Calculus 2 Workbook
Version 4

A Companion to *Active Calculus*

Matthew Boelkins, et. al., ©2012-2019

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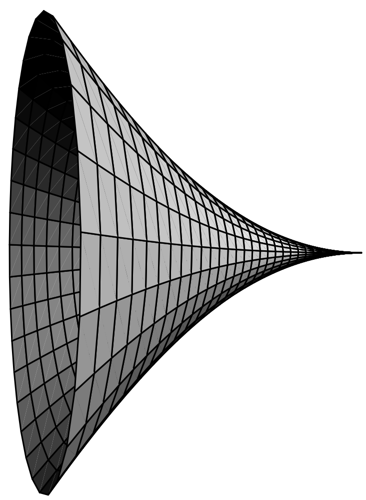


Note to Students: (Please Read) This workbook contains examples and exercises that will be referred to regularly during class. Please purchase or print out the rest of the workbook before our next class and bring it to class with you every day.

1. **To Print Out the Workbook.** Go to the Canvas page for our course and click on the link "Math 211 Workbook", which will open the file containing the workbook as a .pdf file. Most modern printers should be able to print the workbook correctly.

2. **To Purchase the Workbook.** Go to the Sonoma State University campus bookstore, where the workbook is available for purchase. The copying charge will probably be between \$10.00 and \$20.00. Please be aware of the following issues: (1) The bookstore may not have the workbooks on their shelves during the first couple of days of classes, and (2) The bookstore's printed version of the workbook sometimes contains errors, particularly in the graphs. You are strongly encouraged to compare the graphs in my electronic file with the bookstore's printed copies before your purchase.

Table of Contents

[1] This material appears in Section 4.3 of the text.

Math 211 – Preliminary Review Problems

Note. The following problems deal with review topics that should already be familiar to you from previous algebra and calculus courses. Being able to do problems similar to those below without the aid of a calculator will be important to your success in Math 211.

1. A ball is thrown upward from the top of a building at time $t = 0$ seconds. Its height above the ground after t seconds is given by $s(t) = -4.9t^2 + 20t + 10$ meters. Find the height of the ball when it is moving upward at 10 meters per second.

2. The population of the world is given approximately by $f(t) = 7.38e^{0.0119t}$ billion people, where t is the number of years after the beginning of 2015. According to this model, when will the world's population be growing by 100 million people per year, and what will be the world's population at that time?

3. Find y' in parts (a) - (f) below.

   (a) $y = e^2 + e^{2x}$

   (b) $y = x^e \cdot e^x$

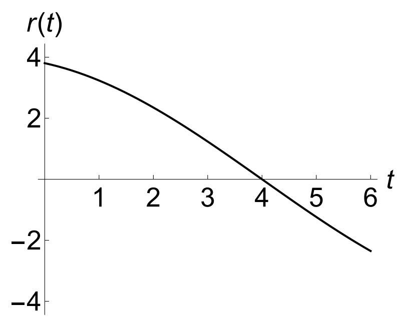
   (c) $y = \cos\sqrt{1 + x^2}$

   (d) $y = \dfrac{2 + \ln x}{2 - \ln x}$

   (e) $y = xg(x^2)$

   (f) $y = x \arctan x$

4. On a winter day in Fort Collins, Colorado, the temperature is $0°$ Celsius at 12:00 noon. The rate at which the temperature is changing t hours after noon is given by $r(t) = 4\cos\left(\frac{\pi}{10}(t+1)\right)$ degrees Celsius per hour. Answer the following questions about the Fort Collins weather between the hours of 12:00 noon and 6:00 p.m.



   (a) When is the temperature decreasing the fastest, and how fast is it decreasing at this time?

   (b) What is the maximum temperature reached, and when does it occur?

   (c) What is the temperature at 6:00 p.m.?

5. Calculate the following integrals.

   (a) $\displaystyle\int \sin x \cos x \, dx$

   (b) $\displaystyle\int x \sin(x^2 + 5) \, dx$

   (c) $\displaystyle\int_1^3 \frac{e^{1/x}}{x^2} \, dx$

   (d) $\displaystyle\int_1^4 \frac{2x - 3}{x^2} \, dx$

   (e) $\displaystyle\int \frac{e^x}{e^x + 1} \, dx$

Section 5.3 – Integration by Substitution

The Substitution Rule. If f is a function with a known antiderivative, we can choose $u = g(x)$ to obtain the following formula:

$$\int f(g(x))g'(x)\,dx = \int f(u)\,du$$

Comments on the Substitution Rule:

1. When making a substitution, the goal is to transform an existing integral into an ____________________ integral that we can then evaluate.
2. While there is no set method that will always work when it comes to choosing your u, here are some guidelines that will often work:
   (a) Choose "u" so that ______ appears somewhere in your integrand.
   (b) Choose "u" to be the ____________ portion of some composite function.

Example 1. Calculate $\displaystyle\int x\sin(x^2+5)\,dx$.

Example 2. $\displaystyle\int \frac{e^x}{e^x+1}\,dx$

Exercises

1. $\displaystyle\int x^2\sqrt{x^3+1}\,dx$

2. $\displaystyle\int \sin x \cos^2 x\,dx$

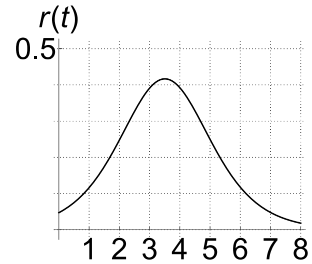
3. $\displaystyle\int \tan x \, dx$

4. $\displaystyle\int_1^3 \frac{e^{1/x}}{x^2} \, dx$

5. At t seconds, the speed of a particle traveling along a straight line is given by $f(t) = \dfrac{20}{1+4t^2}$ meters per second. Calculate $\int_0^3 f(t)\,dt$, and give a complete sentence interpretation of your answer in the context of this problem.

6. $\displaystyle\int \frac{x}{1-x}\,dx$

7. At time t years, the rate of growth of a fish population in a lake is given by $r(t) = \dfrac{500e^t}{(100+3e^t)^2}$ thousand fish per year. A graph of the function $r(t)$ is shown to the right.



(a) Which of the following statements best describes the fish population over the interval $0 \le t \le 8$ years?

(i) The fish population grows for a few years and then begins to decline.
(ii) The fish population is always growing, but at varying rates.
(iii) The fish population grows at a faster and faster rate.

(b) How much does the fish population increase between $t = 0$ and $t = 5$ years?

Section 5.4 – Integration by Parts

Recall the Product Rule for differentiation:

$$\frac{d}{dx}[u(x)v(x)] \quad =$$

Example 1. Evaluate $\displaystyle\int x \cos x \, dx$.

Example 2. Evaluate $\displaystyle\int x \ln x \, dx$.

Integration by Parts Formula.

$$\int u\,dv \;=\; uv - \int v\,du$$

Comments:

1. ______ and ______ together must give *everything* inside your original integral. The "dv" part will contain the differential from your original integral.

2. If possible, we choose u and dv so that the integral $\int v\,du$ can be evaluated directly. Otherwise, we try to choose u and dv so that $\int v\,du$ is ________________________ than the integral we started with.

Example 3. Evaluate $\displaystyle\int_0^1 \arcsin x\,dx$.

Exercises

Evaluate each of the following integrals.

1. $\displaystyle\int xe^x\,dx$

2. $\displaystyle\int_1^e \ln x\,dx$

3. $\int x^2 e^{2x}\, dx$

4. $\int x^5 e^{x^3}\, dx$

Section 5.5 – Other Options for Finding Algebraic Antiderivatives

Example 1. Calculate $\int \frac{2}{(x-1)(x+1)}\,dx$.

Strategy for Integrating a Rational Function, $\frac{P(x)}{Q(x)}$

Assuming that $Q(x)$ has a greater degree than $P(x)$:

- If $Q(x)$ contains a linear factor $x-c$, use a partial fraction of the form

$$\frac{A}{(x-c)}. \tag{1}$$

- If $Q(x)$ contains a repeated linear factor $(x-c)^n$, use partial fractions of the form

$$\frac{A_1}{(x-c)} + \frac{A_2}{(x-c)^2} + \cdots + \frac{A_n}{(x-c)^n}. \tag{2}$$

- If $Q(x)$ contains a factor x^2+cx+d that is irreducible (i.e., it cannot be factored), use partial fractions analogous to those in (1) and (2) above except that the constant numerators are replaced with linear functions of the form ______________ .

Example 2. For each of the following, find the form of the partial fractions decomposition. Do NOT solve for the constants.

(a) $\dfrac{3x}{(x+1)(x-2)^2}$

(b) $\dfrac{3x}{(x+1)(x-2)^3}$

(c) $\dfrac{4x^2+1}{(x^2+1)(x-2)}$

(d) $\dfrac{4x^2+1}{(x^2-9)(x-2)}$

(e) $\dfrac{3}{x^2(x^2+1)^2}$

Example 3. Evaluate $\displaystyle\int \frac{2x^2+4x-9}{x^3+9x}\,dx$.

Exercises

1. For each of the following, find the form of the partial fractions decomposition. Do NOT solve for the constants.

   (a) $\dfrac{10}{x^2 - x - 6}$

   (b) $\dfrac{3 - 2x}{(x^2 - 6x + 9)(x^2 + 1)}$

   (c) $\dfrac{5x^3 - 2}{(x^5 - x^3)(x^2 + 4)^2}$

2. Use the method of partial fractions to evaluate $\displaystyle\int \frac{14x^2 - 17x + 1}{x(2x-1)^2}\,dx$.

Integration Using Tables

Example 4. By completing the square and substituting, evaluate $\displaystyle\int \frac{1}{x^2+6x+13}\,dx$.

Example 5. Evaluate each of the following, using integral tables as appropriate.

(a) $\displaystyle\int \sqrt{4x^2-5}\,dx$

(b) $\int x^2 \sin x \, dx$

Examples and Exercises

3. Evaluate each of the following, using integral tables where appropriate.

   (a) $\int \cos^3(4x) \, dx$

   (b) $\int e^{5x} \sin(3x) \, dx$

(c) $\displaystyle\int x\sqrt{16-x^4}\,dx$

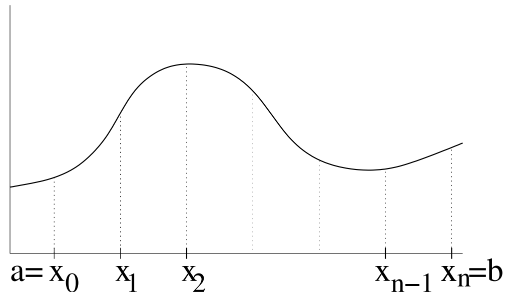
(d) $\displaystyle\int \sqrt{x^2-8x+25}\,dx$

(e) $\displaystyle\int \frac{e^{2x}}{3+4e^x}\,dx$

(f) $\displaystyle\int e^x\sqrt{1+2e^x}\,dx$

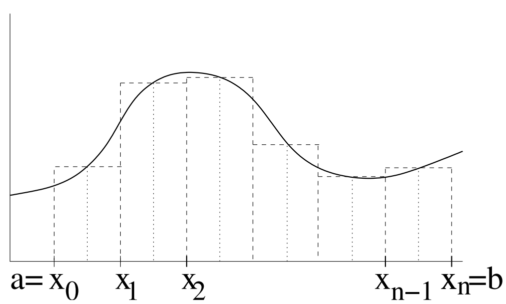
Section 5.6 – Numerical Integration

1. **The Trapezoid Rule**
   $\Delta x =$ width of one subdivision
   $n =$ number of subdivisions



$$T_n = \text{approximation of } \int_a^b f(x)\,dx \text{ using } n \text{ subdivisions}$$
$$=$$

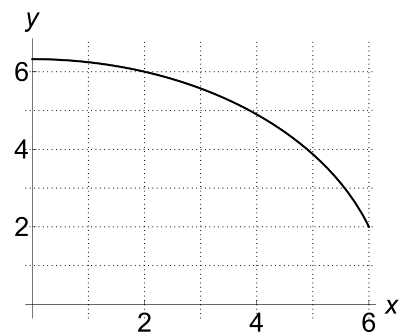
2. **The Midpoint Rule**
   $\Delta x =$ width of one subdivision
   $n =$ number of subdivisions

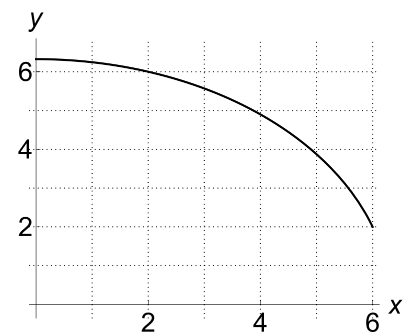


$$M_n =$$

Example 1. Let $f(x) = \sqrt{40 - x^2}$.

(a) Use the Midpoint Rule and the Trapezoid Rule, both with $n = 3$, to find two estimates of $\int_0^6 \sqrt{40 - x^2}\, dx$. Also draw the approximating rectangles and trapezoids in the figures below.



