless, the spectrum of masses that comes out at recombination is dictated in a linear fashion by the initial spectrum put in. The simplest power-law spectra (which have been tried most frequently) seem to yield only large-mass ($\sim 10^{15}\ M_\odot$) condensations, not smaller galactic masses. One way out of this dilemma has recently been put forth by Zel'dovich and his collaborators (Zel'dovich 1970; Sunyaev and Zel'dovich 1972; Doroshkevich *et al.* 1973): The large-mass condensations may collapse to flat, irregular "pancakes," form shocks, and fragment into much smaller lumps. In this picture small-mass objects form from nonlinear processes in large-mass ones. Here, we want to investigate a possible opposite process: Can larger-mass objects form from the nonlinear interaction of smaller masses?

A first observational clue is that condensations in the present Universe are not "small disturbances." They may be closer to an opposite limit, that of point masses in a self-gravitating gas. The view is rather commonly held that this discreteness evolved after the present mass distribution had been determined by the linear theory of small disturbances—even strong condensations were once weak (cf. Weinberg 1972, p. 576).

We do not adopt this view. Instead, we start with two ideas suggested by Peebles (1965, 1972): that the development of larger scales of condensation may have been sequential; and specifically, that statistical randomness in an "incoherent dust" model induces the growth of larger instabilities with increasing time. We outline in this paper a model which is initially grainy and remains so at all times, starting with an initial mass spectrum of "small" masses which are supposed to have been formed by other processes. The initial time is taken as recombination or soon after. As the expanding Friedmann cosmology evolves, the mass points condense into aggregates which (when they are themselves sufficiently bound) we identify as single particles of a larger mass. In this way, the condensation proceeds to larger scales. We will insist that our model contain no ad hoc information about an initial spectrum of long-wavelength density perturbations: When condensations has proceeded to lumpiness on a certain scale, the statistical randomness in the positions of the discrete lumps is itself a perturbation to all larger scales, and this causes condensation of increasingly large masses at later and later times. We take these statistical fluctuations as the only source of long-wavelength perturbations. We will see in § II that there are dimensional grounds for suspecting that the behavior of the system at late times depends only very weakly—perhaps not at all—on the initially assumed spectrum of seed masses and their statistical distribution. When the condensation has proceeded to scales much larger than the seed scale, the gas should have essentially no memory of its initial scale, and the condensation process should approach a *self-similar* solution, where the functional form of the mass distribution is maintained even as the characteristic scale of condensations gets larger and larger. The main content of this paper consists of linearized analytic estimates of the self-similar distribution as a function of time (§ III), together with the results of nonlinear numerical N-body experiments (§ IV) which show some features of a self-similar condensation process which cannot be treated in a linearized framework.

Neither the analytic treatment nor the numerical treatment given here is a fully satisfactory treatment of the problem. However, in spite of their rather different biases, they give very similar results. We take the attitude that in the absence of a more rigorous treatment, it is reasonable to suppose that these overlapping results are genuine predictions of the model, and—even at this stage—they can be reasonably compared with observation.

The model makes a number of predictions that are directly testable: namely, the masses and densities of condensed structures of various scales, e.g., the scales of galaxies and clusters of galaxies. In § V we compare these predictions with the data and find some striking agreements, despite the relative crudeness of our treatment. There is effectively only one adjustable parameter in the model, the characteristic mass of seed condensations soon after recombination (which, however, comes out uncomfortably large).

In § VI we discuss this and other shortcomings of the model, and speculate on how it might be fitted into a cosmological picture valid before recombination. Even if galaxies are really formed by some other process, clusters of galaxies can easily be formed by statistical condensation; the size and spacing of clusters gives in this instance a direct measure of the deceleration parameter q_0. We discuss the value that is deduced.

Some of our results involve only a slight change in point of view from previous work, e.g., that of Peebles (1965) and Saslaw (1968). Other authors have considered hierarchical models which do not seem to be related to this paper; see, e.g., Layzer (1969) and Wertz (1971).

II. COLD "GAS" IN A FRIEDMANN COSMOLOGY

The problem considered here is rather more idealized than is cosmologically plausible in the real Universe. In idealizing the model we hope to make it well-posed mathematically, and to sharpen its observational predictions. A justification of the idealization is that the agreement of these predictions with the data is quite remarkable in some particulars.

The problem is this: What is the evolution in time of a self-gravitating "gas" of point particles in Friedmann cosmology? We may take the gas as characterized, by a number density of particles n_*, or equivalently by a typical interparticle distance $l \sim n_*^{-1/3}$. On some scale $L \gg l$, the gas is supposed to be statistically uniform; by hypothesis, uniformity is reached well inside the light horizon L_h where general-relativistic effects become important, so

~"Hubble horizon"

$$L_h \gg L \gg l . \tag{1}$$

As long as the gas does not condense to a highly relativistic object (e.g., local collapse to a black hole), its dynamics on scales less than $\sim L$ is effectively Newtonian. (We will show below that if condition [1] is satisfied at some initial time, it remains satisfied at all later times.)

Particles in the gas have some characteristic mass m_*. The gas is expanding cosmologically, and we can identify a Hubble constant h by measuring recession velocities over varying distances. The particles have some typical peculiar velocity v with respect to the expansion, $v \ll c$. The dimensional parameters n_*, m_*, h, v (and we should include G, the gravitational constant) define the characteristic state of the gas. The velocity of light c, or L_h ($\sim c/h$) does not enter since equation (1) is satisfied.

The dimensional parameters having been identified, it is possible to identify dimensionless combinations of them. The gas has two such, which we take to be

$$q = \tfrac{4}{3}\pi m_* n_* G/h^2 , \qquad N_J = n(v/h)^3 . \tag{2}$$

These parameters are measures of competing phenomena in the gravitational condensation of the gas. The first, q, a deceleration parameter, describes the Universe's ability to impede local condensation by its own expansion. The second, N_J, measures a quantity related to the number of particles in a Jeans mass. For reasonable values of q, aggregates with fewer than about N_J particles are stabilized against gravitational condensation by the peculiar velocities ("temperature") of the masses.

Proceeding from some initial time, the details of the condensation process will depend on q and N_J. Two systems with the same q and N_J ought to condense similarly, regardless of differences in the scale of their dimensional parameters. If q and N_J are functions of time, then condensation in a single system proceeds differently at different epochs. Correspondingly, if q and N_J are constant in time, one might hope for a *self-similar* condensation, which looks the same at all times except for a change of scale.

We focus our attention on this latter possibility. ? Whatever the value of q_0 at present, q approaches $\frac{1}{2}$ asymptotically at earlier times; present limits suggest that its value was very nearly constant at $\frac{1}{2}$ for some period following recombination. Further, drag forces prior to recombination are likely to have made the peculiar velocities of any condensations small in comparison with the Hubble expansion, so $N_J \ll 1$ ("cold" gas of particles). Our dynamical calculations below suggest that starting small, N_J also remains small. If we take the current observational value of N_J from galaxies not in clusters, and from clusters of galaxies, we find that it is indeed small:

$$N_J = 0.008\left(\frac{n^{-1/3}}{10\ \mathrm{Mpc}}\right)^{-3}\left(\frac{v}{100\ \mathrm{km\ s^{-1}}}\right)^{3} \times \left(\frac{h}{50\ \mathrm{km\ s^{-1}\ Mpc^{-1}}}\right)^{-3} . \tag{3}$$

Thus, there is a natural possibility for a self-similar condensation, with the constant values $q = \frac{1}{2}$, $N_J \approx 0$.

