Supporting Information for ??

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1 Anisotropic random network

Distance-dependent connection probability

Theorem 1.1. *Let $G_{n,w} = (G, \Phi, a)$ be an anisotropic random graph. Define $C : [0, \sqrt{2}] \to [0, 1]$ as the distance-dependent connection probability profile of (G, Φ), that is such that $C(x)$ is the probability that for a vertex pair $(v, v') \in V(G)^2 \setminus \Delta_{V(G)}$ in distance $x = \|\Phi(v) - \Phi(v')\|$ the vertex v projects to vertex v'. Then*

$$C(x) = \begin{cases} \frac{1}{2} & \text{for } x \leq w/2 \\ \frac{1}{\pi} \arcsin\left(\frac{w}{2x}\right) & \text{for } x > w/2. \end{cases}$$

Proof. Let v, v' be a pair of vertices in $V(G)^2 \setminus \Delta_{V(G)}$ in Euclidean distance x of each other. The vector difference $\Phi(v') - \Phi(v)$ may then be written as $xe^{i\theta}$, with $0 \leq \theta < 2\pi$. We have

$$R_{-\alpha(v)} x e^{i\theta} = x e^{i(\theta - \alpha(v))}.$$

Only for suitable combination of θ and $\alpha(v)$ an edge from v to v' exists. Assuming $\alpha(v)$ fixed, we calculate the probability of connection depending on the random choice of θ. We can assume $\alpha(v) = 0$, otherwise the same argument holds for $\theta' = \theta - \alpha(v)$.

From the definition of the anisotropic random graph we obtain the necessary and sufficient conditions

$$x \cos\theta \geq 0 \quad \text{and} \quad |x \sin\theta| \leq \frac{w}{2}.$$

Choosing uniformly at random θ in the range of $[0, 2\pi)$, the first condition is satisfied with a probability of $\frac{1}{2}$. Consider for the second condition $\theta \in [0, \pi)$. We have

$$\sin\theta \leq \frac{w}{2x},$$

and for $x \leq \frac{w}{2}$ the inequality holds for all θ by definition of $\sin\theta$. In the case of $x > \frac{w}{2}$, we note that for the first condition to hold θ must already be in $[0, \frac{\pi}{2})$ and can thus write the second condition θ as

$$\theta \leq \arcsin\frac{w}{2x},$$

yielding $C(x)$ by combining the considerations above and using the symmetry of sine for θ in the third and fourth quadrant. $\square$

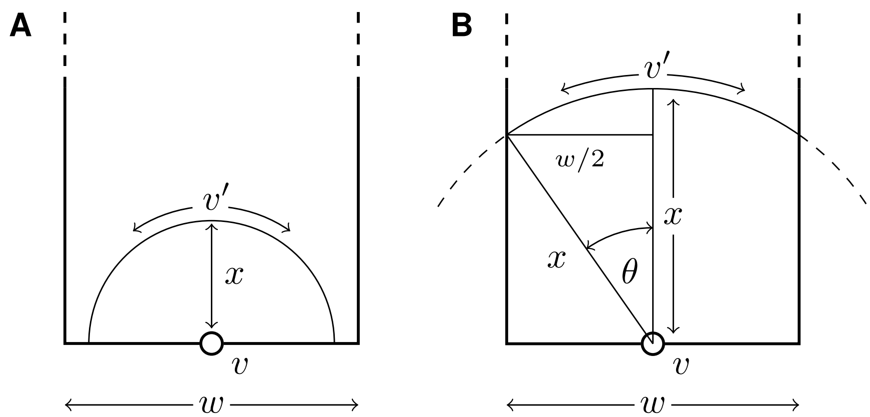

Figure S1: **Illustrating the proof of Theorem 1.1** Distance-dependent connectivity profile $C(x)$ in an anisotropic geometric graph calculated from geometric dependencies. **A)** In the case of $x \leq {}^{w}/_{2}$, target v' may be located anywhere on the shown semicircle and therefore receives input from v with probability $^{1}/_{2}$. **B)** For $x > {}^{w}/_{2}$, suitable positions for target v' are dependent on x. The geometric relation $\sin\theta = {}^{w}/_{2x}$ leads to the distance-dependent connectivity profile as described in Theorem 1.1.

2 Rewiring algorithm

Choice of rewiring margin ε

The margin ε of the rewiring algorithm determines the number of new targets available for a single to be rewired edge. The higher ε, the more targets are available (Fig. S2A). To rewire effectively, ε should be as large as possible. Similarly, for larger ε it is less likely that the rewiring algorithm is not be able to add at the current edge back into the graph without creating a parallel edge (Fig. S2B). However, to maintain the relative distribution of connected targets at any given distance in the rewired graph ε should be small (Fig. S2C). To rewire effectively while still maintaining the distance-dependent connection probability distribution, we chose $\varepsilon/E = 0.05$ as the relative rewiring margin throughout the study. Here it is $E = 296$ (??) the length of the square network area.