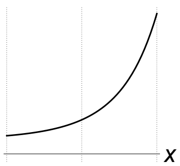


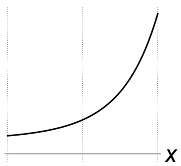
(b) Is M_3 an over or underestimate of $\int_0^6 f(x)\, dx$? How about T_3? Use these observations to give an improved estimate of $\int_0^6 f(x)\, dx$.

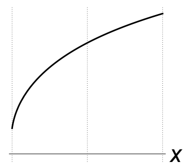
Trapezoid and Midpoint Rule Estimation. Let f be a continuous function defined on $[a, b]$.

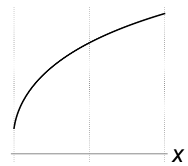
- If the graph of f is concave up on $[a, b]$, then $\underline{\hspace{1.5cm}} \leq \displaystyle\int_a^b f(x)\,dx \leq \underline{\hspace{1.5cm}}$.
- If the graph of f is concave down on $[a, b]$, then $\underline{\hspace{1.5cm}} \leq \displaystyle\int_a^b f(x)\,dx \leq \underline{\hspace{1.5cm}}$.
- The **error** is the difference between the exact value of $\int_a^b f(x)\,dx$ and its estimate using either the Trapezoid or Midpoint Rule. Specifically, we define

$$\underline{\hspace{3cm}} = \int_a^b f(x)\,dx - T_n \qquad \text{and} \qquad \underline{\hspace{3cm}} = \int_a^b f(x)\,dx - M_n.$$









Example 2. The growth rate of a city is given by $f(t) = \dfrac{1}{1 + \ln(t+1)}$ thousand people per year, where t is the number of years after 2010.

(a) Use the Midpoint Rule with $n = 4$ to estimate $\int_0^8 f(t)\,dt$.

(b) Use the Midpoint Rule and technology to estimate $\int_0^8 f(t)\,dt$ accurate to 2 decimal places. Then, give a complete sentence interpretation of your estimate in the context of this problem.

Example 3. Let $f(x) = 1/x$. Then $\int_1^2 \frac{1}{x}\,dx$ = ______________________________.

n	T_n	M_n	$E_{T,n} = \int_1^2 \frac{1}{x}\,dx - T_n$	$E_{M,n} = \int_1^2 \frac{1}{x}\,dx - M_n$
1	0.75	0.6666666667	-5.69×10^{-2}	2.65×10^{-2}
10	0.6937714032	0.6928353604	-6.24×10^{-4}	3.12×10^{-4}
100	0.6931534305	0.6931440556	-6.25×10^{-6}	3.12×10^{-6}
1000	0.6931472431	0.6931471493	-6.25×10^{-8}	3.12×10^{-8}

Some Notes and Observations:

- Each time n is increased by a factor of 10, we gain about ______ decimal places of accuracy in the Trapezoid and Midpoint Rule approximations.
- The Trapezoid Rule underestimates the integral by about ________ the amount by which the Midpoint Rule overestimates the integral. This leads to Simpson's Rule, a more accurate method of estimating integrals:

  $$S_n =$$

  When S_n is used to approximate the integral, it can be shown that each time n is increased by a factor of 10, we gain about ______ decimal places of accuracy.

Examples and Exercises

1. Let $f(x) = \sin(x^2/4)$.

   (a) Use the Midpoint Rule with $n = 4$ to estimate $\int_0^4 f(x)\,dx$.

   (b) Use the Midpoint Rule and technology to estimate $\int_0^4 f(x)\,dx$ accurate to three decimal places.

2. The amount of electrical power drawn by a household is given by $f(t)$ kilowatts, where t is the number of hours after 6:00 p.m. Values of the function f are given in the table to the right.

t	0	1	2	3
$f(t)$	2.15	1.52	1.10	1.02

   (a) Use the Trapezoid Rule to estimate $\int_0^3 f(t)\,dt$, which gives the energy used by the household between 6:00 and 9:00 p.m. What are the units of the energy used?

   (b) Do you think your answer to part (a) is an over or an underestimate of the actual amount of energy used by the household? Explain.

3. Let $f(x) = e^x$.

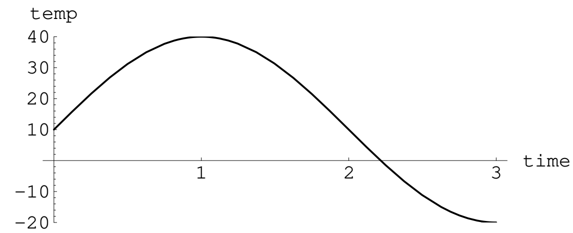
   (a) Find the exact value of $\int_0^2 f(x)\,dx$, and then write down a decimal approximation of this value rounded to 6 decimal places.

   (b) Use technology to complete the table with the indicated approximations of $\int_0^2 f(x)\,dx$ using the Trapezoid, Midpoint, and Simpson's Rules. Give all answers accurate to 6 decimal places, and compare with your answer to part (a).

n	T_n	M_n	S_n
5			
10			
15			

Section 6.0 – The Average Value of a Function

Problem. The temperature in Rapid City, South Dakota on a winter day is given by the function $f(t) = 30\sin((\pi/2)t) + 10$ degrees Fahrenheit, where t measures time (in hours) after 12:00 noon. We are interested in determining the average temperature in Rapid City between 12:00 noon and 3:00 p.m.



Goal: Derive a general formula for finding the average value of a function like the one above on an interval $[a, b]$.

Examples and Exercises

1. Use your formula to find the average temperature in Rapid City between 12:00 noon and 3:00 p.m. on the day described on the previous page.

2. Consider the function $f(x) = 4 - x^2$ on the interval $[0, 2]$.

   (a) Compute the average value of $f(x)$ on $[0, 2]$.

   (b) Find c in $[0, 2]$ such that $f_{\text{avg}} = f(c)$.

   (c) Sketch the graph of f and a rectangle whose width is 2 and whose area is the same as the area under the graph of f. What is the height of your rectangle?

Section 6.1 – Using the Definite Integral to Find Area

Goal. Derive a general formula for the area between the functions $y = f(x)$ and $y = g(x)$ on the interval $a \leq x \leq b$.

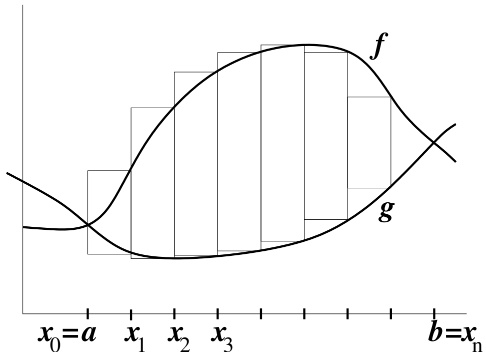


Figure 1.

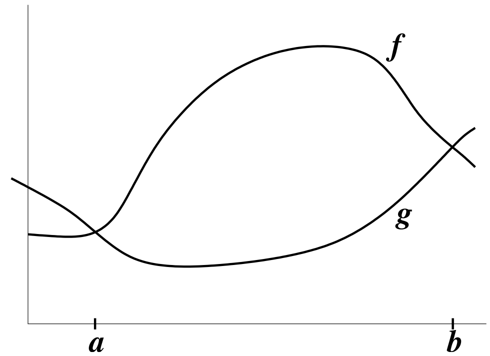
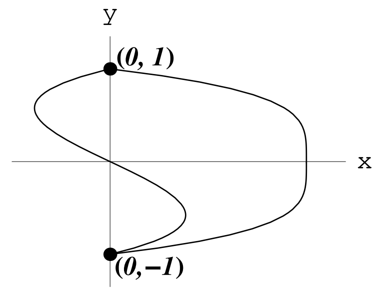


Figure 2.

Example 1. Find the area bounded between $y = \sqrt{x}$ and $y = x^2/8$.

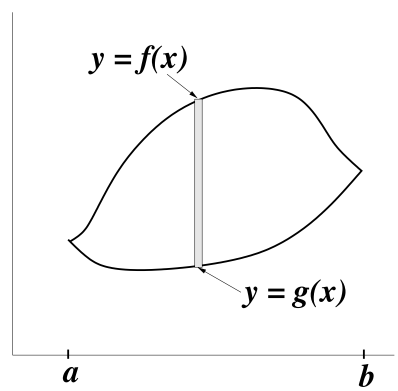
Example 2. Find the area of the region bounded by the curves $x = 1 - y^4$ and $x = y^3 - y$ (see diagram to the right).



Summary of Area Formulas

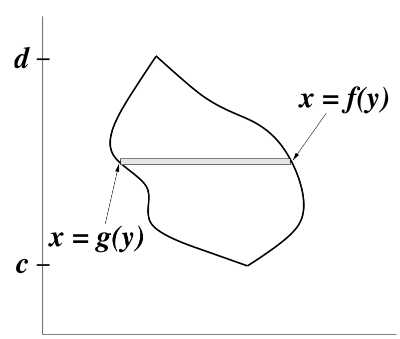
(A) Area between functions of x:

Area $=$



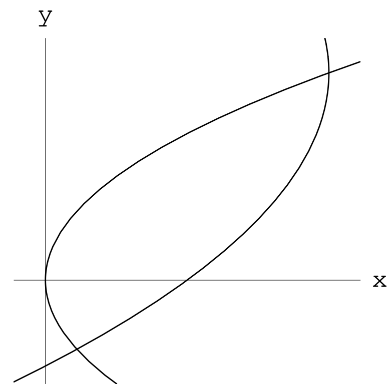
(B) Area between functions of y:

Area $=$

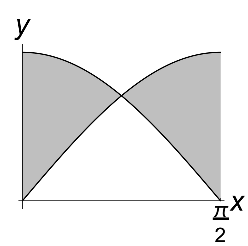


Examples and Exercises

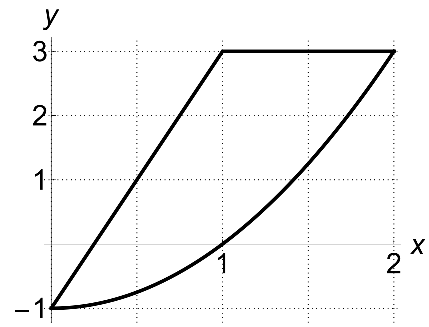
1. To the right, the graphs of $2y^2 - x = 0$ and $y^2 - 6y + x = 9$ are shown. Find the exact area of the bounded region between these curves.



2. Find the area between $y = \sin x$ and $y = \cos x$ on the interval $[0, \frac{\pi}{2}]$.

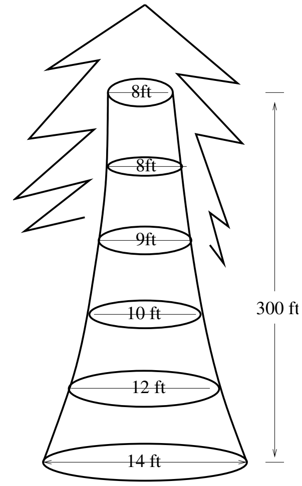


3. Consider the region bounded by the curves $y = x^2 - 1$, $4x - y = 1$, and $y = 3$ shown to the right. Set up integrals that represent the area of this region in two different ways: (a) in the "dy" direction, (b) in the "dx" direction. Then, calculate the area by choosing the easier of the two methods.



Section 6.2 – Using Definite Integrals to Find Volume

Preliminary Example. Colonel Armstrong is a 1300 year old Redwood tree whose trunk is 300 feet tall. Every 60 feet, a diameter measurement of Colonel Armstrong has been taken (see diagram to right). Use this information to estimate the volume of wood in Colonel Armstrong's trunk.



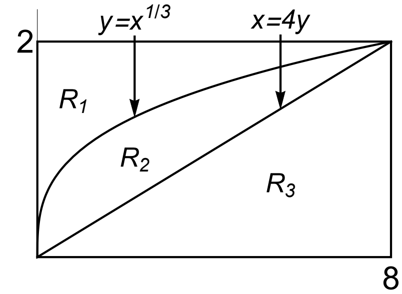
Example 1. Find the volume of the solid region generated by rotating the curve $y = \sin x$ about the x-axis on the interval $[0, \pi]$.

Example 2. Find the volume of the solid region generated by rotating the region bounded by $y = 2,\ x = 0$, and $y = \sqrt[3]{x}$ about the x-axis.

Example 3. Set up, but DO NOT EVALUATE, an integral that gives the volume of the solid region generated by rotating the region bounded between $y = x$ and $y = x^2$ about the line $x = 2$.

Exercises

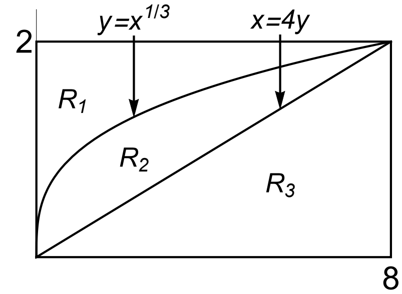
1. Consider the figure given to the right. For each of the following, set up, but do not evaluate, an integral that represents the volume obtained when the specified region is rotated around the given axis.