This argument does not prove that a self-similar condensation *must* occur. The reason is that identifying a "characteristic" mass m_* and number density n_* does not completely determine the microstatistical properties of the gas. First, the gas has a distribution of particle masses $n(m)$, where $n(m)dm$ is the number density of particles with mass between m and $m + dm$. This can be related to the old parameters n_* and m_* by the definitions

$$n_* \equiv \int_0^\infty n(m)dm , \tag{4}$$

$$m_* \equiv n_*^{-1}\int_0^\infty mn(m)dm \equiv \rho/n_* \tag{5}$$

(but see discussion follow eq. [25] below).

Second, the gas will have (or will develop) correlations in space, described (e.g.) by the N-point correlations for all N. Two gases can in principle have the same "characteristic" parameters but nevertheless be microscopically different—and they might condense differently. This is the main point which must be investigated in the balance of this paper. However, we can state here the physical reasons for our hope that the details of condensation do *not* depend on microstatistical initial properties: Since the gravitational force is long-range and attractive, it acts primarily to develop positive N-body correlations, i.e., to form bound clusters of masses which do not subsequently participate in the universal expansion. Naïvely, this would seem to rule out the possibility of self-similarity, because the degree of clumping will measure how far the system has evolved from a "random" initial state. Further reflection shows the loophole: the high correlations of a bound lump are not physically relevant to the further condensation; they only restate the obvious fact that highly bound clusters act roughly like point masses to larger scales. The large correlations in the gas must be interpreted as changes in $n(m)$, by treating highly correlated groups of particles as single more massive particles. In this case, the relevant question is to determine the statistical properties of the current scale of large lumps, not the initial small particles which have long since coagulated into bound groups.

It is well known that the self-gravitating N-body problem is formally unstable (in the sense that an infinitesimal perturbation in initial conditions leads to an arbitrarily large difference at later times). The problem is also highly nonlinear. If as we expect, the evolution at any stage is determined only by the statistics of the current scale, it is not unreasonable to expect that the evolution is dominated by the stochastic properties of the unstable nonlinear dynamical equations, rather than by any memory of its previous history. In this case, a self-similar development seems rather likely.

It will be useful to introduce a statistical measure of correlation which is invariant under the identification of bound clumps as single particles. An obvious choice is the mass variance Σ^2 in a volume V,

$$\Sigma^2(V) \equiv \langle M \rangle^2 - \langle M^2 \rangle \,, \tag{6}$$

where angular brackets denote ensemble averaging over different volumes V, and $M = \int\int mn(m)dmdV$. (The definition is slightly fuzzy because we have not specified the shape of V; this is discussed below.)

The information that we choose to express in $n(m)$ and $\Sigma^2(V)$ can also be expressed, as is more conventionally done, in terms of the Fourier spectrum of the mass density function, $\rho(\boldsymbol{k})$. However, all of the information is *not* in the power spectrum $|\rho(\boldsymbol{k})|^2$, since phase information cannot be neglected on the scale of the inverse interparticle (or interclump) distance. For this reason, it is quite unwieldy to translate our model into the Fourier-spectrum formalism, and we do not do so.

cosmo-sim

At the initial time, we want to require that $\Sigma^2(V)$ should arise *only* from the inherent randomness of the point-mass density, and not from any larger-scale superposition of an *a priori* perturbation spectrum. This gives upper and lower bounds on the behavior of $\Sigma^2(V)$ as V becomes large.

The lower bound is obtained by imagining the gas particles to be as ordered as they can be, so that the mass variance on larger scales is maximally suppressed. It is not plausible for the mass points to be in a regular crystal lattice; such a gravitational structure is unstable against randomization in the order of one expansion time (or Kepler time). We can assume, however, that every particle "belongs" to some regular lattice site, and is no more distant from the site than a typical lattice dimension for particles of comparable mass, $\sim[mn(m)]^{-1/3}$. A particle is random in its own cell, then, but there is never more or less than one particle per cell.

The upper bound on the variance stems from the requirement that the gas be statistically homogeneous. The lumpiest gas allowed is one in which the positions of all masses are independently random, so that the variance is linear in the volume of a region, and independent of the shape of the volume.

The function $n(m)$ is initially arbitrary, subject only to the requirement that the total mass density be finite. As the gas expands and develops condensations, which are identified as new point particles, the mass spectrum $n(m)$ changes. The task facing us is to calculate the evolution of $n(m)$ as a function of time or expansion parameter. Further, we want to substantiate our physical guess, that at large times the details of $n(m)$ do not depend on its original functional form; rather, that $n(m)$ develops to a self-similar solution which generates itself on all subsequent scales.

III. A LINEAR ANALYSIS OF THE GROWING MODE

The problem of statistical condensation is fundamentally nonlinear, because the statistically random properties of the smallest scales arise in the first instance from their (nonlinear) N-body interactions as point particles. These randomizing interactions are a source for perturbations on all larger scales. Given this source, however, one might in principle compute the evolution of a much larger scale by the standard linearized theory. This is not an easy calculation, because the "discreteness" perturbation projects into all of the linearized modes (e.g., a growing and dying radial mode, and two rotational modes). Furthermore, the calculation is not guaranteed to give correct results, because it is quite possible (see § IV) that the nonlinear effects on small scales "bootstrap" their way up in scale faster than the linear effects grow on large scales. For these reasons, we give here only a semiquantitative analysis, with the following assumptions: (i) only the growing radial mode is treated; and (ii) its behavior is assumed to be predicted accurately by an energy argument (see note added in proof).

a) *Case of Maximum Variance*

Here we assume that the positions of all particles in the gas are independently random, at least on scales larger than the typical interparticle distance. In this case, the variance of mass in a given volume varies linearly with the volume V, so that the variance per volume is

$$\sigma^2 \equiv \Sigma^2(V)/V \,, \tag{7}$$

a constant. The relation of σ^2 to $n(m)$ is easily shown to be

2nd moment

$$\sigma^2 = \int_0^\infty m^2 n(m) dm \,. \tag{8}$$

Let $p(\delta, V)$ be the probability that a given volume V contains a mass whose fractional deviation from the ensemble average is between δ and $\delta + d\delta$. Then for reasonably large volumes and small δ's, $p(\delta, V)$ is approximately Gaussian:

$$p(\delta, V) = (2\pi)^{-1/2}\delta_*^{-1} \exp\left[-\tfrac{1}{2}\delta^2/\delta_*^2\right] , \tag{9}$$

where δ_* is the standard deviation

$$\sigma(M) = \delta_* \equiv \frac{[\Sigma^2(V)]^{1/2}}{M} = \frac{(\sigma^2 V)^{1/2}}{\rho V} = \frac{\sigma}{\rho} V^{-1/2} \,. \tag{10}$$

(The mean density ρ is defined by eq. [5].) Since the

v = δ_c / σ(M)
= 1.686 / σ(M)

Differentiate wrt. M to find fraction in range dM and multiply by ρm/M the number density of halos if all of the mass were composed of such halos → differential number density of halos
⇒ usual form of the PS: dn/dlnM = …

cosmology is parabolic ($q_0 \approx \frac{1}{2}$), a large volume which is overdense by a fraction δ at expansion scale R_1 will expand only a finite amount, and reach its maximal size at an expansion parameter of about $R_2 = R_1/\delta$. After this, the lump recollapses, violently relaxes (Lynden-Bell 1967), virializes. For our purpose we can consider it as a self-bound "point mass" from expansion scale R_2 on. The Appendix describes quantitative details of all this for the case of spherical lumps.