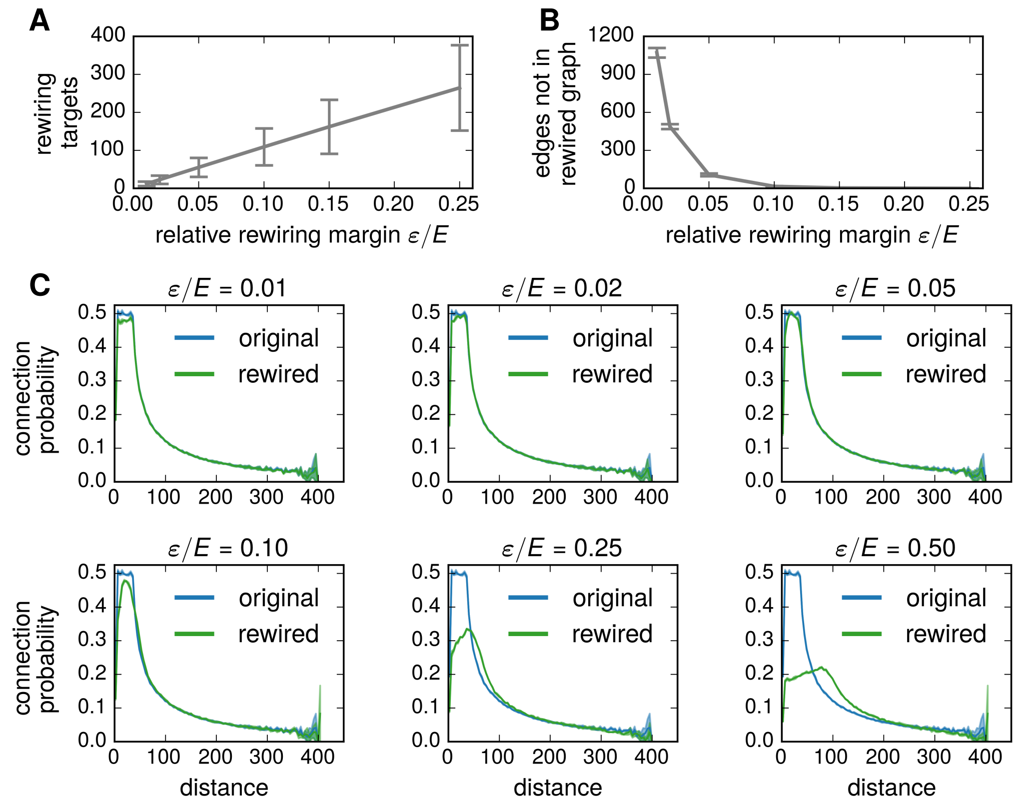

Figure S2: **Determining the rewiring margin ε.** For three anisotropic networks ($N = 1000, w =??, E = 296(??)$), the statistics of the rewiring algorithm were recorded for different rewiring margins ε. **A**. With increasing ε, the average number of available rewiring targets for a single edge increases. The number of rewiring targets was recorded for each edge and then averaged for every graph. Error bars show the average standard deviation for the number of targets within a network. **B**. Number of edges that couldn't be rewired and are not included in the rewired network decreases with increasing rewiring margin. Error bars show the standard error of the mean. **C**. Distance-dependent connection probability before (blue) and after rewiring (green). The larger ε, the more the distance-dependent connection probabilities differ in the rewired network from the original profile. For each graph, distances between connected were extracted and binned ($n = 100$) and divided by the frequency of the distance occurring in the graph. The lighter areas around the curves represent SEMs across the three networks.

3 Network motifs of three neurons

Three-neuron motif statistics

There are 16 non-isomorphic 3-motifs in simple directed graphs. In accordance with the study of Song et al., the patterns were labeled 1 to 16. Assuming independence, the expected distribution for the random variable X, that maps three random vertices $v_1 \neq v_2 \neq v_3$ in a graph G to the $n \in \{1, 2, \ldots, 16\}$, labeling the isomorphism class of the full subgraph with vertex set $\{v_1, v_2, v_3\}$ in G as above, can be obtained from pair connection statistics. In anisotropic networks we found that the probabilities of occurrence are

$p_u = 0.791336$	for unconnected pairs,
$p_s = 0.184151$	for pairs with a single connection and
$p_r = 0.024513$	for reciprocally connected pairs.

We denote with $p_{\bar{s}} = p_s/2$ the probability to find a single connection from v_1 in v_2 in a vertex pair (v_1, v_2). The probability of occurrence of the motif with label "8" is then the product of the probabilities of it constituents multiplied with a factor

$$\mathbf{P}(X = 8) = 6\, p_u p_s p_r,$$

where the factor 6 is determined by the number of different *labeled* graphs belonging to the isomorphism class. The distribution of X for the remaining motifs is given by

$$\begin{array}{lll}
\mathbf{P}(X=1) = p_u^3 & \mathbf{P}(X=\ \ 6) = 6p_{\bar{s}}^2 p_u & \mathbf{P}(X=12) = 3p_{\bar{s}}^2 p_r \\
\mathbf{P}(X=2) = 6p_u p_u p_{\bar{s}} & \mathbf{P}(X=\ \ 7) = 6p_{\bar{s}} p_u p_r & \mathbf{P}(X=13) = 6p_{\bar{s}}^2 p_r \\
\mathbf{P}(X=3) = 3p_u p_u p_r & \mathbf{P}(X=\ \ 9) = 3p_r^2 p_u & \mathbf{P}(X=14) = 3p_{\bar{s}}^2 p_r \\
\mathbf{P}(X=4) = 3p_{\bar{s}}^2 p_u & \mathbf{P}(X=10) = 6p_{\bar{s}}^3 & \mathbf{P}(X=15) = 6p_{\bar{s}} p_r^2 \\
\mathbf{P}(X=5) = 3p_{\bar{s}}^2 p_u & \mathbf{P}(X=11) = 2p_{\bar{s}}^3 & \mathbf{P}(X=16) = p_r^3.
\end{array}$$

Verification of the distribution

There $4^3 = 64$ possible combinations and summing up the coefficients for motifs 1-16 yields exactly this number.

It is

$$\sum_{i=1}^{16} \mathbf{P}(X = i) = (p_u + 2p_{\bar{s}} + p_r)^3 = 1$$

as expected.