   (a) R_1 about the y-axis.

   (b) R_2 about the x-axis.

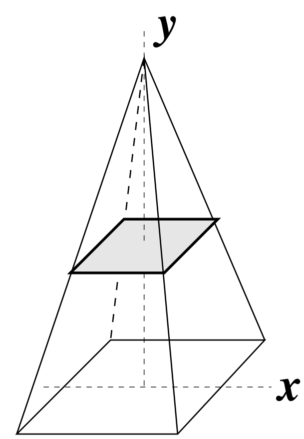
(c) R_2 about the y-axis.



(d) R_2 about the line $x = 9$.

(e) R_3 about the line $y = 3$.

2. Find the volume of a right pyramid whose base is a square with side L and whose height is h. This solid region is pictured to the right. Notice that a suggestive slice has been drawn in for you.



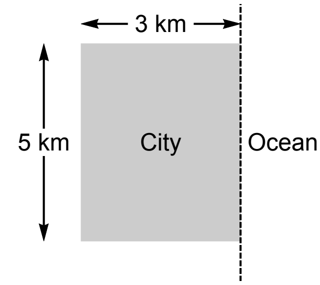
Section 6.3 – Density and Mass

Example 1. Consider a thin rod of length 10 cm.

(a) Find the mass of the rod if it has a uniform density of $\rho = 5$ grams per cm.

(b) Find the mass of the rod if its density is given by $\rho(x) = x/2$ grams per cm, where x is the distance from the left side of the rod, in cm.

Example 2. A rectangular city having a length of 5 km and a width of 3 km borders along an ocean as shown in the diagram to the right. The population density of the town at a distance of x km from the ocean is $\sigma(x) = 500 - 30x^2$ people per square kilometer.



(a) Where is the population density of the city the greatest?

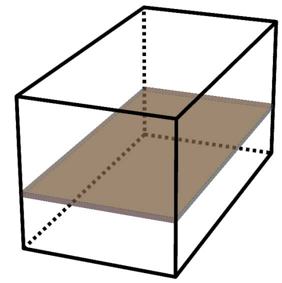
(b) Find the population of the city.

Examples and Exercises

1. Find the mass of a thin, straight rod of length 30 cm whose density at a distance of x cm from the left end of the rod is $\rho(x) = (x^2/300)$ grams/cm.

2. A cloth banner 10 meters wide and 5 meters high is placed on the side of a building and left out over night during a rain storm. The next morning, the banner is heavier on the bottom due to gravity's effect on the moisture in the banner; specifically, the density of the banner at a distance of y meters from its top edge is $\rho(y) = (0.5 + 0.1y)$ kg/m^2. Find the mass of the wet banner.

3. A rectangular aquarium having a length of 3 meters, a width of 2 meters, and a height of 2 meters is filled to the top with a mixture of water and sediment (see the diagram to the right, which illustrates the tank together with a typical "slice" of liquid). At a depth of y meters, the liquid in the tank has a density of $\rho(y) = 1000e^{0.05y}$ kg/m^3. Find the mass of the liquid in the tank.



Section 6.4 – Physics Applications: Work, Force, and Pressure

Some Preliminary Facts About Work:

Work =

Weight is the force that gravity exerts on an object of a particular mass. In particular,

Weight = .

Quantity	Metric Unit	English Unit
Mass		
Force		
Work		

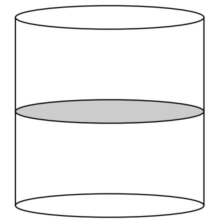
Example 1. A rock climber has a mass of 60 kg. If she climbs a vertical distance of 100 meters, how much work is done against the force of gravity?

Example 2. A 45-meter rope with a mass of 30 kg is dangling over the edge of a cliff. Ignoring friction, how much work is needed to pull the rope up to the top of the cliff?

(a) Explain what is wrong with the following solution to the above problem.

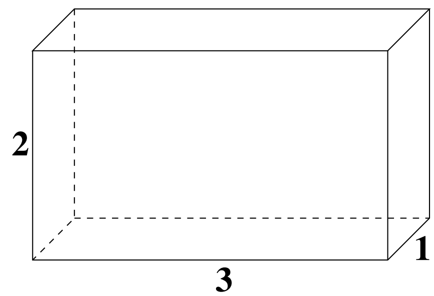
```
Work = (Force)· (Distance) = (294 N)(45 m) = 13230 Joules
```

(b) Give a correct solution to this problem.



Example 3. A cylindrical tank having a radius of 0.5 meters and a height of 2 meters is half full of kerosene, which has a density of 810 kg per m^3. Find the work needed to pump the kerosene over the top edge of the tank.

Example 4. A rectangular aquarium having a length of 3 meters, a width of 1 meter, and a height of 2 meters is filled to the top with water. Given that water has a density of 1000 kg per m^3, find the work required to pump all of the water over the top edge of the tank.



Some Preliminary Facts About Pressure:

Pressure =

Some common units of pressure are ______________________ and ______________________ .

Example 5. A rectangular swimming pool has a length of 25 meters, a width of 10 meters, and is filled with water to a depth of 2 meters. Given that the density of water is 1000 kg/m^3, find the hydrostatic pressure at the bottom of the pool.

Hydrostatic Force and Pressure. We use the term ______________________ to refer to forces and pressures related to immersion in a fluid. The hydrostatic pressure present at a depth of h in a fluid of uniform density ρ is given by

$$P = \qquad\qquad ,$$

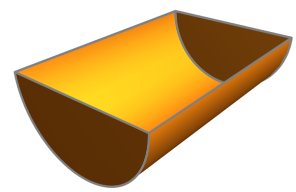
where g is the acceleration of gravity. The hydrostatic force on an object experiencing a constant pressure of P over its entire surface is given by

$$F = \qquad\qquad ,$$

where A is the surface area of the object.

Example 6. The roof of an underwater observatory is a flat rectangle, parallel to the ocean surface, having a total area of 200 square meters. Given that the roof is 15 meters below the ocean surface, and that the density of seawater is 1020 kg/m^3, find the hydrostatic force on the roof of the observatory.

Example 7. The figure to the right shows a trough that consists of a curved bottom in the shape of a half-cylinder of radius 1 meter, and two side panels in the shapes of semicircular disks of radius 1 meter. Suppose that the trough is filled with water, which has a constant density of 1000 kg/m^3. Find the hydrostatic force on one of the two side panels of the trough.



(a) Explain what is wrong with the following solution to this problem:

`The pressure of the water on one side panel of the tank is`

$$P \;=\; \rho g h \;=\; (1000)(9.8)(1) \;=\; 9800\ \texttt{N/m}^2.$$

`Therefore, the force on the side of the tank is`

$$F \;=\; PA \;=\; 9800 \cdot \frac{\pi(1)^2}{2} \;=\; 4900\pi \;\approx\; 1.539 \times 10^4\ \texttt{Newtons}.$$

(b) Give a correct solution to this problem.

Exercises

1. A rope that is 20 meters long and and having a uniform density of 1.5 kg per meter is hanging over the edge of a steep sea cliff so that its bottom edge barely reaches the ground.

   (a) How much work is done in pulling the rope to the top of the cliff?

   (b) A surfer of mass 85 kg is stranded by high surf on a beach beneath the same cliff. How much work would be required to rescue the surfer by pulling him up over the edge of the cliff (assuming that he is firmly attached to the bottom end of the rope)?

2. (Adapted from *Stewart*) A circular swimming pool has a diameter of 8 meters, and the sides of the pool are 4 meters high.

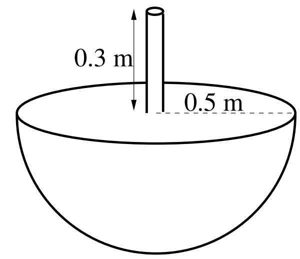
   (a) If the pool is initially full of water, how much work is done in pumping the water out over the top edge? (Use the fact that water has a density of 1000 kg per m^3).

   (b) If the pool is initally half full of water, how much work is done in pumping the water out over the top edge?

   (c) Explain why your answer to part (b) is *not* half of your answer to part (a).

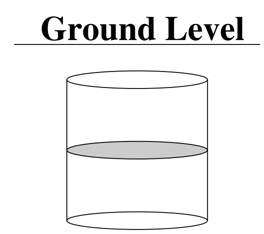
3. A thick rope of length 30 m is hanging over the edge of a tall building. During the night, it rains, and after the rain is over, gravity causes the water in the rope to gradually sink, resulting in a rope that is heavier at the bottom than at the top. Specifically, the density of the wet rope at a distance of y meters from the top of the rope is $\rho(y) = 1 + 0.03y$ kg/m. Find the work necessary to lift the wet rope to the top of the building.

4. A hemispherical extended family sized punch bowl has a radius of 0.5 meters and is full of punch with a density of 1000 kg per m^3. How much work would it take to pump all of the punch out of the bowl through the outlet? (See picture to the right.)



5. A heavy rectangular banner measuring 5 meters by 10 meters is hanging over the edge of a tall building. (Assume that the banner is oriented so that the 10 meter side is vertical and the 5 meter side is horizontal; that is, parallel to the ground.) If the entire banner has a mass of 25 kilograms, find the work required to pull the banner up over the side of the building.

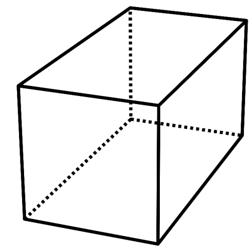
6. A gas station stores its gasoline in a tank underground. The tank has the shape of an upright circular cylinder with its top 2 feet below ground level, and the tank is half full of gasoline (see diagram). The radius of the tank is 4 feet, and its height is 8 feet. Find the work necessary to pump all of the gasoline to ground level. (Gasoline weighs 42 pounds per ft^3.)



7. A rectangular aquarium has a length of 3 meters, a width of 2 meters, and a height of 2 meters. The aquarium is completely filled with water, which has a density of 1000 kg/m^3. Find the hydrostatic force on the following portions of the aquarium:

   (a) On the 2×3 bottom panel of the aquarium.

   (b) On one of the 2×3 side panels of the aquarium.



Section 6.5 – Improper Integrals

Example 1. A customer call center discovers that the fraction of their customers who wait between $t = a$ and $t = b$ minutes to have their call answered is given by $\int_a^b 0.2e^{-0.2t}\,dt$.

(a) Fill in the chart below:

Wait Time	Fraction of Customers
Between 5 and 10 minutes	
Between 5 and 15 minutes	
Between 5 and 20 minutes	
Between 5 and 25 minutes	

(b) What fraction of the customers wait at least 5 minutes to have their call answered?

Definition. We define the improper integral $\int_a^\infty f(x)\,dx$ as follows:

(*I*) $$\int_a^\infty f(x)\,dx \ =$$

If the above limit exists and is finite, we say that the improper integral ________________ . Otherwise, we say that the improper integral ________________ .

Example 2. To the right, you are given the graphs of $y = 1/x^2$ and $y = 1/x$. Calculate the improper integrals $\int_1^\infty \frac{1}{x^2}\,dx$ and $\int_1^\infty \frac{1}{x}\,dx$.

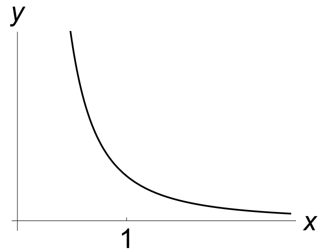


Figure 1. Graph of $y = 1/x^2$

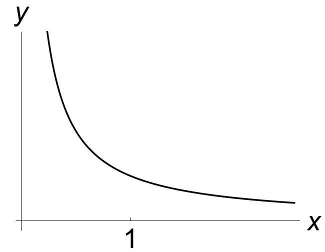


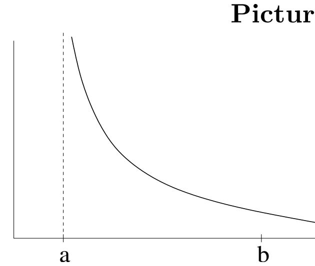
Figure 2. Graph of $y = 1/x$

Fact. For $a > 0$, the improper integral $\int_a^\infty \frac{1}{x^p}\,dx$ converges if ________ and diverges if ________ .

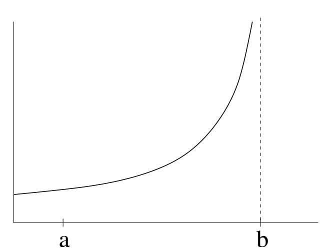
Other Types of Improper Integrals

Improper Integral Definition **Picture**

(II) $\int_a^b f(x)\,dx =$

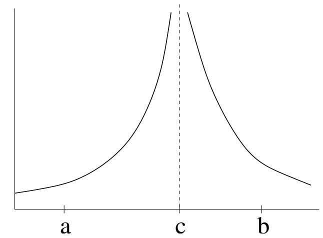


(III) $\int_a^b f(x)\,dx =$



Improper Integral Definition **Picture**

(IV) $\int_a^b f(x)\,dx =$



Note. We say that an improper integral converges if all of the defining limits exist and are finite. If any of the involved limits are infinite or do not exist, we say that the improper integral diverges.