The probability P that a given volume will *already* be self-bound at expansion scale R_2 is thus

Integrate from smallest δ capable of causing a collapse

$$P = \int_{\delta = R_1/R_2 = 1.686}^{\infty} p(\delta, V)d\delta = \frac{1}{2}\,\mathrm{erfc}\left(\frac{R_1 \rho V^{1/2}}{2^{1/2} R_2 \sigma}\right), \tag{11}$$

$= \frac{1}{2}\mathrm{erfc}(\nu/\sqrt{2})$

where equations (9) and (10) have been used, and erfc is the complementary error function. Since for the volume V, δ is small at scale R_1, the mass in all volumes V is very nearly $\rho_1 V$. (Here and henceforth, a subscript 1 denotes a quantity evaluated at scale R_1; similarly for subscript 2, etc.) The fact that a volume has condensed before scale R_2 does not prejudice it against being part of a larger volume which has also condensed. Thus the fraction of independent volumes (or of mass) which have become self-bound in lumps with mass between M and $M + dM$ is the derivative of P,

$$\frac{dP}{dM} = 2^{-3/2}\pi^{-1/2} \frac{R_1}{R_2} \frac{\rho_0^{1/2}}{\sigma_0} \times M^{-1/2} \exp\left(-\frac{1}{2}\frac{R_1^2}{R_2^2}\frac{\rho_0}{\sigma_0^2} M\right). \tag{12}$$

The mass density due to lumps of this mass is obtained by multiplying this fraction by the total density at scale R_2, $\rho_1(R_1/R_2)^3$; converting mass density to number density brings in an M^{-1}. We have not yet accounted for half of the total mass—that which is in initially *underdense* regions. It eventually accretes onto neighboring lumps in overdense regions. A spherical calculation (see Appendix) suggests that the net effect is to double the mass of condensed lumps without changing their mass spectrum. We have not rigorously accounted for the fact that condensations may occur on the border of two adjacent volumes. Here we assume that this introduces an error of only order unity; see Schechter (1974) for another point of view.

Fudge factor of 2 (!!!)

We have now obtained the number density of lumps (new point particles) at scale R_2,

$$n_2(M) = \frac{2}{M}\rho_1\left(\frac{R_1}{R_2}\right)^3 \frac{dP}{dM}$$
$$= (2\pi)^{-1/2}\left(\frac{R_1}{R_2}\right)^4 \frac{\rho_1^{3/2}}{\sigma_1} \times M^{-3/2}\exp\left[-\frac{1}{2}\left(\frac{R_1}{R_2}\right)^2 \frac{\rho_1}{\sigma_1^2} M\right]. \tag{13}$$

as σ(M)→inf, ν→0, so collapsed frac. P→½erfc(0)=½

The initial mass distribution $n_1(m)$ enters only through its moments ρ_1 and σ_1. It is straightforward to calculate ρ_2 and σ_2 from equation (13). The results are

$$\rho_2 \equiv \int_0^\infty m n_2(m)dm = \rho_1\left(\frac{R_1}{R_2}\right)^3, \tag{14a}$$

$$\sigma_2^2 \equiv \int_0^\infty m^2 n_2(m)dm = \sigma_1^2\left(\frac{R_1}{R_2}\right). \tag{14b}$$

Notice that if $R_1 = R_2$, equations (14) say that the form of the distribution (13) "reproduces its own parameters" ρ_1 and σ_1. Also notice that formal substitution of equations (14) into the distribution (13)—eliminating σ_1^2 and ρ_1 in favor of σ_2^2 and ρ_2—gives the same result as setting $R_1 = R_2$ and relabeling the dummy subscripts. This shows that the distribution (13) depends only on the current values of its moments ρ_2 and σ_2^2, which scale by equation (14). The choice of initial scale R_1 is quite arbitrary: the distribution $n(m)$ reproduces itself and the condensation is self-similar.

To sum up: Starting with an arbitrary initial spectrum $n_1(m)$ we considered large volumes of small overdensity and derived a new spectrum $n_2(m)$ valid at a much later time, expansion scale $R_2 \gg R_1$; extrapolating the functional form back to the original scale R_1, we have two seemingly independent miracles, (i) self-consistency of ρ_1 and σ_1^2, and (ii) self-similarity, where the distribution $n_2(m)$ does not actually depend on R_1.

b) Case of Minimum Variance: Intermediate Cases

What happens when we assume that each initial particle belongs to a regular lattice site, and its position is randomly only on a scale less than a lattice dimension $\sim [mn(m)]^{-1/3}$? The variance of mass is now no longer a sum over the entire volume, since the interior of a volume contains a predictable number of cells. Only cells intersecting the surface, where the particle might be inside or outside, contribute to the variance sum. The thickness of the surface area varies as the lattice dimension with m. The variance of a volume V much larger than a lattice cell is readily seen to be

$$\Sigma^2(V) = \xi^2\left[\int_0^\infty m^{5/3} n^{2/3}(m)dm\right] V^{2/3}, \tag{15}$$

where ξ is a number of order unity which characterizes details of the shape of V, precise definition of lattice cells, etc. (We will see below that ξ takes on a canonical value.) The dependence $\Sigma^2(V) \propto V^{2/3}$ represents a lower bound on the variance, as $\Sigma^2(V) \propto V$ represented an upper bound. It is mathematically convenient to derive a whole family of intermediate self-similar distributions at the same time as we obtain the lower-bounding case, although only the two bounding cases have physical interpretations. Suppose that $\Sigma^2(V) \propto V^{2\alpha}$, so that the bounded range of α is $\frac{1}{3} \le \alpha \le \frac{1}{2}$. On

Lower bound on Σ: particles are confined in "cells"
⇒ Σ² ~ V^(2/3) (α = ⅓)

Upper bound on Σ: particles completely random
⇒ Σ² ~ V (α = ½)

dimensional grounds, $\Sigma^2(V)$ must be related to $n(m)$ by

$$\Sigma^2(V) = \xi^2\left[\int^{\infty} m^{1+2\alpha}n(m)^{2\alpha}dm\right]V^{2\alpha} \tag{16a}$$

$$\equiv \sigma^2 V^{2\alpha}\,, \tag{16b}$$

which includes equations (7)–(8) and (15) as special cases; here again ξ is a constant of order unity. The derivation now goes almost exactly as in § II*a*, equations (10)–(13). We have

$$\delta_* \equiv \frac{[\Sigma^2(V)]^{1/2}}{M} = \frac{(\sigma^2 V^{2\alpha})^{1/2}}{\rho V} = \frac{\sigma}{\rho^\alpha}M^{\alpha-1}\,, \tag{17}$$

whence

$$P = \tfrac{1}{2}\,\mathrm{erfc}\left[\frac{R_1\rho^\alpha M^{1-\alpha}}{2^{1/2}R_2\sigma}\right] \tag{18}$$

and

$$n_2(M) = \frac{2}{M}\rho_1\left(\frac{R_1}{R_2}\right)^3\frac{dP}{dM}$$

$$= (2\pi)^{-1/2}2(1-\alpha)\left(\frac{R_1}{R_2}\right)^4\frac{\rho_1{}^{1+\alpha}}{\sigma_1}M^{-1-\alpha}$$

$$\times \exp\left[-\frac{1}{2}\left(\frac{R_1}{R_2}\right)^2\frac{\rho_1{}^{2\alpha}}{\sigma_1{}^2}M^{2(1-\alpha)}\right]. \tag{19}$$

Calculating ρ_2 and σ_2 from definitions in equations (14a) and (16b) gives

$$\rho_2 = \rho_1(R_1/R_2)^3\,, \tag{20a}$$

$$\sigma_2{}^2 = \xi^2 f(\alpha)\sigma_1{}^2(R_1/R_2)^{2(3\alpha-1)}\,, \tag{20b}$$

where

$$f(\alpha) = (2\pi)^{-\alpha}[2(1-\alpha)]^{2\alpha-1}\alpha^{-(1+\alpha)}\Gamma(1+\alpha)\,. \tag{21}$$

We now see that in a self-similar distribution, the constant ξ takes on a canonical value $\xi = [f(\alpha)]^{-1/2}$, so that equation (20b) reads

$$\sigma_2{}^2 = \sigma_1{}^2(R_1/R_2)^{2(3\alpha-1)}\,. \tag{22}$$

In this case ξ varies smoothly from 3.1370 to 1.0 in the range $\frac{1}{3} \le \alpha \le \frac{1}{2}$. We have not found a physical interpretation for the constant ξ.