Example 3. Determine if the integral $\int_0^2 \frac{1}{\sqrt{x}}\,dx$ converges or diverges. If it converges, find its value.

Examples and Exercises

1. For each of the following, decide whether the improper integral converges or diverges, and, if it converges, find its value.

   (a) $\displaystyle\int_1^\infty \frac{1}{\sqrt{x+1}}\,dx$

   (b) $\displaystyle\int_2^\infty \frac{1}{x(\ln x)^2}\,dx$

2. The probability that a particular brand of light bulb will last more than c hundred hours is given by $\displaystyle\int_c^{\infty} 0.05e^{-0.05t}\,dt$. Find the probability that the bulb will last more than 500 hours.

3. The expected lifetime of a light bulb, in hundreds of hours, is given by $\displaystyle\int_0^{\infty} 0.1te^{-0.1t}\,dt$. Find the expected lifetime of this light bulb.

Sections 7.1 & 7.2 – An Introduction to Differential Equations

Preliminary Example.

Physical Situation	An object is taken out of an oven and placed in a room where the temperature is 75°F. Let $T(t)$ represent the temperature of the object, in °F, after t minutes.
Modeling Differential Equation	$\dfrac{dT}{dt} = k(75 - T)$, where $k > 0$ is a constant.
Description of the Physical Law Modeled by the Differential Equation	

(a) For the remainder of this example, let $k = 0.5$, so that $dT/dt = 0.5(75 - T)$. Complete the following table for the given values of T.

T	25	75	125	175
dT/dt				

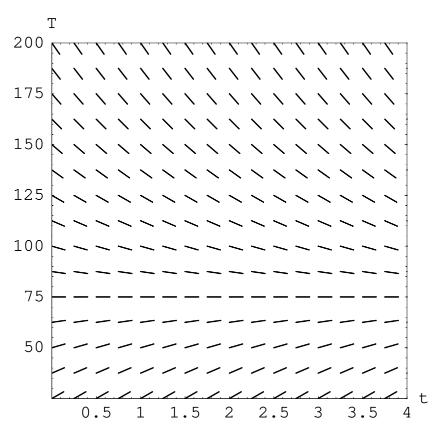
(b) To the right, you are given a slope field for the differential equation $\dfrac{dT}{dt} = k(75 - T)$ for $k = 0.5$. For each initial condition shown below, draw the corresponding temperature function $T(t)$ on the slope field.

(i) $T(0) = 200$

(ii) $T(0) = 125$

(iii) $T(0) = 75$

(iv) $T(0) = 25$

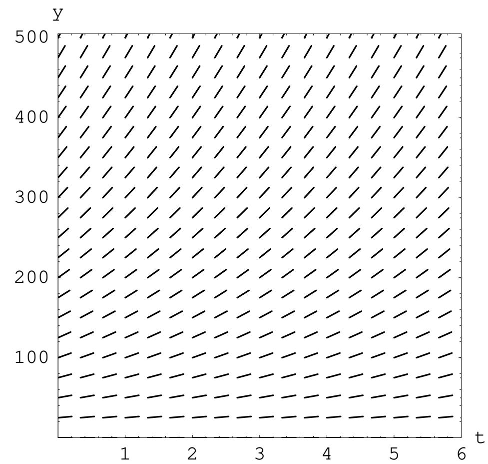


Example 1. Let $P(t)$ represent the population of a bacteria colony after t hours. Assume that this population is modeled by the differential equation $\frac{dP}{dt} = 0.3P$, whose slope field is given below.

(a) Draw in the population functions that are solutions to the following "initial value problems":

(i) $\frac{dP}{dt} = 0.3P,\ P(0) = 100$

(ii) $\frac{dP}{dt} = 0.3P,\ P(0) = 200$



(b) Estimate the amount of time it takes for the population of a bacteria colony to double if it starts with 200 bacteria.

(c) For each of the following functions, use the slope field to guess whether the function could be a solution to the differential equation $\frac{dP}{dt} = 0.3P$. Then, use algebra and calculus to confirm your guesses.

(i) $P = e^{0.3t}$ (ii) $P = 100e^{0.3t}$ (iii) $P = 200\sin t$ (iv) $P = 0$

Example 2. The amount Q of the radioactive substance Calcium-47 remaining in a sample is governed by the differential equation $dQ/dt = -0.15Q$, where t represents elapsed time in days.

(a) Show that $Q = Ce^{-0.15t}$ is a solution to this differential equation for any constant C.

(b) Solve the initial value problem $dQ/dt = -0.15Q,\ Q(0) = 100$. Then, interpret your solution in the context of this problem.

(c) If we start with a 100-gram sample of of Calcium-47, how much of this sample will remain after 6 days?

Definition. A *differential equation* is an equation that describes the derivative, or derivatives, of a function that is unknown to us. A *solution* to a differential equation is a __________________ that satisfies this description.

Note: There are many different techniques for finding and analyzing solutions to differential equations, depending on their form. For some differential equations, solutions can be ...

- graphically approximated using ________________________ .
- numerically approximated using iterative techniques such as ____________________________________ .
- determined exactly using analytic methods such as ____________________________________ .

Exercises

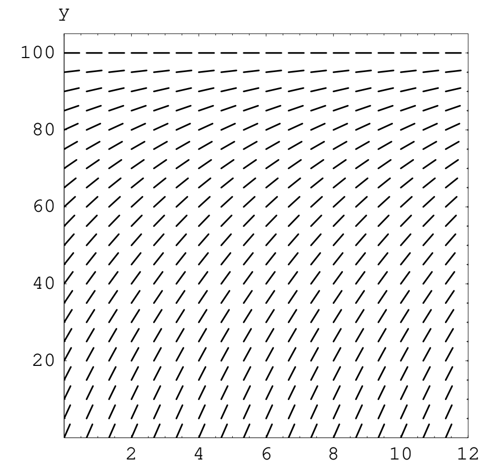
1. Let $y(t)$ represent the percentage of a particular task that has been learned after t months. Then y can be modeled using the differential equation $\frac{dy}{dt} = k(100 - y)$, where k is a positive constant.

   (a) The slope field to the right represents the differential equation given above with $k = 0.2$. Draw in the solutions to the following initial value problems:

      i. $\frac{dy}{dt} = 0.2(100 - y),\ y(0) = 20$

      ii. $\frac{dy}{dt} = 0.2(100 - y),\ y(0) = 60$

   (b) Who learns faster at the beginning, a person who starts out knowing 20% of a task or a person who starts out knowing 60% of a task? Explain.

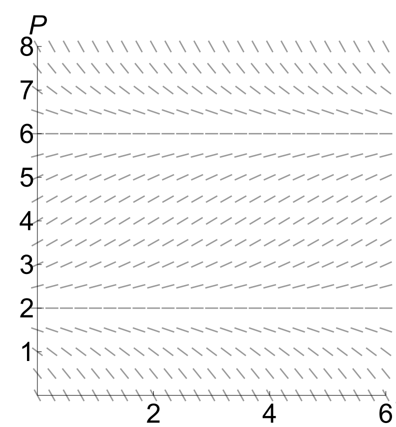


   (c) Explain, in the context of this situation, what the modeling differential equation $\frac{dy}{dt} = k(100 - y)$ is saying.

2. Wildlife biologists are studying the fish population of a lake in which a specific annual amount of fishing takes place. They discover that the fish population, P (in thousands), is well-modeled by the differential equation

$$\frac{dP}{dt} = 1.6P - 0.2P^2 - 2.4,$$

   where t represents time, in years. A slope field for this differential equation is shown to the right.



   (a) Draw the solution curves that correspond to the initial conditions (i) $P(0) = 2.5$ and (ii) $P(0) = 8$. Then, describe the behavior of the fish population in each case.

   (b) Are there any scenarios in which the entire fish population of the lake dies off? If so, describe specifically what they are and how you reached your conclusion.

3. Let C be any constant, and consider the differential equations given to the right. For each of the functions given below, decide whether the function is a solution to (I), to (II), or to both (I) and (II).

   (I) $\dfrac{dx}{dt} = 2x$ (II) $x'' = -x$

   (a) $x = Ce^{2t}$ (b) $x = 2\cos t$ (c) $x = 3\cos t - 4\sin t$ (d) $x = 0$

4. Consider the differential equations (I) $\dfrac{dx}{dt} = 2x$ and (II) $x'' = -x$ given in Exercise 3. In one of these differential equations, x represents the position of an oscillating spring as a function of time, and in the other, x represents an exponentially growing population as a function of time. Which is which, and why do you think so?

5. For what nonzero values of k does the function $y = e^{kt}$ solve the differential equation $y'' - y' - 6y = 0$?

6. (a) Show that all members of the family $y = x^3 + \frac{c}{x^2}$ are solutions of the differential equation $xy' = 5x^3 - 2y$.

(b) Find the solution to the initial value problem $xy' = 5x^3 - 2y,\ y(1) = 5$.

7. Below, you are given 8 differential equations. Match each differential equation with the appropriate direction field (given on page 69). Then, complete parts (a)–(c) below.

(I) $y' = 1$	(II) $y' = y$	(III) $y' = x - y$	(IV) $y' = \cos(x)$
(V) $y' = -xy$	(VI) $y' = \frac{x}{y}$	(VII) $y' = 0$	(VIII) $y' = x^2$

(a) For each of the direction fields, draw in the solution that satisfies the initial condition $y(0) = 1$.

(b) Can you guess a formula for any of the solutions you drew in for part (a)? Check to see if you're right!

Definition. A constant solution to a differential equation is called an *equilibrium solution.* Graphically, an equilibrium solution is a horizontal line, that is, a solution of the form $y = c$ for some constant c.

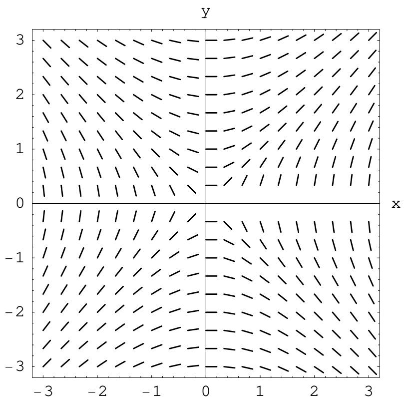
8. Use the above definition of equilibrium solution to answer the following:

   (a) Does the differential equation from the Preliminary Example on page 60 have any equilibrium solutions? If so, what are they?

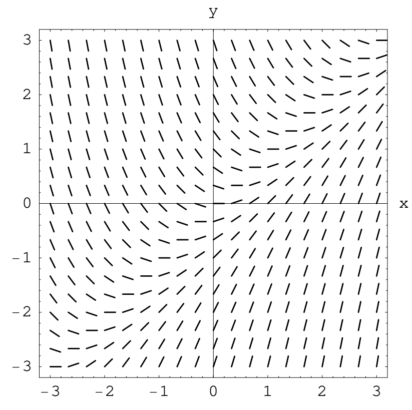
   (b) Does the differential equation from Example 1 on page 61 have any equilibrium solutions? If so, what are they?

   (c) What are the equilibrium solutions for the differential equation from Exercise 2 on page 63?