It is easily verified that the distribution (19) has the same self-similar properties that we found in the previous special case $\alpha = \frac{1}{2}$; at $R_1 = R_2$ the parameters ρ_2 and σ_2 reproduce themselves; and substitution of equations (20a) and (22) into (19) show that the choice of initial scale R_1 is totally arbitrary.

c) Properties of the Distributions

The functional form (19) of the linear analytic candidate for a self-similar distribution distills to

$$n_R(M) \propto M^{-1-\alpha}\exp\left[-\text{const.}\times\left(\frac{M^{1-\alpha}}{R}\right)^2\right] \tag{23}$$

~ { M^(-1-α) for small M
exp for large M

hi Σ
lo Σ

(subscript R now denotes the expansion scale). The parameter α is determined by the degree of order at some initial scale, the dependence of mass variance on volume.

The exponential gives a "typical" mass at any expansion scale,

$$M \propto R^{1/(1-\alpha)} \propto \begin{cases} R^2\,, & \alpha = \frac{1}{2} \quad \text{high } \Sigma \\ R^{3/2}\,, & \alpha = \frac{1}{3} \quad \text{low } \Sigma \end{cases} \tag{24}$$

low Σ: n ~ M^-1.33
high Σ: n ~ M^-1.5

(cf. Peebles 1965). Below the exponential cutoff the number density varies as a power law, $M^{-1-\alpha}$. The total number of particles diverges at the low mass end, but the cumulative total mass $\mathfrak{M}(M)$ below a given mass M is quite finite,

$$\mathfrak{M}(M) \propto \int_0^M mn(m)dm \propto M^{1-\alpha}\,, \quad M < M_{\text{cutoff}}\,. \tag{25}$$

The case of minimal variance $\alpha = \frac{1}{3}$ thus gives a cumulative spectrum which is slightly more concentrated near the cutoff mass than does the maximum variance case $\alpha = \frac{1}{2}$. The "median" mass $[\mathfrak{M}(M) \equiv 0.5\mathfrak{M}(\infty)]$ is in both cases roughly a quarter the cutoff value. (The "mean" mass is not well defined due to the low mass number divergence; equations [4] and [5] are not literally applicable.)

How does the linear size l of a condensation vary with its mass—or equivalently, how does the binding energy U vary? We must distinguish two different cases: First, condensations which were formed at the same expansion scale form with the same density, roughly the average density of the Universe at that epoch. For these "synchronic" condensations we have

$$l \propto M^{1/3}\,, \qquad U \equiv GM/lc^2 \propto l^2\,. \tag{26}$$

Second, if the comparison is between typical condensations of different expansion scales (i.e., between a current condensation and its older constituent subunits), we have for these "diachronic" condensations $M \propto R^{1/(1-\alpha)}$ and $\rho \propto R^{-3}$ whence

$$l \propto (M/\rho)^{1/3} \propto M^{(4-3\alpha)/3} \propto \begin{cases} M^{5/6}\,, & \alpha = \frac{1}{2} \\ M\,, & \alpha = \frac{1}{3}\,; \end{cases}$$

$$U \propto M^{\alpha-1/3} \propto \begin{cases} \text{const.}\,, & \alpha = \frac{1}{3} \\ l^{1/5}\,, & \alpha = \frac{1}{2}\,. \end{cases} \tag{27}$$

A final point is to verify that the condensation remains Newtonian, i.e., that the scale of clumpiness does not overtake the light horizon: From equation (27) we have $l \propto R^{(4/3-\alpha)/(1-\alpha)}$ while $L_h \propto ct \propto R^{3/2}$. Thus, if initially $l \ll L_h$, the relation continues to hold for any choice of $\alpha \ge \frac{1}{3}$.

IV. NUMERICAL *N*-BODY EXPERIMENTS

How reliable are the linearized model predictions of the previous section? This is not an easy question, since

① At a given time, ℓ ~ M^1/3

② At a given mass, ℓ of the typical mass M* evolves with time as ℓ ~ M^5/6 or M the assumptions which went into the calculation are not easily justified, especially the use of energy arguments to predict the formation of large clusters at distant times.[1]

The most testable predictions of the model are those of equations (24) and (25), how the typical—or cutoff—mass increases with expansion factor, and at a given expansion factor what the detailed shape of the mass spectrum is below the cutoff. The detailed functional form above the cutoff is not readily testable because the amount of mass there is negligible. Both of the testable predictions are dependent on the value of α, and one caveat should be introduced here: The value of α was defined by the variance on asymptotically large volumes V; but a given scale does "most" of its condensing when it is the "next" scale to condense, and its statistical properties at this time are dominated by the nonlinear effects of a scale only slightly smaller than its own. Thus, in the full nonlinear problem, the effective value of α might not be tied to the large-scale statistics. It is an artifact of the linear approach which does so tie it in the results of § III.

Our attitude toward the analytic results is this, then: A reasonable working hypothesis is that the condensation process proceeds with some effective α between $\frac{1}{3}$ and $\frac{1}{2}$; but we should turn to numerical experiments to see what this effective value is, or indeed if any value of α gives viable results, qualitatively consistent with the analysis predictions.

A physical interpretation of the "effective α" is this: On dimensional grounds one expects the peculiar velocities of condensation to remain of order $GM^{4/3}n(M)^{1/3}$ (crossing time to nearest neighbor a dynamical Kepler time). However, the dimensionless number in front of this quantity is crucial: a value substantially greater than 1 will tend to randomize positions on larger than the *next* condensation scale, and might favor a change in α toward $\frac{1}{2}$; a value substantially less than 1 would decrease randomization and might favor $\alpha \rightarrow \frac{1}{3}$.

We have done several numerical 300-body experiments, and two experiments with 1000 bodies which are described here. At an initial time (which, like all quantities, can be scaled arbitrarily since there are no free dimensionless parameters) the bodies, all of equal mass, are distributed in a unit sphere. Initial velocities are taken to be linearly related to position by a perfect Hubble law, so that initial peculiar velocities vanish. Gravitational forces are obtained by summing over pairs of particles (however, large-angle scatters are eliminated by cutting off the force at small radii). Positions and velocities are updated alternately, so that the differencing is effectively second order. The spherical boundary is expanded as if in a homogeneous Friedmann cosmology. Particles encountering the boundary are specularly reflected; however, the resulting edge effects introduced seem to be unimportant.

Condensations are defined as roughly spherical regions whose mass density exceeds the smooth average by a factor of 10 (see Appendix for a "derivation" of this criterion). Specifically, the following algorithm produces the list of condensations: (1) For the ith mass point a list of radii to all particles is produced and ordered by radius. (2) The largest radius containing the required overdensity is the radius of the ith "candidate." The radius may be zero. (3) The list of candidates is sorted by radius. (4) Any candidates whose centers lie within the radius of a larger-radius candidate is eliminated. (5) The surviving candidates are the identified condensations. They are sorted by mass into the final list.

The algorithm will not correctly identify condensations which are too aspherical. However, its "carving up" of these into overlapping spheres will count many mass points twice. Thus the ratio of summed condensation masses to total mass is a measure of the algorithm's consistency; this ratio is typically less than 1.1, and the calculations is halted if it exceeds 1.2. A condensation analysis takes $\sim 200N^2$ operations and is a major contribution to running time.

The two 1000-body experiments differed only in the initial positions of the mass points. In Experiment A, the points were distributed randomly and independently in the unit sphere. According to the calculation of § III, this should bias the system toward a self-similar distribution with $\alpha = \frac{1}{2}$. For Experiment B, a cube enclosing the unit sphere was divided into cubic cells. Each cell received one mass point, whose position was randomized within the cell; particles outside the unit sphere were then deleted. This initial condition favors the slowest possible development of condensations.

Figures 1*a* and 1*b* show the computed evolution of the two experiments, the cumulative mass distribution $\mathfrak{M}(M)$ as determined by the condensation-finding algorithm at subsequent expansion scales. In both experiments, the sharp corner of the initial mass distribution (all points of same mass) quickly erodes away to a more or less straight line on a log-log plot. The line subsequently evolves parallel to itself as the distribution moves to larger and larger clusters. (Attention should be focused on the mass range $\sim 5 < M < \sim 50$; smaller masses continue to be influenced by the discrete points of fixed unit mass, while larger masses start to encroach on the $N = 1000$ limit.) halo finder

The overall behavior seems to indicate that some kind of self-similar distribution is evolved quite rapidly in both experiments. But is it of the predicted functional form? Figures 2*a* and 2*b* show how the typical mass of condensations increases with expansion factor. Shown is the mass M such that $\mathfrak{M}(M)/\mathfrak{M}(\infty)$ takes on one of the three values 0.3, 0.5, 0.8. In figure 2*a*, straight lines are drawn with the slope 2 predicted

30, 50, and 80 percentiles

[1] See note added in proof, below.

Exp. A: completely random ⇒ α → ½
Exp. B: particles in cells ⇒ α → ⅓

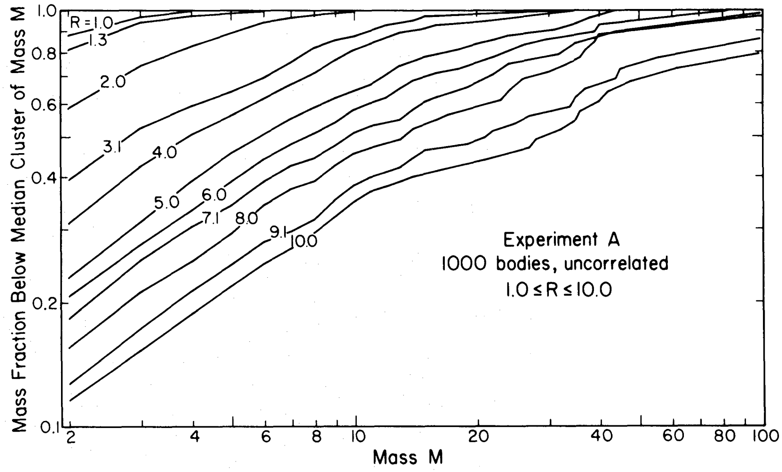

FIG. 1*a*

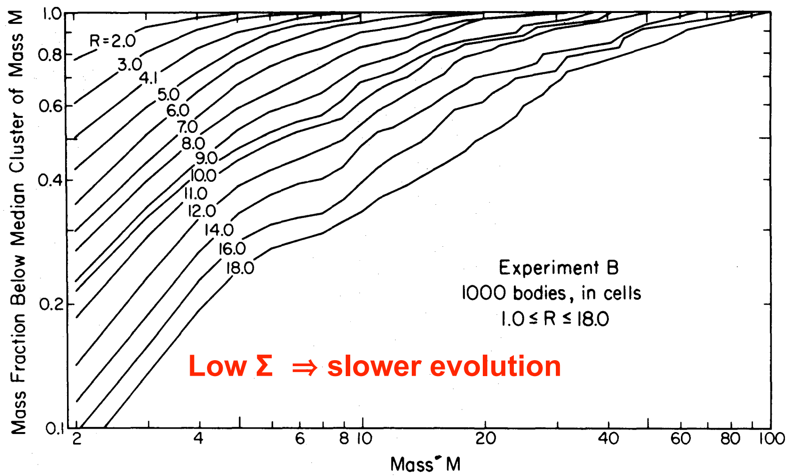

FIG. 1*b*

FIG. 1.—Numerical experiments: Mass spectrum of condensed clusters as a function of cosmological expansion factor R. Starting with 1000 particles of unit mass in an expanding Friedmann cosmology, both experiments develop a spectrum of larger condensations. At large expansion factors, the shape of the spectra seems to become fixed, and the curves move in parallel to the right (increasing cluster mass); this indicates that a self-similar condensation process has set in. The two experiments shown differ in the initial statistical distribution of masses. Experiment A has all points randomly in the volume; Experiment B starts in a more highly ordered state. (See text for details.)

by the theory (eq. [24]). It is fairly clear that the asymptotic evolution is consistent with the $\alpha = \frac{1}{2}$ analytic theory. The case of Experiment B (fig. 2*b*) is not so clear. The initial evolution is clearly slower than in A; however, for late times and large clusters, the curve appears to have turned up to a slope of about 2, corresponding to an "effective" α of $\frac{1}{2}$.

What about the predicted power-law behavior of $\mathfrak{M}(M)$ below the exponential cutoff? Here we have computed power-law fits of the form $\mathfrak{M}(M) \propto M^{\gamma}$ at the four largest expansion factors of each experiment (and for $6 \le M \le 80$). The average values of γ, rms

cum. mass

$\gamma = (1-\alpha)$

deviation of one fit from the mean of four, and predicted values are shown in table 1.

We cannot be certain that our numerical calculations are free from systematic error. Nevertheless, we draw the following tentative conclusions:

i) Experiment A seems to have developed consistently with the linearized analytic prediction. In particular, the increase of typical cluster mass with expansion factor $M \propto R^2$ seems quite accurately determined (fig. 2*a*), and the shape of the mass spectrum below the cutoff is at least qualitatively right.

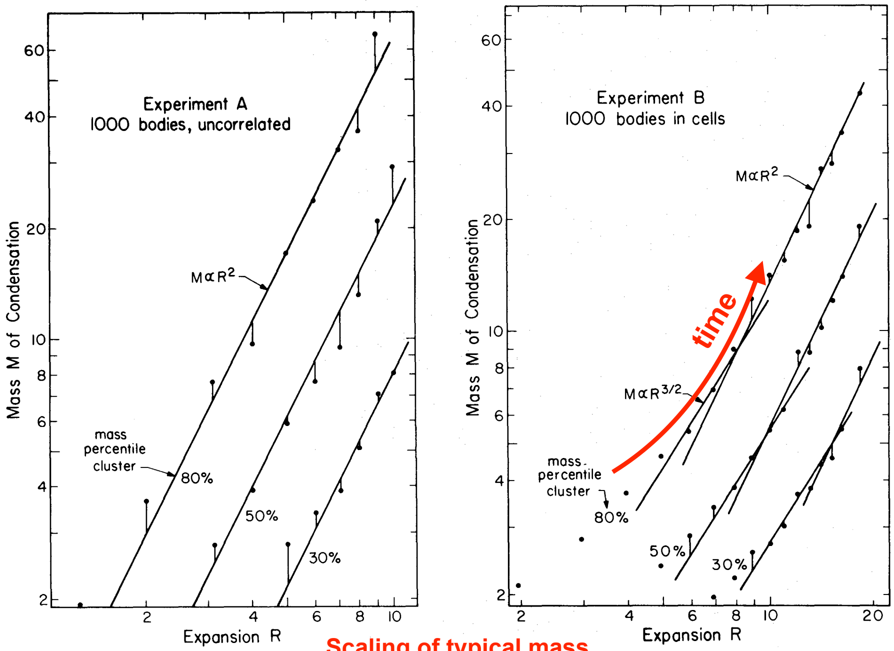

FIG. 2*a* FIG. 2*b*

FIG. 2.—Numerical experiments: Growth of the mean, 80th percentile and 30th percentile cluster with expansion factor. Experiment A appears consistent with the relation $M \propto R^2$ which is predicted by the linear perturbation theory. Experiment B does not follow its slower linear prediction; rather, it appears to evolve toward $M \propto R^2$. If correct, this shows that nonlinear interactions on the current condensation scale can "bootstrap" the condensation scale up in mass faster than the linear perturbation theory predicts. (See text for details.)