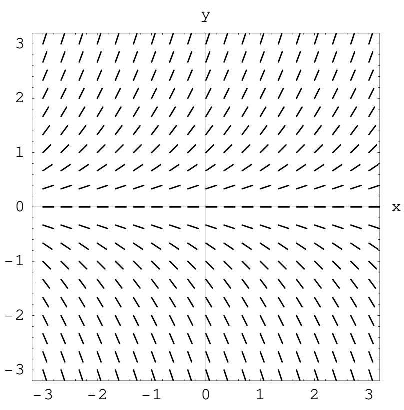
(A)


y
x
3 2 1 0 −1 −2 −3
−3 −2 −1 0 1 2 3

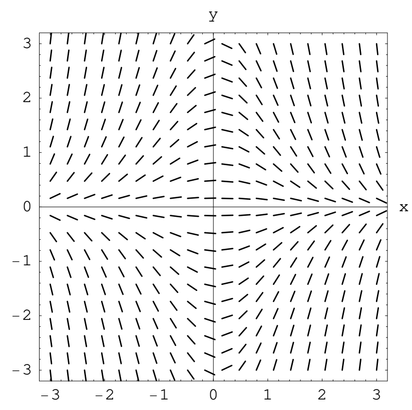
(B)


y
x
3 2 1 0 −1 −2 −3
−3 −2 −1 0 1 2 3

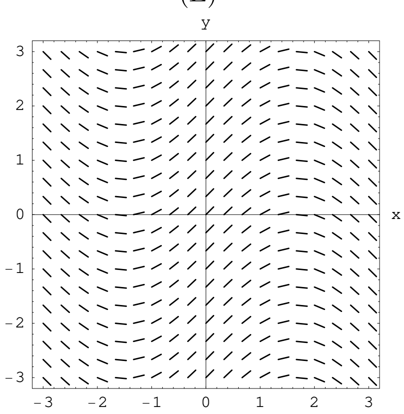
(C)


y
x
3 2 1 0 −1 −2 −3
−3 −2 −1 0 1 2 3

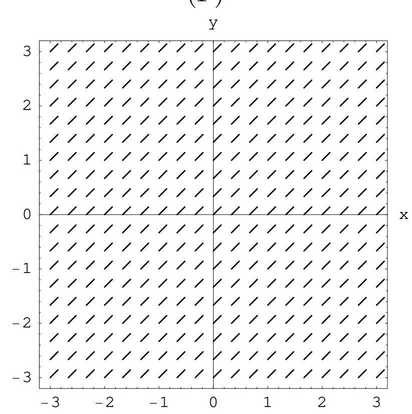
(D)


y
x
3 2 1 0 −1 −2 −3
−3 −2 −1 0 1 2 3

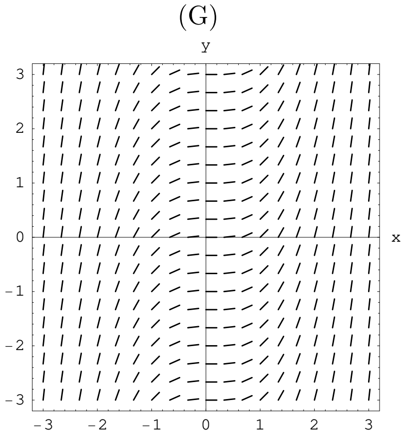
(E)


y
x
3 2 1 0 −1 −2 −3
−3 −2 −1 0 1 2 3

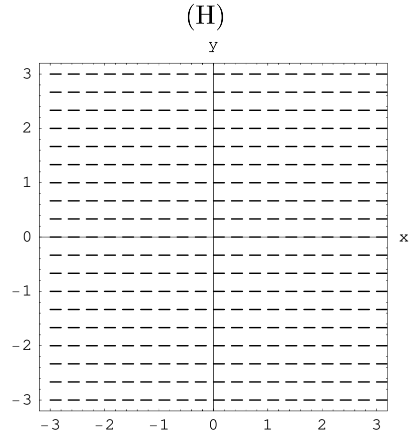
(F)


y
x
3 2 1 0 −1 −2 −3
−3 −2 −1 0 1 2 3

(G)


y
x
3 2 1 0 −1 −2 −3
−3 −2 −1 0 1 2 3

(H)

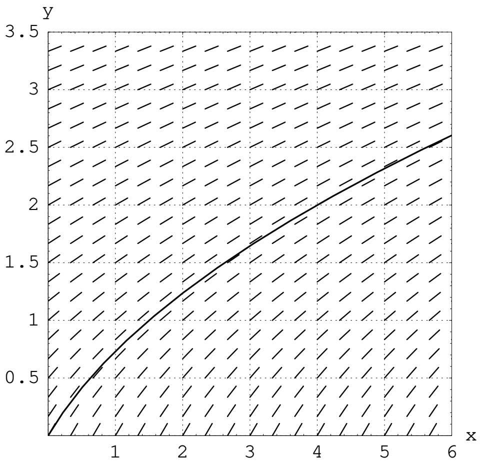

y
x
3 2 1 0 −1 −2 −3
−3 −2 −1 0 1 2 3

Sections 7.3 – Euler's Method

Preliminary Example. In the figure below, you are given the slope field for an initial value problem of the form

$$\frac{dy}{dx} = F(x, y), \qquad y(0) = 0.$$

Derive a method for approximating the solution curve $y(x)$ for this initial value problem.



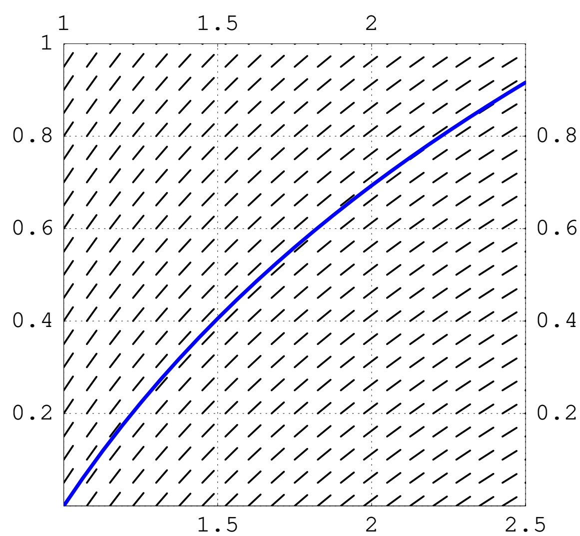
Euler's Method Formulas:

Examples and Exercises

1. Consider the initial value problem

$$\frac{dy}{dx} = \frac{1}{x}, \quad y(1) = 0.$$

To the right, you are given a slope field and a graph of the unknown solution to this problem, $y(x)$. Use Euler's Method with step size 0.5 to estimate $y(2.5)$. Sketch your solution curve on the slope field to the right and compare with the exact solution, $y(x)$.



2. The slope field for the differential equation

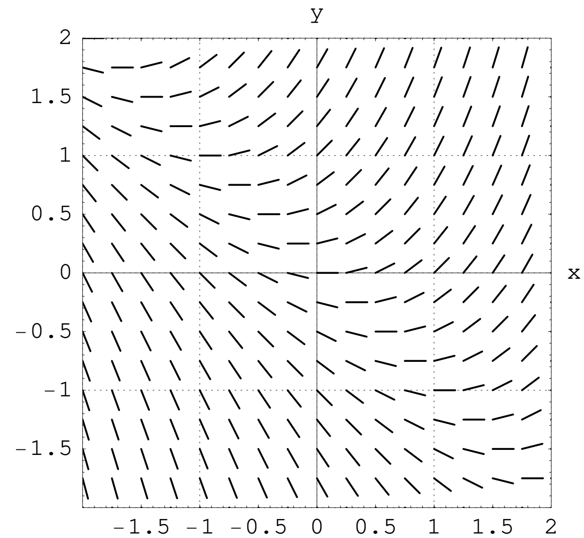
$$y' - y = x$$

is shown to the right. Use Euler's method to approximate solution curves to the following initial value problems:

(a) $y' - y = x, \quad y(-1) = 0$

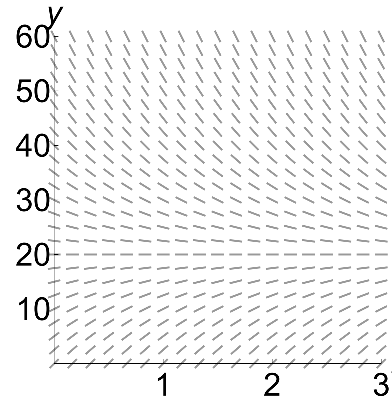
(b) $y' - y = x, \quad y(-1) = 0.5$

Use a step size of 0.5 in both cases.



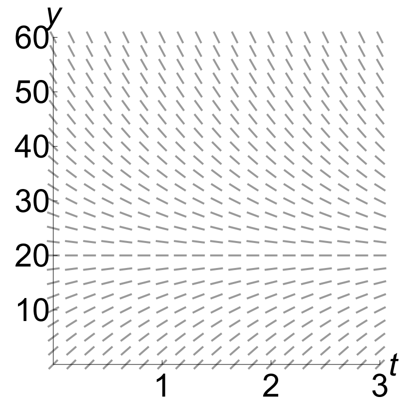
Sections 7.4 – Separable Differential Equations

Example 1. Let y represent the temperature, in degrees Celsius, of a metal bar t minutes after being placed in a room having a constant temperature of 20°C.

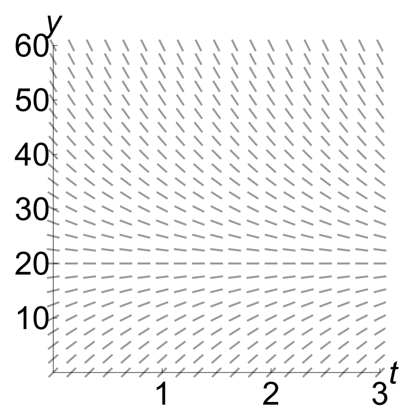


(a) Solve $dy/dt = 20 - y$, the differential equation that governs the temperature of the bar. Then, draw in several of your solution curves on the slope field to the right.

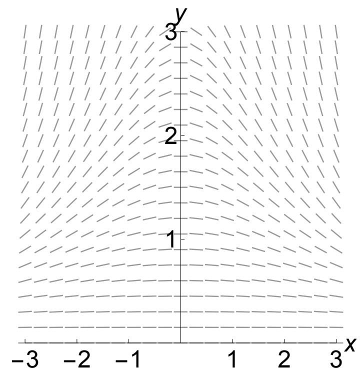
(b) Solve the initial value problem $dy/dt = 20 - y, \quad y(0) = 5$, and sketch this solution on the slope field to the right. What is the significance of your answer in the context of this problem?



(c) If the starting temperature of the bar is 60°C, find the temperature of the bar after 2 minutes.



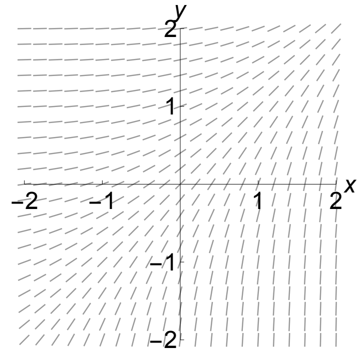
Example 2. Solve the initial value problem $8\frac{dy}{dx} = -xy^2,\ y(0) = 2$, and then draw your solution curve on the slope field given to the right.



Example 3. The population of a bacteria colony is governed by the differential equation $dP/dt = 0.2P$, where t is measured in hours. Assuming that the starting population of the colony is 500 bacteria, what is the population of the colony after 2 hours?

Examples and Exercises

1. Solve the initial value problem $\frac{dy}{dx} = e^{x-y}$, $y(0) = 0$, and then draw your solution curve on the slope field given to the right. (Hint: $e^{x-y} = e^x/e^y$)



2. (Based on *Hughes-Hallett, et. al.*) As heat from the water in a lake escapes into the air, ice forms on the surface of the lake. The thickness of the ice on the surface of a lake after t weeks is given by $y(t)$ cm. Assume that the ice thickness is 0 cm at $t = 0$ and is governed by the differential equation

$$\frac{dy}{dt} = \frac{3}{2y}.$$

   (a) Find a formula for the thickness of the ice on the lake after t weeks.

(b) As the thickness of the ice increases, does the rate at which the ice accumulates increase or decrease? Explain.

3. Find the general solution to the differential equation $x^2y' + y = 0$.

4. Let $v(t)$ represent the speed of a falling parachutist (before the chute opens), in feet per second, after t seconds. The motion of the parachutist is governed by the differential equation

$$\frac{dv}{dt} = 32 - 0.2v.$$

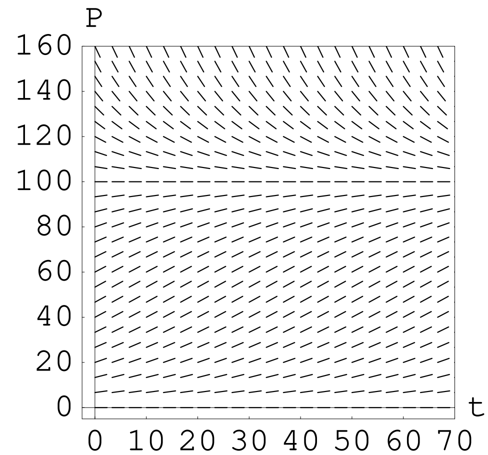
You may assume that the parachutist has an initial speed of zero when jumping out of the plane; that is, $v(0) = 0$.

(a) Find a formula for the speed of the parachutist as a function of time.

(b) What happens to the speed of the parachutist as $t \to \infty$? Use your answer from part (a) to justify your answer.

Sections 7.6 – Population Growth and the Logistic Equation

1. Suppose that a population develops according to the logistic equation
$$\frac{dP}{dt} = 0.05P - 0.0005P^2,$$
where t is measured in weeks. A direction field for this differential equation is given to the right.



(a) Draw in the solution to the following initial value problems: (i) $dP/dt = 0.05P - 0.0005P^2$, $P(0) = 25$ (ii) $dP/dt = 0.05P - 0.0005P^2$, $P(0) = 150$. Describe the behavior of the population in each case.

(b) If a population modeled by this differential equation started at 25, give a very rough estimate of the population after 40 weeks.

(c) Are there any equilibrium solutions to this differential equation? If so, what are they and what do they mean in the context of this problem?

(d) Show that the function $P(t) = \dfrac{100}{1 + Ce^{-t/20}}$ is a solution to this differential equation, where C is any constant.

(e) Use the result of part (d) to find exact solutions to the initial value problems from part (a) and a more accurate answer to the question asked in part (b).

Definition. A differential equation that can be written in the form

$$\frac{dP}{dt} = kP(N - P),$$

where k and N are constants, is called a *logistic* differential equation.

Notes.

1. Logistic differential equations are often used to model population growth. The constant N is called the *carrying capacity* of the population.