ii) Experiment B, which was biased initially toward the slower-developing case $\alpha = \frac{1}{3}$, clearly seems to have bootstrapped itself to a larger "effective" value, consistent with $\alpha \rightarrow \frac{1}{2}$. This is most clearly evident in figure 2*b*. If true, this indicates that the growth of the "current" scale by nonlinear clumping effects dominated the linear growth of density perturbations of larger scales: a given region condensed when the current scale had bootstrapped its way up to it in size, rather than when the density contrast in the linear theory would have been order unity (which would have been much later). This is a startling result, and requires careful further investigation.

Unfortunately we cannot push the evolution farther in expansion scale in these experiments, because the largest clusters are not negligibly small compared with the total of 1000 points. In the future, we hope to pursue a more definitive program of related numerical work.

V. COMPARISON WITH OBSERVED FEATURES OF GALAXIES AND CLUSTERS

What observed condensations might be part of a self-similar hierarchy like the one of our model? Equations (26) and (27) make it possible to say something about this question, but they also point out a shortcoming of the model. In comparing the sizes and binding energies of condensations we must know whether the condensations formed at the same time or at different times. Here we must make a rather dangerous assumption: We assume that the different observed classes of condensations (globular clusters, galaxies, clusters of galaxies) are suitable for diachronic comparisons, but that the objects within a single class formed synchronically. This assumption is dangerous largely because it is outside the framework of the model that is to be tested. The model does not explain why the different classes should be as distinct as they

TABLE 1

FITTED PARAMETERS FROM FIGURES 1 AND 2

	LINEARIZED MODEL		NUMERICAL EXPERIMENT	
	α	γ	α	γ
Experiment A.....	1/2	1/2	≈1/2	0.44 ± 0.02
Experiment B.....	1/3	2/3	≈1/2	0.52 ± 0.04

Better to compare the most luminous, in order to be sure to see all objects.

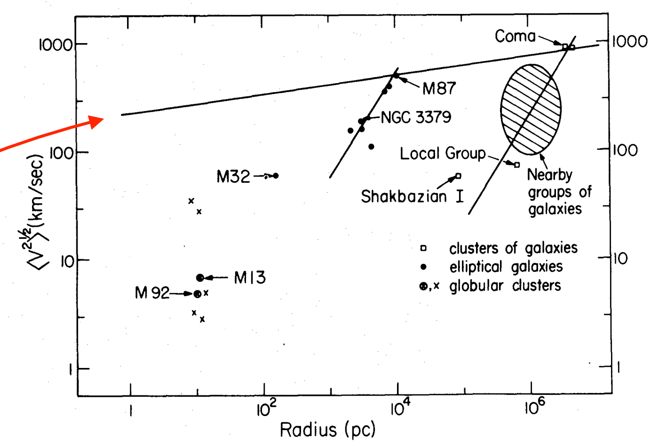

FIG. 3.—Measured velocity dispersions for condensed objects as a function of their radius. The predictions of the self-similar condensation model are shown. Steep lines have slope 1, indicating the constant density characteristic of synchronic condensations. The flat line has a slope of 1/10, and should connect the upper cutoff masses characteristic of condensations at different epochs (assuming $\alpha = 1/2$; see text for details). Data have been taken from Oemler (1973), Rood *et al.* (1972), Robinson and Wampler (1973), Fish (1964), Minkowski (1962), Morton and Chevalier (1973), Richstone and Sargent (1972), Griffin (1972), Wilson and Coffeen (1954), and Allen (1963). The velocity dispersion for globular clusters marked by $\times$'s was computed assuming the same mass-to-luminosity ratio as M92 and M13. The globular clusters in any case do not fit the model, but the fit for galaxies and clusters of galaxies seems satisfactory.

are observed to be. Our point of view is to suppose that physical processes beyond the scope of the model prevented or disrupted condensations on scales between the observed classes, but that the observed classes fossilize at least *some* epochs of a self-similar development.

Figure 3 tests the model against observed radii and velocity dispersions (equivalent to binding energies or to masses) of globular clusters, elliptical galaxies, and clusters of galaxies. The line through each class has the slope predicted by equation (26). In the case of galaxies, the existence of a power law relating size to mass was first pointed out by Fish (1964). Fish's suggested relation is $M \propto l^2$ corresponding to $U \propto l$. The model here predicts a constant density relation $M \propto l^3$ or $U \propto l^2$. The present data do not seem to exclude either possibility.

The line connecting the classes in figure 3 has the slope predicted by equation (27) with $\alpha = \frac{1}{2}$, and is drawn through the most massive galaxy, M87. The line passes rather well through the most massive clusters of galaxies, and we count this as a success for the model.

The globular clusters do not fit the model; they are not massive enough. An alternative hypothesis—that they have evolved in size—is not plausible for two reasons: (i) They would have had either smaller radii or much larger velocity dispersions in the past, and (ii) the epoch of their condensation would have to be before recombination, which is probably impossible.

Next, we look at the detailed mass spectrum predicted by the model. For galaxies we compare this with the observed luminosity distribution of galaxies in the central region of the Coma cluster. (The assumption of a constant mass-to-luminosity ratio is not strictly correct, but this should not introduce a large systematic error in a system composed primarily of elliptical galaxies.) Figure 4 shows results of recent measurements by Oemler (1974) along with best fits to the self-similar functional form (23) for the extreme values $\alpha = \frac{1}{2}$ and $\alpha = \frac{1}{3}$. The two values of α fit about

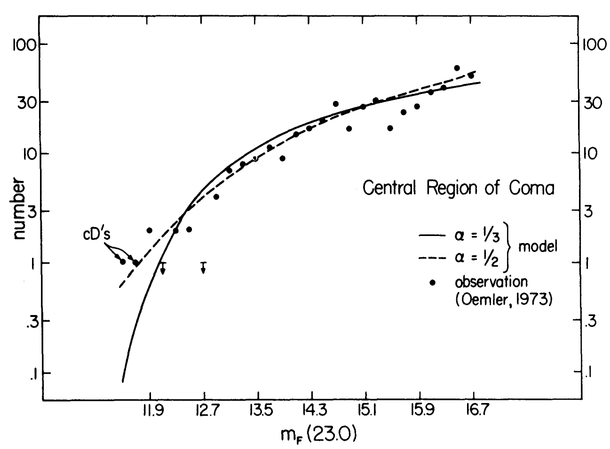

FIG. 4.—Luminosity distribution of galaxies in the central 90′ square of the Coma cluster. The data (Oemler 1973) plots the number of galaxies per 0.2-mag interval, corrected for background, against red (F) magnitude to 23.0-mag isophotes. The solid and dashed lines are the predictions of the self-similar condensation model for the two extreme values of the parameter α, as derived from the linearized treatment of § III. The predictions are normalized in total number, and in mass-to-luminosity ratio, to obtain the best fit, but have no other free parameters. Both data and model show a sharp cutoff. See text.

equally well: the value $\alpha = \frac{1}{2}$ accounts better for the slope at faint magnitudes, while the value $\alpha = \frac{1}{3}$ fits better at brighter magnitudes. The sharp exponential cutoff which is characteristic of the self-similar distribution can explain Abell's (1965) observation of a characteristic cutoff magnitude for rich clusters of galaxies. It also supports the statistical explanations of Peebles (1969) and Peterson (1970) regarding the narrow magnitude distribution of the brightest cluster members.