2. Using separation of variables, it can be shown that the general solution to the above logistic differential equation is given by

$$P(t) = \frac{N}{\left(\frac{N - P_0}{P_0}\right) e^{-kNt} + 1},$$

where P_0 is the population at time $t = 0$; that is, the initial population. See our text, pp. 440-442, for details.

Section 8.1 – Sequences

Example 1. Suppose that you invest \$10000 in an account that earns 5% annual interest, compounded monthly. At the end of the n month, let A_n be the total amount of money in your account. Complete the table shown to the right. Then, describe the behavior of the sequence $A_1, A_2, A_3, \ldots$ of account balances.

n	Interest	A_n
0	\$0.00	\$10000.00
1	\$41.67	\$10041.67
2	\$41.84	\$10083.51
3	\$42.01	\$10125.52
4	\$42.19	\$10167.71
5		
6		

Definition. A *sequence* is a list of numbers written in a definite order:

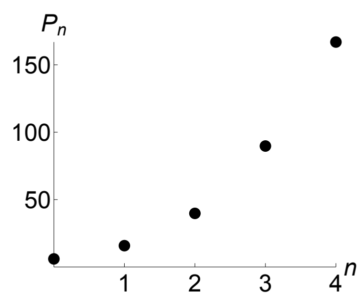
$$s_1,\ s_2,\ s_3,\ \ldots,\ s_n,\ \ldots\ =\ \{s_n\}$$

We say that an infinite sequence *converges to* L or *has the limit* L if we can make s_n arbitrarily close to L by taking n sufficiently large. If the sequence $\{s_n\}$ converges to L, we write ____________________.

Example 2. Suppose that the population of fish in a lake after n years is modeled by the formula

$$P_n\ =\ \frac{1002e^n}{3e^n + 164}.$$

A portion of the graph of the sequence $\{P_n\}$ is shown to the right.



(a) Find the values of P_0, P_1, P_2, P_3, and P_4. What do these numbers mean in the context of this problem?

(b) Does the sequence $\{P_n\}$ converge, and if so, to what value?

Example 3. For each of the following, determine whether the sequence converges or diverges. If it converges, find the limit. Justify your answers.

(a) $\left\{\dfrac{\ln n}{n}\right\}$

(b) $\left\{\dfrac{e^{2n}}{5e^n + 1}\right\}$

(c) $\left\{ \dfrac{2n\sqrt{n}}{3n\sqrt{n}+\sqrt{n}-2} \right\}$

Examples and Exercises

1. For each of the following, determine whether the sequence converges or diverges. If it converges, find the limit. Justify your answers.

   (a) $\left\{ \dfrac{n^2-6n+20}{n^2-6n+10} \right\}$

   (b) $\left\{ \dfrac{2n+n^{5/2}}{3n^2(\sqrt{n}+1)} \right\}$

(c) $\left\{\dfrac{n^2+1}{3e^{3n}-4}\right\}$

(d) $\left\{\dfrac{2^n-1}{3^n+1}\right\}$

(e) $\left\{\dfrac{\sin n}{n^2}\right\}$

Section 8.2 – Geometric Series

Example 1. A patient taking an anti-inflammatory drug takes one 100 mg tablet every three hours. It is known that, at the end of three hours, about 10% of the drug is still in the body.

(a) Find an expression for Q_n, the amount of the drug present in the body right after the nth tablet is taken.

(b) After a long period of time on the medication, find Q, the amount of the drug present in the body right after a tablet is taken.

Geometric Series Definition and Facts. Let a and r be constant. The infinite sum

$$\sum_{n=0}^{\infty} ar^n =$$

is called a geometric series. The number a is called the ______________ of the series, and r is called the ______________ of the series. The series has a finite sum if $-1 < r < 1$; in particular, we have the following:

1. $\displaystyle\sum_{n=0}^{\infty} ar^n =$ ______________ if $-1 < r < 1$.

2. $\displaystyle\sum_{n=0}^{\infty} ar^n$ ______________ if $|r| \geq 1$.

Example 2. Use the formula for the sum of a geometric series given above to confirm your answer to part (b) of Example 1 on page 84.

Examples and Exercises

1. For each of the following series, determine if the series is geometric. If it is geometric, determine whether or not it converges, and if it converges, find the sum of the series.

   (a) $\displaystyle\sum_{n=0}^{\infty} \frac{2^{n+1}}{3^n}$

   (b) $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^2}$

(c) $\displaystyle\sum_{n=1}^{\infty} \frac{1}{2} \cdot \left(\frac{4}{3}\right)^{n}$

(d) $\displaystyle\sum_{n=1}^{\infty} 3\left(\frac{1}{2}\right)^{n+1}$

(e) $1 - \frac{1}{3} + \frac{1}{9} - \frac{1}{27} + \frac{1}{81} - \cdots$

(f) $\displaystyle\sum_{n=1}^{\infty} \frac{e^{2n}}{e^{2n+1}}$

(g) $\displaystyle\sum_{n=1}^{\infty} \frac{n+1}{3n+1}$

2. A patient takes one 200 mg ibuprofen tablet every 8 hours, and it is known that 6% of the drug is still in the body at the end of 8 hours. After taking the medication for a long period of time, find the amount of the ibuprofen present in the patient's body right after a tablet is taken.

Section 8.3 – Series of Real Numbers

Example 1. What is the value of the infinite sum $\sum_{n=1}^{\infty} \frac{1}{n^3}$?

Definition. We define the *infinite series* $\sum_{n=1}^{\infty} a_n$ to be the limit of the sequence $\{S_n\}_{n=1}^{\infty}$ of *partial sums*, where $S_n = a_1 + a_2 + a_3 + \cdots + a_n$. Therefore,

$$\sum_{n=1}^{\infty} a_n = \lim_{n\to\infty} S_n =$$

Note. If $\lim_{n\to\infty} S_n = S$, where S is a finite number, we write $\sum_{n=1}^{\infty} a_n = S$, and we say that the series

______________________________ to S.

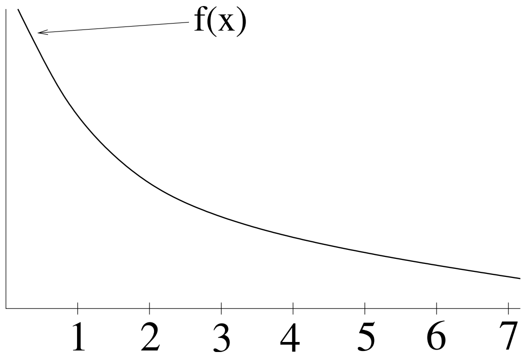
Example 2. Discuss the convergence of $\displaystyle\sum_{n=1}^{\infty} \frac{n}{n+1}$.

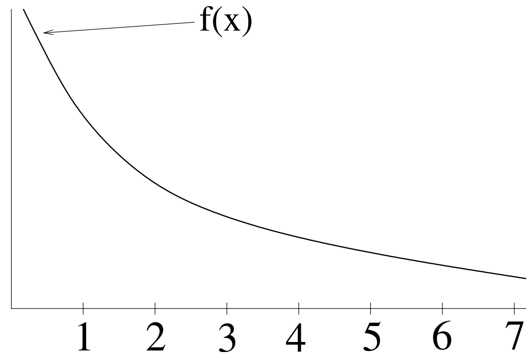
Theorem (The Divergence Test). If $\displaystyle\lim_{n\to\infty} a_n \neq 0$, then the series $\displaystyle\sum_{n=1}^{\infty} a_n$ ____________________.

Example 3. Give a formal proof that the series $\displaystyle\sum_{n=1}^{\infty} \frac{n}{n+1}$ from Example 2 diverges.

Example 4. Give a formal proof that the series $\displaystyle\sum_{n=1}^{\infty} \frac{e^n+1}{3e^n+n}$ diverges.

Illustration. Consider the series $\sum_{n=1}^{\infty} a_n$, where $a_n = f(n)$, and f is continuous, positive, and decreasing.





Theorem (The Integral Test). Suppose f is continuous, positive, and decreasing on $[c, \infty)$, and let $a_n = f(n)$.

1. If ____________________ diverges, then ____________________ diverges.

2. If ____________________ converges, then ____________________ converges.

Example 5. Give a formal proof that the series $\sum_{n=1}^{\infty} \frac{1}{n^3}$ from Example 1 converges.

Theorem (p-series Test). Let p be a constant. The series $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^p}$ converges if __________ and diverges if __________ .

Example 6. Prove whether or not the series $\displaystyle\sum_{n=2}^{\infty} \frac{1}{n \ln n}$ converges.

Example 7. Discuss the convergence of $\displaystyle\sum_{n=1}^{\infty} \frac{1}{2^n - 1}$.

Theorem (Limit Comparison Test). Let $\sum a_n$ and $\sum b_n$ be series with positive terms. If

$$\lim_{n\to\infty} \frac{b_n}{a_n} = c$$

for some constant c that is ____________________ and ____________________ , then $\sum a_n$ and $\sum b_n$ either both converge or both diverge.

Notes:

1. The purpose of the Limit Comparison Test is to choose a "nice" series $\sum a_n$ (whose convergence or divergence is known) to decide whether or not the given series $\sum b_n$ converges.

2. Generally, we choose $\sum a_n$ to be either a ____ or a ____ that is somehow similar to $\sum b_n$.

Example 8. Use the Limit Comparison Test to test each of the following series for convergence or divergence. Write your answers as formal proofs.

(a) $\displaystyle\sum_{n=1}^{\infty} \frac{1}{2^n - 1}$

(b) $\displaystyle\sum_{n=1}^{\infty} \frac{3n}{2n^2+4n+1}$

Theorem. (Arithmetic Operations) Suppose that $\displaystyle\sum_{n=1}^{\infty} a_n$ and $\displaystyle\sum_{n=1}^{\infty} b_n$ converge. Then we have

1. $\displaystyle\sum_{n=1}^{\infty} (a_n + b_n) = \sum_{n=1}^{\infty} a_n + \sum_{n=1}^{\infty} b_n$.
2. $\displaystyle\sum_{n=1}^{\infty} (a_n - b_n) = \sum_{n=1}^{\infty} a_n - \sum_{n=1}^{\infty} b_n$.
3. If c is constant, then $\displaystyle\sum_{n=1}^{\infty} (ca_n) = c\sum_{n=1}^{\infty} a_n$.

Theorem (The Ratio Test). Suppose that $\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = r$.

(a) If $0 \leq r < 1$, then the series $\sum a_n$ ____________________ .

(b) If $r > 1$, then the series $\sum a_n$ ____________________ .

(c) If $r = 1$, then the test is ____________________ .

Note. Observe that if $\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = r$, then, for large n, the series is approximately ____________________ with a ______________________________ of r.

Example 9. Prove whether or not the series $\sum_{n=1}^{\infty} \frac{3n^2}{n!}$ converges.

Examples and Exercises

1. Test each of the following series for convergence. Express your answers as proofs that follow the proof-writing guidelines we have discussed in class.

   (a) $\displaystyle\sum_{n=1}^{\infty} \frac{2n^3}{3n^3+1}$

   (b) $\displaystyle\sum_{n=1}^{\infty} \frac{n}{n^2+1}$

2. Test each of the following series for convergence. Express your answers as proofs that follow the proof-writing guidelines we have discussed in class.

   (a) $\displaystyle\sum_{n=1}^{\infty} \frac{2n^2 - 1}{3n^4 + 2}$

   (b) $\displaystyle\sum_{n=1}^{\infty} \frac{2^n}{n3^n}$

(c) $\displaystyle\sum_{n=1}^{\infty} \frac{2}{\sqrt{n+1}}$

(d) $\displaystyle\sum_{n=1}^{\infty} \frac{4}{n+e^{2n}}$

(e) $\displaystyle\sum_{n=1}^{\infty} \frac{e^n - n}{e^n + 1}$

3. Suppose a_n, b_n, c_n, $d_n \geq 0$ for all n, and that we are given all of the following:

$$\lim_{n\to\infty} \frac{(4/3)^n}{a_n} = 49, \quad \lim_{n\to\infty} \frac{1/n^3}{b_n} = 0, \quad \lim_{n\to\infty} \frac{a_n}{c_n} = 6, \quad \lim_{n\to\infty} \frac{2^{-n}}{d_n} = \frac{1}{5}$$

(a) Which of the series $\sum a_n$, $\sum b_n$, $\sum c_n$, and $\sum d_n$ definitely converge? Justify your answer.

(b) Which of the series $\sum a_n$, $\sum b_n$, $\sum c_n$, and $\sum d_n$ definitely diverge? Justify your answer.

4. Test each of the following series for convergence. Express your answer as a proof that follows the proof-writing guidelines we have discussed in class.

   (a) $\displaystyle\sum_{n=1}^{\infty} \frac{3n}{e^{2n+1}}$

   (b) $\displaystyle\sum_{n=1}^{\infty} \frac{2n^2-1}{3n^3+2}$

(c) $\displaystyle\sum_{n=1}^{\infty} \frac{e^n}{n!}$

(d) $\displaystyle\sum_{n=1}^{\infty} \frac{n}{2n^3 - 1}$

(e) $\displaystyle\sum_{n=1}^{\infty} \frac{n^2-1}{3n^2+4}$

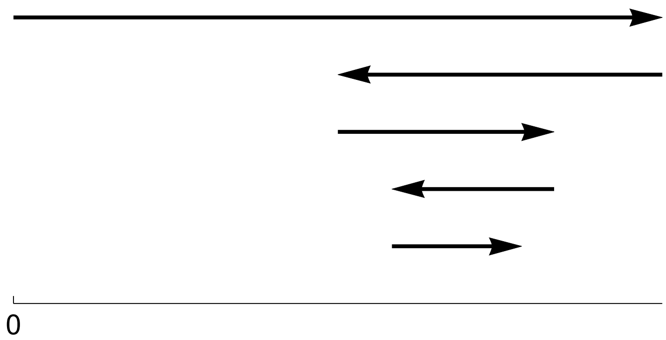
(f) $\displaystyle\sum_{n=1}^{\infty} \frac{1}{6^n-1}$

(g) $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n(1+\ln n)^2}$

Section 8.4 – Alternating Series

Example 1. Discuss the convergence of the "alternating" series

$$\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n}$$



Theorem (Alternating Series Test). Let $b_n > 0$ for all n. A series of the form $\sum_{n=1}^{\infty} (-1)^{n-1} b_n$ or $\sum_{n=1}^{\infty} (-1)^{n} b_n$ is called an *alternating series.* If an alternating series satisfies

(a) $\lim_{n\to\infty} b_n = 0$ and (b) $b_{n+1} \leq b_n$ for all n,

then the series ______________________________ .

Theorem (Alternating Series Estimation). If S is the sum of an alternating series that satisfies conditions (a) and (b) in the above theorem, then

$$|S - S_n| \leq \underline{\hspace{3cm}},$$

where S_n is the sum of the first n terms in the series.

Note. This says that if we use the first n terms to approximate the sum of the series, then the maximum error in the approximation is __________ .

Example 2.

(a) Prove that the series $\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n}$ from Example 1 converges.

(b) Use the first 4 terms to approximate the infinite sum $\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n}$. What is the maximum possible error in your approximation?

Example 3.

(a) Prove that $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{2n^3-1}$ converges.

(b) Estimate $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{2n^3-1}$ accurate to within 0.005 of its true value.

Theorem (Absolute Convergence Test). If the series $\sum_{n=1}^{\infty} |a_n|$ converges, then the series $\sum_{n=1}^{\infty} a_n$ also converges.

Example 4. Does $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n}{n^2}$ converge? Justify your answer.

Examples and Exercises

1. Test each of the following series for convergence. Express your answers as proofs that follow the proof-writing guidelines we discussed in class.

   (a) $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt{n}}$

(b) $\displaystyle\sum_{n=1}^{\infty} \frac{n+5}{5^n}$

(c) $\displaystyle\sum_{n=1}^{\infty} \frac{2n+5}{3+n\sqrt{n}}$

(d) $\displaystyle\sum_{n=1}^{\infty} \frac{e^n}{e^n+1}$

(e) $\displaystyle\sum_{n=1}^{\infty} \frac{6}{5^n-1}$

(f) $\displaystyle\sum_{n=2}^{\infty} \frac{1}{n(\ln n)^2}$

(g) $\displaystyle\sum_{n=1}^{\infty} \frac{2n}{8n-5}$

(h) $\displaystyle\sum_{n=1}^{\infty} \frac{3^n}{(2n)!}$

(i) $\displaystyle\sum_{n=1}^{\infty} \frac{(-2)^n}{e^{2n}}$

(j) $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n\sqrt{n}}$

2. How many terms would one have to add up in order to approximate $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{\sqrt{n}}$ to within 0.1 of its true value? Justify your answer. (<u>Note</u>: You need not actually add up all the terms, just determine how many you would need to achieve the desired accuracy.)

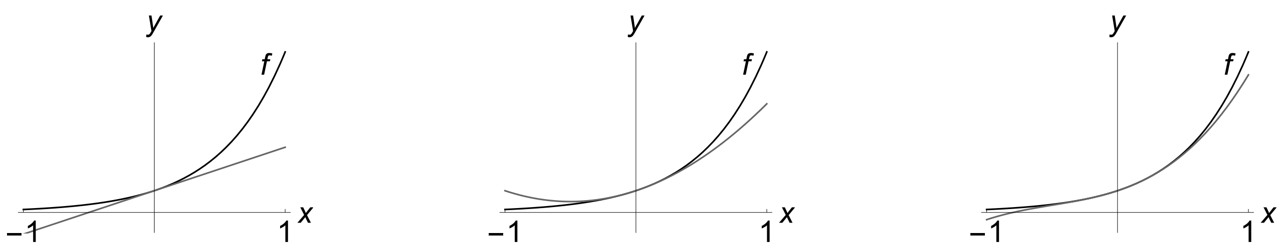
Section 8.5 – Taylor Polynomials and Taylor Series

Preliminary Example. (a) Describe how to find a series $P(x) = c_0 + c_1x + c_2x^2 + c_3x^3 + c_4x^4 + \cdots$ such that $P(x) = e^{2x}$.

(b) Below are graphs of $f(x) = e^{2x}$ together with the following polynomials from part (a):

$$P_1(x) = 1 + 2x, \quad P_2(x) = 1 + 2x + 2x^2 \quad \text{and} \quad P_3(x) = 1 + 2x + 2x^2 + \frac{4}{3}x^3$$

Label the graphs of P_1, P_2, and P_3 in the diagrams below.



Definition. Let $f(x)$ be a function. Then the *Taylor Series* for $f(x)$ centered at a is given by the formula

$$\begin{aligned} P(x) &= \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n \\ &= f(a) + \frac{f'(a)}{1!}(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \frac{f'''(a)}{3!}(x-a)^3 + \cdots . \end{aligned}$$

The sum of the 1st n terms of the series, $P_n(x)$, is called the *nth degree Taylor Polynomial* of $f(x)$; that is,

$$P_n(x) = f(a) + \frac{f'(a)}{1!}(x-a) + \cdots + \frac{f^{(n)}(a)}{n!}(x-a)^n.$$

Note. When the series is centered at $a = 0$, the Taylor series for $f(x)$ is often called the *Maclaurin series* for $f(x)$, and it has the following form:

Example 1.

(a) Complete the process we began in the Preliminary Example by finding a formula for the Taylor series for $f(x) = e^{2x}$ centered at $a = 0$.

(b) Use the Taylor polynomials P_1, P_2, and P_3 to approximate $f(0.1)$. Compare your approximations with your calculator's value of $f(0.1)$.

Example 2.

(a) Build the Taylor series for $f(x) = \ln x$ centered at $a = 1$.

(b) Use the Taylor polynomials P_1, P_2, and P_3 to approximate $\ln(1.05)$. Compare your approximations with your calculator's value of $\ln(1.05)$.

Error Approximations for Taylor Polynomials. Let f be a continuous function with $n+1$ derivatives, and let P_n be the nth degree Taylor Polynomial for f centered at a. For a number c, the *error* that results from approximating $f(c)$ by $P_n(c)$ is $E_n(c) =$ ______________________________. It can be shown that

$$|E_n(c)| \leq M \cdot \frac{|c-a|^{n+1}}{(n+1)!},$$

where __________________ $\leq M$ on the interval $[a, c]$.

Note: The Taylor series for $f(x)$ will converge to $f(x)$ if $\lim_{n\to\infty} |E_n(x)| = 0$.

Example 3. In Exercise 2 on page 120, you used P_3, P_5, and P_7 for the function $f(x) = \sin x$ to estimate the value of $\sin 2$. Find the maximum error in each of your estimates, and complete the table shown.

Calculator: $\sin 2 \approx$

n	$P_n(2)$	Maximum Error
3		
5		
7		

Example 4. This example refers to the Taylor series for $f(x) = \ln x$ from Example 2 on page 115.

(a) Find the maximum errors your estimates of $\ln(1.05)$ using P_1, P_2, and P_3.

(b) Estimate $\ln(1.5)$ accurate to within 0.003 of its true value.

Examples and Exercises

1. Let $f(x) = e^{-x}$.

   (a) Find the Maclaurin series of $f(x) = e^{-x}$.

   (b) Use P_2, P_3, and P_4 to approximate $f(1)$. Compare your approximations with your calculator's value of $f(1)$.

2. Let $f(x) = \sin x$.

   (a) Find the Maclaurin series of $f(x) = \sin x$.

   (b) Use P_3, P_5, and P_7 to approximate $\sin 2$. Compare your approximations with your calculator's value of $\sin 2$.

3. Let $f(x) = x^{-2}$.

   (a) Find the Taylor series for $f(x) = x^{-2}$ at $a = 1$.

   (b) Use P_2 to estimate the value of $f(1.1)$.

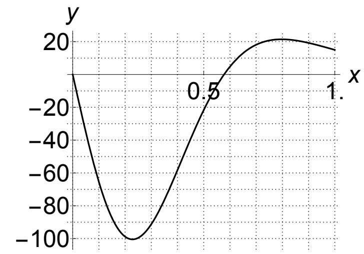
   (c) Use the Taylor error approximation formula to find the maximum error in your estimate from part (b).

4. Let $f(x) = \arctan x$. Using WolframAlpha, a student calculates the 5th Maclaurin polynomial of $f(x)$ to obtain

$$P_5(x) \;=\; x - \frac{x^3}{3} + \frac{x^5}{5},$$

and she obtains the graph of $f^{(6)}(x)$, the sixth derivative of f, shown to the right.

$f^{(6)}(x)$



(a) Use P_5 to estimate arctan 1, and use the Taylor error formula to find the maximum possible error in your estimate.

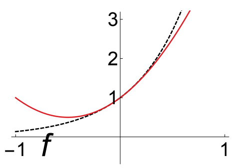
(b) Use P_5 to estimate arctan 0.1, and use the Taylor error formula to find the maximum possible error in your estimate.

Section 8.6 – Power Series

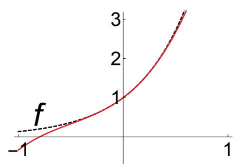
Example 1. Let $P(x) = \sum_{n=0}^{\infty} \frac{2^n}{n!} x^n$, which is the MacLaurin series for $f(x) = e^{2x}$ from Example 1 on page 114.

(a) Write out the partial sums $P_2(x), P_3(x)$, and $P_4(x)$.

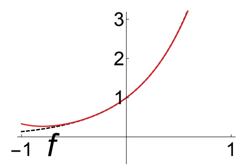
$f(x)$ and $P_2(x)$



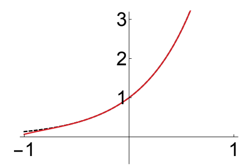
$f(x)$ and $P_3(x)$



$f(x)$ and $P_4(x)$



$f(x)$ and $P_5(x)$



(b) Determine the values of x for which $\sum_{n=0}^{\infty} \frac{2^n x^n}{n!}$ converges.

Definition. A *power series* centered at a is a function of the form

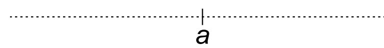
$$\sum_{n=0}^{\infty} c_n(x-a)^n \;=\; c_0 + c_1(x-a) + c_2(x-a)^2 + c_3(x-a)^3 + \ldots,$$

where $c_0, c_1, c_2, \ldots$ is a sequence of real numbers and x is an independent variable.

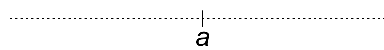
Note: One common example of a power series is a ______________________________ .

Convergence of Power Series. Consider the power series $\sum_{n=0}^{\infty} c_n(x-a)^n$. The set of all x values for which this series converges is called the ____ of the power series. For such a series, there are three possible convergence behaviors:

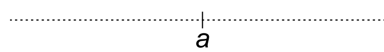
1. The series converges only when $x = a$.



2. The series converges for all x.



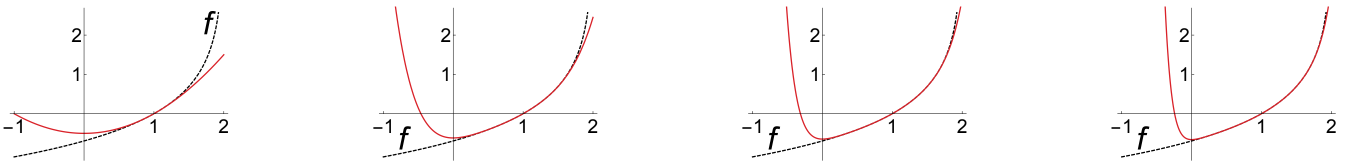
3. There is a positive number p such that the series converges if $|x-a| < p$ and diverges if $|x-a| > p$.



Note: To find the interval of convergence of a power series, we start by applying the _______________ Test. Other convergence tests will be needed to determine convergence at ____________________, if there are any.

Example 2. The series $\displaystyle\sum_{n=1}^{\infty} \frac{(x-1)^n}{n}$ is a power series for the function $f(x)$ that appears in the diagrams below.