A similar comparison with the distribution of clusters of galaxies would be desirable, but greater care must be taken to ensure a complete sample free from systematic errors, and a consistent definition of cluster mass. Very recent data on rich clusters by Oemler (private communication) are consistent with luminosity varying as l^3.

VI. SHORTCOMINGS OF THE MODEL, COSMOLOGICAL IMPLICATIONS, DISCUSSION

A notable omission in the predictions of the self-similar model (already mentioned above) is an explanation of the distinctness of observed condensation scales. One would expect to observe only the initial seed condensations and clusters of them. Galaxies should be composed of smaller subunits. However, there is an obvious loophole, the appeal to more realistic astrophysics: gas dynamics, radiative cooling, etc. It is possible that one does not see the subunits of galaxies because star formation had not yet occurred in them and—once condensed to galactic volumes—they collided and dissipated themselves. One then does not see single condensations of a mass between galaxies and clusters of galaxies for the same reason: these scales were collisionless and preserved the identity of their subunits, which we observe as galaxies. This raises one interesting observational question: Are there bound subclusters within rich cluster of galaxies? (Projection onto the sky would tend to hide such subclustering, and it does not seem to be ruled out at present.) These matters are rather clearly beyond the scope of this paper. The suggestion that the mass scale of galaxies is marked by a cooling process is not new: Fish (1964) (and see Lynden-Bell 1969) first pointed out that typical sizes l and densities ρ of satisfy $\rho l \approx 2$ g cm^{-2}, suggestive of a process involving free particle opacity.

We turn now to the most serious flaw in the model. Since recombination at $z \sim 1500$, a self-similar distribution can have increased its typical mass only by 6×10^4 ($\alpha = \frac{1}{3}$) to 2×10^6 ($\alpha = \frac{1}{2}$). If we take the current fully condensed scale to be clusters of ~ 100 to ~ 300 galaxies, the "seeds" at recombination must have been in the range of 3×10^7 to 3×10^9 $M_\odot$, with our numerical experiments favoring the smaller value. These masses are unpleasantly large. The seeds must have been isothermal, rather than adiabatic, since adiabatic perturbations smaller than $\sim 10^{12}$ $M_\odot$ will have damped out before recombination (Silk 1968; Peebles and Yu 1970). The large seeds are therefore probably not ruled out by the isotropy of the cosmic microwave background. (In any case the angular size of the seeds, seen at redshift ~ 1500 from the present, will be of order seconds of arc to $\lesssim 1'$.)

Assumes Mgal ~ 3e11

How are these seeds to have formed? It is tempting to invoke the (largely unknown) properties of an early chaotic universe. Strong shocks could have generated isothermal perturbations; perhaps large-scale gravitational waves can be important (Rees 1971). If we take seriously another suggestion by Rees (1972) that the observed radiation entropy might have been created by chaotic processes at rather late epochs, we might go so far as to imagine that self-similar condensation was possible at an earlier epoch; early condensations would subsequently become frozen into the developing photon-dominated gas; if not attenuated (e.g., if isothermal), they would be available as seed masses at later recombination. Another possibility is suggested by the work of Kundt (1970), Carlitz (1972), and others on the condensation properties of a gas with the very soft Hagedorn equation of state, for which pressures might be unimportant at the earliest epochs and self-similar condensation might be possible.

The point is that, other than their necessary large characteristic mass, there are no restrictions on the details of the mass spectrum of seed masses: any spectrum will generate the self-similar mass distribution which (we suggest) agrees rather well with the observed mass distribution of galaxies. This decoupling of early processes (which make seeds) from late processes (which make galaxies and clusters) is an important feature of the model.

We can also apply this argument at later epochs: Suppose that the seeming self-similarity of the galactic mass spectrum is coincidental, and that galaxies are actually formed by an entirely different process (but at redshifts ~ 20). In this case, a self-similar development would start from the galaxies; since the seeds have roughly the correct spectrum to start with, the development would be indistinguishable from one which had already condensed through several scales. The predictions of the model about clusters of galaxies, therefore, depend only on the existence of dynamically independent protogalaxies at $z \sim 20$, not on the process of their formation.

Thus far we have assumed that the constant q defined by equation (2) is ≈ 0.5, as it would be early enough in any Friedmann cosmology. In concluding, we examine briefly the effects that would accompany the transition to a $k = 1$ ($q \to \infty$) or $k = -1$ ($q \to 0$) cosmology. Qualitatively, a value of $q < 0.5$ tends to retard the current scale of condensation and subsequent scales. This is because the expansion time h^{-1} becomes shorter with respect to the Keplerian collapse time $(Gm_* n_*)^{-1/2}$. Likewise, a value $q > 0.5$ accelerates the condensation of the largest scales. This interaction of q with the condensation of the largest current scale leads to a means, in principle, of determining q_0, the present value of q: comparison of the size of the largest ("fully condensed") objects with the distance between them is essentially a measure of q_0. (This method is somewhat related to that of Gunn and Gott 1972, where the infall of matter onto a single cluster of galaxies was considered.)

If $q_0 > 0.5$, fully then condensed (e.g., virialized) objects should be almost space-filling. Although the extent of superclustering is controversial (de Vaucouleurs 1971), it seems observationally unlikely that this scale can be fully condensed in our sense; we turn our attention to the case $q_0 < 0.5$.

Let R_c be the maximum radius of expansion of the largest condensed scale, and let t_c be the epoch of the maximum. If t_0 is the present epoch, we have $\sim\frac{1}{2} < t_c/t_0 < 1$; if the lower limit were violated, a *larger* scale would have had time to condense. If R is the current radius of the condensation, we have

$$R \approx R_c\,, \qquad \text{for} \qquad t_c/t_0 \approx 1\,;$$
$$R \approx R_c/2\,, \qquad \text{for} \qquad t_c/t_0 \approx \tfrac{1}{2}\,. \tag{28}$$

In the first case the condensation is just now at maximum expansion; the second case approximates the subsequent effects of virialization, violent relaxation, etc. (Gott 1972). The characteristic distance between condensations is twice the radius to which the cluster lump would have expanded in a perfect Friedmann universe. For the universe we have (e.g., Weinberg 1972)

$$t_0 = \frac{1}{H_0}\int_0^1 \left(1 - 2q_0 + \frac{2q_0}{x}\right)^{-1/2} dx = \frac{1}{H_0} f(q_0) \tag{29}$$

where

$$f(q_0) = (1 - 2q_0)^{-1} - q_0(1 - 2q_0)^{-3/2}\cosh^{-1}(1/q_0 - 1)\,. \tag{30}$$

The condensation evolves by

$$t = \text{const.} \times \int_0^r \left(\frac{1}{x} - \frac{1}{R_c}\right) dx\,. \tag{31}$$

The constant is determined by the requirement that the cluster be comoving with the universe at the earliest times. One easily obtains

$$t_c = \frac{\pi}{2}\frac{(2q_0)^{-1/2}}{H_0}\left(\frac{R_c}{R_0}\right)^{3/2}. \tag{32}$$