$f(x)$ and $P_2(x)$ $\quad$ $f(x)$ and $P_6(x)$ $\quad$ $f(x)$ and $P_{10}(x)$ $\quad$ $f(x)$ and $P_{14}(x)$



(a) Use the diagrams to guess the interval of convergence of the series.

(b) Find the interval of convergence of $\displaystyle\sum_{n=1}^{\infty} \frac{(x-1)^n}{n}$.

Example 3. Find the interval of convergence of the power series $\displaystyle\sum_{n=1}^{\infty} \frac{(x-2)^n}{2^n\sqrt{n}}$.

Power Series Representations of Some Common Functions

	Power Series		**Valid For**
$\sin x$	$= x - \dfrac{x^3}{3!} + \dfrac{x^5}{5!} - \dfrac{x^7}{7!} + \dfrac{x^9}{9!} - \cdots$	$= \displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!}$	$-\infty < x < \infty$
$\cos x$	$= 1 - \dfrac{x^2}{2!} + \dfrac{x^4}{4!} - \dfrac{x^6}{6!} + \dfrac{x^8}{8!} - \cdots$	$= \displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!}$	$-\infty < x < \infty$
e^x	$= 1 + x + \dfrac{x^2}{2!} + \dfrac{x^3}{3!} + \dfrac{x^4}{4!} + \cdots$	$= \displaystyle\sum_{n=0}^{\infty} \frac{x^n}{n!}$	$-\infty < x < \infty$
$\dfrac{1}{1-x}$	$= 1 + x + x^2 + x^3 + x^4 + \cdots$	$= \displaystyle\sum_{n=0}^{\infty} x^n$	$-1 < x < 1$

Example 4.

(a) Find a power series for $f(x) = e^{-x^2}$ centered at $a = 0$.

(b) Use the first three nonzero terms of your power series to approximate the value of $\displaystyle\int_0^1 e^{-x^2}\,dx$. What is the maximum possible error in your approximation?

Power Series Differentiation and Integration Theorem. Suppose $f(x)$ has a power series expansion

$$f(x) \;=\; \sum_{n=0}^{\infty} c_n x^n \;=\; c_0 \,+\, c_1 x \,+\, c_2 x^2 \,+\, c_3 x^3 \,+\, c_4 x^4 \,+\, \cdots\,,$$

so that the series converges absolutely to $f(x)$ on the interval $-r < x < r$. Then, for $-r < x < r$, we have

- $f'(x) \;=$

- $\displaystyle\int f(x)\,dx \;=$

Example 5. Find a power series centered at $a = 0$ for each of the following functions, and find the interval of convergence.

(a) $f(x) = \dfrac{1}{1-2x}$

(b) $f(x) = \dfrac{1}{1+2x^4}$

(c) $f(x) = \dfrac{8x^4}{(1-x^4)^2}$

Example 6.

(a) Find a power series expansion for $f(x) = \arctan x$ about $a = 0$, and give an interval on which the series converges.

(b) Use $P_5(x)$ (from your series in part (a)) to approximate the value of $\arctan 0.5$, and compare your approximation with the value your calculator gives for $\arctan 0.5$.

Examples and Exercises

For Exercises 1 and 2 below, find the interval of convergence of the given power series.

1. $\displaystyle\sum_{n=0}^{\infty} \frac{x^n}{n!}$

2. $\displaystyle\sum_{n=1}^{\infty} \frac{x^n}{n^2 3^n}$

3. Find a power series for each of the following centered at $a = 0$. Also give an interval of validity for each series.

   (a) $\dfrac{4}{1+2x}$

   (b) $\dfrac{1}{1-3x^2}$

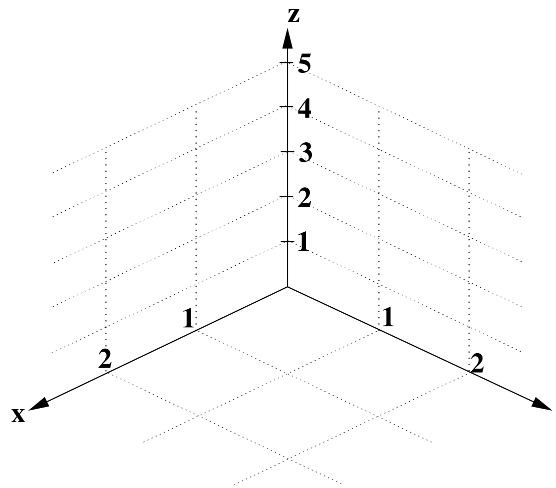
(c) $\dfrac{1}{(1-x)^2}$

4. Use the first three nonzero terms from the series we found in part (b) of Example 5 on page 128 to estimate $\displaystyle\int_0^{0.5} \frac{1}{1+2x^4}\,dx$. What is the maximum error in your approximation?

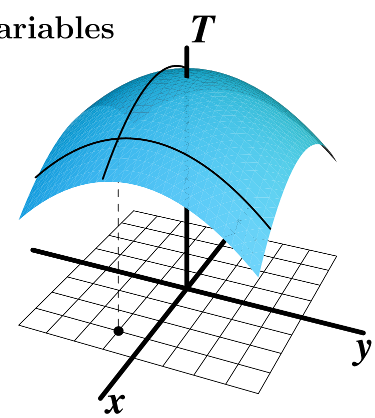
5. Use a power series to approximate $\int_0^1 \sin(x^2)\,dx$ to within 0.0001 of its true value.

Functions of Two Variables: An Introduction

Example 1: The Graph of a Function of Two Variables. Let $z = f(x, y) = x^2 + y^2$. Make a table and plot a few points on the graph of this function.



Derivatives of Functions of Several Variables



Example 2. The temperature, T, in degrees Celsius, at a point (x, y) on an 8 meter by 8 meter metal plate is given by the function $T = f(x, y)$, whose graph is shown to the right. Our goal is to discuss the problem of finding the rate at which the temperature of the plate changes, at a particular point, as you move away from the point in either the x-direction or the y-direction.

Partial Derivatives of f With Respect to x and y

Let $z = f(x, y)$. Then for all points at which the limits exist, we define the **partial derivatives at the point (a, b)** by

$$\begin{aligned} f_x(a,b) &= \text{rate of change of } f \text{ with respect to } x \text{ at } (a,b) = \lim_{h\to 0} \frac{f(a+h,b)-f(a,b)}{h}, \\ f_y(a,b) &= \text{rate of change of } f \text{ with respect to } y \text{ at } (a,b) = \lim_{h\to 0} \frac{f(a,b+h)-f(a,b)}{h}. \end{aligned}$$

If we let a and b vary, we have the **partial derivative functions** $f_x(x, y)$ and $f_y(x, y)$.

Some Alternate Notations for Partial Derivatives

Again, let $z = f(x, y)$. Then

$$\begin{aligned} f_x &= \\ f_y &= \\ (f_x)_x &= \\ (f_y)_y &= \\ (f_x)_y &= \\ (f_y)_x &= \end{aligned}$$

Examples and Exercises

1. For each of the following functions, find the first partial derivatives with respect to both independent variables.

   (a) $f(x, y) = 3x + 5y + 2x^3 + 4y^3$

(b) $f(x, y) = 4x^3 + 6xy^2 - 2x^2y + 3$

(c) $f(x, y) = 5x^3 - 3y^2 + 4xy - 8$

(d) $f(x,y) = x^4 + y^4 - 6x^3y^2 + 1$

2. For each of the following functions, find the first partial derivatives with respect to both independent variables.

   (a) $f(x,y) = ye^{xy}$

(b) $f(x, y) = \dfrac{x}{x^2 + y^2}$

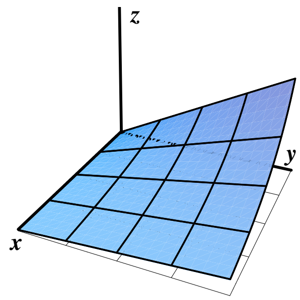
(c) $f(x, t) = (x + t^2)^4$

(d) $f(s, t) = \sin t \cos(st)$

3. For each of the following functions, find the following first and second partial derivatives: f_x, f_y, f_{xx}, f_{yy}, and f_{xy}.

   (a) $f(x, y) = x \ln y$ (b) $f(x, y) = x^2 + xy + y^2$ (c) $f(x, y) = e^{x^2+y^2}$

4. Given to the right is the graph of the function $f(x,y) = \frac{1}{2}ye^{-x/2}$ on the domain $0 \leq x \leq 4$ and $0 \leq y \leq 4$.



   (a) Use the graph to the right to rank the following quantities in order from smallest to largest: $f_x(3,2),\ f_x(1,2),\ f_y(3,2),\ f_y(1,2),\ 0$

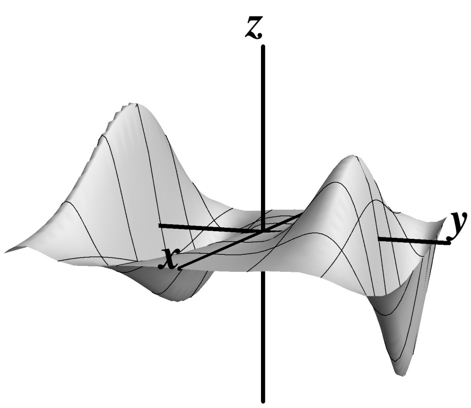
   (b) Use algebra to calculate the numerical values of the four derivatives from part (a) and confirm that your answer to part (a) is correct.

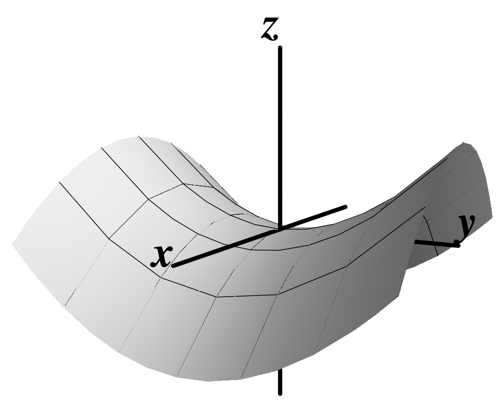
Maximum and Minimum Values of Functions of Two Variables

Definitions. Let $z = f(x, y)$ be a function of two variables.

1. We say that $f(x, y)$ has an *absolute maximum* at (a, b) if $f(a, b) \geq f(x, y)$ for all (x, y) on the entire graph of f.
2. We say that $f(x, y)$ has an *absolute minimum* at (a, b) if $f(a, b) \leq f(x, y)$ for all (x, y) on the entire graph of f.
3. We say that $f(x, y)$ has a *local maximum* at (a, b) if $f(a, b) \geq f(x, y)$ for all (x, y) near (a, b). In this case, $f(a, b)$ is called a *local maximum value* of f.
4. We say that $f(x, y)$ has a *local minimum* at (a, b) if $f(a, b) \leq f(x, y)$ for all (x, y) near (a, b). In this case, $f(a, b)$ is called a *local minimum value* of f.
5. A point (a, b) is called a *critical point* of f if $f_x(a, b) = f_y(a, b) = 0$.

Exercise. For each of the following functions $z = f(x, y)$, label the points on the graph where it appears that f has a maximum or a minimum value (indicate which type) or a critical point.





Theorem. If a differentiable function f has a local maximum or a local minimum at (a, b), then (a, b) is a ________________________ of f. In other words, both $f_x(a, b)$ and $f_y(a, b)$ equal ______.

Question 1. At a critical point, how do we tell if we have a local maximum, a local minimum, or neither?

Theorem (Second Derivative Test). Suppose the second partial derivatives of f are continuous on a disk with center (a,b), and suppose that (a,b) is a critical point of f. (Recall that to be a "critical point," both derivatives f_x and f_y must equal zero at that point.) Let

$$D \;=\; D(a,b) \;=\; f_{xx}(a,b)f_{yy}(a,b) - [f_{xy}(a,b)]^2.$$

(a) If $D > 0$ and $f_{xx}(a,b) > 0$, then $f(a,b)$ is a local minimum.

(b) If $D > 0$ and $f_{xx}(a,b) < 0$, then $f(a,b)$ is a local maximum.

(c) If $D < 0$, then $f(a,b)$ is not a local maximum or minimum.

Note 1. If $D = 0$, then the above test gives no information.

Note 2. If f has a critical point at (a,b) but there is no local maximum or local minimum value at (a,b), then we call (a,b) a *saddle point* of f.

Question 2. How do we find the *absolute* maximum and minimum values of a function?

To find the absolute maximum and minimum values of a continuous function f on a closed, bounded set D:

1. Find the values of f at the ______________________________ of f in D.
2. Find the maximum and minimum values of f on the ______________________________ of D.
3. The largest of the values from steps 1 and 2 is the absolute maximum value; the smallest of these values is the absolute minimum value.

Examples and Exercises

1. Find all local maxima, local minima, and saddle points for each of the following functions.

   (a) $f(x, y) = 2x^3 - 24xy + 16y^3$

(b) $f(x,y) = xy - x^2y - xy^2$

(c) $f(x,y) = \frac{x^2}{2} + 3y^3 + 9y^2 - 3xy + 9y - 9x$