Now from equations (32) and (29), we have

$$\frac{R_c}{R_0} = 2\left[\frac{f(q_0)}{(t_0/t_c)\pi}\right]^{2/3} q_0^{1/3}\,. \tag{33}$$

This equation is applied as follows: the largest fully condensed scale in the universe is roughly that of clusters of galaxies, which may or may not have had time to virialize. If a typical cluster has a diameter of 3 Mpc, and a typical distance between clusters is 10 Mpc, we have (using eq. [28])

$$\frac{R_c}{R_0} \approx \begin{cases} 0.3 & \text{for} \quad t_c/t_0 \approx 1 \\ 0.6 & \text{for} \quad t_c/t_0 = \tfrac{1}{2}. \end{cases} \tag{34}$$

Substitution of equation (34) into (33), and solving for q_0, gives

$$\sim 0.03 < q_0 < \sim 0.5\,. \tag{35}$$

Unfortunately the lower limit is quite sensitive to the ratio 3 Mpc/10 Mpc; if we used 2 Mpc/15 Mpc, it would have come out ~ 0.003. Clearly one needs a rather more precise model to obtain good limits on q_0. Nevertheless, it is interesting that our lower limits lie in the range of the observed matter content—there was no input to the model which would have guaranteed even order-of-magnitude agreement *a priori*. With precise definitions of what a cluster is and how clusters are separated, and with better understanding of statistical condensation, it may be possible to obtain q_0 with some useful precision.

In essentially all aspects, this paper is more a prospectus for future work than a report of work in final form. The condensation behavior of a perfect gravitating gas is an interesting theoretical problem in its own right. The fact that a number of features of the self-similar condensation process seem to mimic the real world may indicate that the problem is astrophysically interesting as well. The model makes a lot of predictions with very little input; this makes it testable; this should also make it improvable (by including more realistic physical processes) without an impossible increase in the number of free parameters. Our main point is that once fully condensed objects have formed, their statistical properties and their N-body influence on larger scales cannot be ignored.

We have benefited from discussions with D. Eardley, J. Gunn, J. Kwan, D. Lee, and S. Teukolsky, and we especially thank A. Oemler for making his unpublished data available to us. We thank J. M. Bardeen and P. J. E. Peebles for comments on the manuscript.

APPENDIX

For the special case of spherical symmetry, it is possible to derive exactly some results which were used above as estimates for the more complicated general case. A spherical mass M within an expanding cosmology evolves according to the Newtonian relation

$$t(r) = \int_0^r dr'\left(\frac{2M}{r'} - \frac{2M}{R}\right)^{-1/2} \tag{A1}$$

(see, e.g., Weinberg 1972). Here $t(r)$ is the time at which the mass has radius r, and R is the maximum radius that it attains in its evolution; we have chosen units with $G = 1$. The average universe evolves by equation (A1) also, by hypothesis with $R = \infty$.

If a comoving lump is overdense by a small fraction δ at expansion scale r_1, we can compute how much the average universe has expanded, and how much the lump has expanded, when the latter is at its maximum size: For the lump we have

$$t(r_1) = \int_0^{r_1} \left(\frac{2M}{r} - \frac{2M}{R}\right)^{-1/2} dr$$

$$\approx \int_0^{r_1} \left(\frac{2M}{r}\right)^{-1/2} = \frac{\sqrt{2}}{3}\left(\frac{r^3}{M}\right)^{1/2} \tag{A2}$$

(since δ is small and $r_1 \ll R$); similarly,

$$t(R) = \int_0^{R} \left(\frac{2M}{r} - \frac{2M}{R}\right)^{-1/2} dr = \frac{\pi}{2^{3/2}}\left(\frac{R^3}{M}\right)^{1/2} . \tag{A3}$$

For the average universe we have

$$t(r) = \frac{\sqrt{2}}{3}\left(\frac{r^3}{M_0}\right)^{1/2} \tag{A4}$$

or

$$r(t) = (\tfrac{9}{2} M_0 t^2)^{1/3} . \tag{A5}$$

Thus,

$$\frac{r[t(R)]}{r[t(r_1)]} = \left(\frac{3\pi}{4}\right)^{2/3} \frac{R}{r_1} . \tag{A6}$$

Since the lump is comoving with the universe at early times, we know that $t'(r_1)_{\text{lump}} = t'(r_1)_{\text{universe}}$, or

$$\frac{2M_0}{r_1} = \frac{2M}{r_1} - \frac{2M}{R} = 2M_0(1 + \delta)\left(\frac{1}{r_1} - \frac{1}{R}\right) , \tag{A7}$$

which gives

$$1 = (1 + \delta)(1 - r/R) \tag{A8}$$

and

$$r_1/R \approx \delta . \tag{A9}$$

From equations (A9) and (A6) we conclude that a lump of overdensity δ reaches a maximum of expansion $R/r_1 = \delta^{-1}$; at that time the average universe has expanded by a factor greater by $(3\pi/4)^2 \approx 5.5$ in volume, so the lump is overdense by this amount. In the text, equation (A9) was used to obtain a lower limit of integration for equation (11). The factor $(3\pi/4)^{2/3}$ was neglected, since the "characteristic" time of a lump's condensation is defined only up to order unity in any case. The overdensity factor 5.5, rounded conservatively up to 10.0, is used as a criterion for cluster formation in the numerical work of § III.

Next suppose that a lump of radius r_1 contains an overdensity δ and is surrounded by an equally underdense spherical region. We want to know how much of the underdense region accretes onto the overdense lump at late expansion scales. Considering the mass out to a radius r_2, analogy with equation (A7) gives,

$$\frac{2M_0}{r_1} = \frac{2M}{r_2} - \frac{2M}{R}$$

$$= 2M_0(1 - \delta)\left(1 + 2\delta\frac{r_1^3}{r_2^3}\right)\left(\frac{1}{r_2} - \frac{1}{R}\right) , \tag{A10}$$

whence

$$1 = (1 - \delta)\left(1 + 2\delta\frac{r_1^3}{r_2^3}\right)\left(1 - \frac{r_2}{R}\right) \tag{A11}$$

and

$$(r_2/r_1)^3 \approx 2/(1 + R_1/R) . \tag{A12}$$

Here $R_1 = r/\delta$ is the expansion scale at which the overdense lump condenses. Equation (A12) says that at large expansion scales $R \gg R_1$ the condensation which originated in an overdense lump of radius r_1 has precisely doubled its original mass by accreting surrounding material out to radius r_2. This result was used in the text in equation (13). justify ×2 fudge factor

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Note added in proof.—In reply to the preprint version of this paper, Peebles (1974) has noted that the validity of the energy argument (§ III above) extends only to our case of maximum variance ($\alpha = \frac{1}{2}$, § III*a*) and is not correct when applied to the more ordered case $\alpha = \frac{1}{3}$ (§ III*b*). As regards the linearized predictions for the growth of condensation masses with expansion scale, his analysis—with reasonable physical assumptions—corrects our law $M \propto R^{3/2}$ to $M \propto R^{6/7}$, for $\alpha = \frac{1}{3}$. Equations (23)–(27) above are then wrong for this case. The equations are correct for the case $\alpha = \frac{1}{2}$, and the relation $M \propto R^2$ in this case is unchanged.

This correction of our analytic error, for which we are grateful, makes the numerical results of § IV above (which do not depend on § III in any case) even more striking. In figure 2*b* we note that the leftmost points of each percentile sequence actually do fit a shallower slope (such as 6/7 or, with different physical assumptions, 6/5) fairly well. The "sudden" change in slope to $M \propto R^2$, then, seems to mark the onset of a nonlinear "bootstrap" with effective $\alpha \simeq \frac{1}{2}$; we continue to favor this value for comparison with observation (as in figs. 3 and 4).