Algorithms and Data Structures

Mini-talk series

About this series

We're going to discuss

- Concepts
- Data structures
- Algorithms
- Algorithmic thinking

Concepts

- **Time & space complexity**
- Memory (stack vs. heap)
- Recursion
- Dynamic programming
- Divide and conquer

Data structures

- Linked lists
- Arrays and vectors
- Stacks and queues
- Heaps (min/max)
- Hash sets
- Hash tables
- Trees and graphs

Algorithms

- Breadth-first search
- Depth-first search
- Binary search

Algorithmic thinking

- How to approach the solution to a coding problem?
- How to choose the right algorithm and data structure?
- How to optimize an algorithm?

Time and space complexity

Time complexity of algorithms

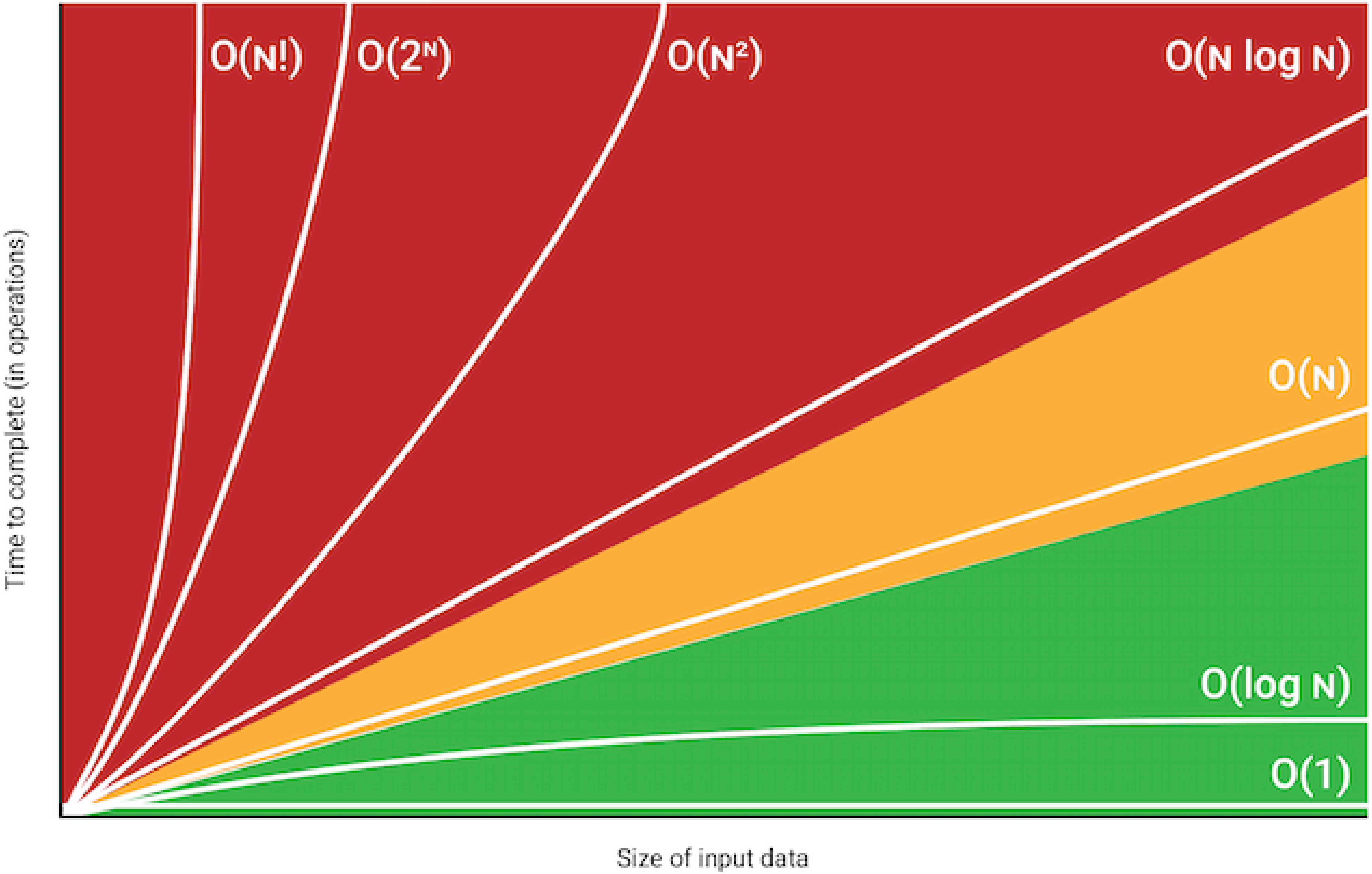
- We don't want the specify the time of an algorithms in seconds because that is machine *dependent*
- Instead: As a measure of how the running time relates to the input size.
- A rough approximation of the number of operations performed by the algorithm.
- "On the order of..."
- The lower the better.
- Represented by the "Big O"-notation.

Big O notation

- Complexity bounds
  - Big O: Upper bound
  - Big omega, Ω: Lower bound
  - Big theta, Θ: Tight bound
  - In pratice, O() is in place of Θ
- Best case, worst case, expected case

Properties of Big O notation

- Constants are ignored
- Only the most important factors are considered
- Add or multiple complexities: $O(A + B)$ vs. $O(A \cdot B)$
- Amortized time: Averaged asymptotic time
- Common runtimes: $O(1)$, $O(\log_2 n)$, $O(n)$, $O(n \cdot \log_2 n)$, $O(n^2)$, $O(n^3)$, $O(n^4)$, …
- Space complexity


O(N!)
O(2^N)
O(N²)
O(N log N)
O(N)
O(log N)
O(1)
Time to complete (in operations)
Size of input data

Example

```cpp
// increment each value in `vec` by `value`
vector<int> increment(const vector<int>& vec, int value) {
  vector<int> result;             // O(1) time, O(1) space
  for (const auto& v : vec) {    // n times:
    result.push_back(v + value); //   O(1) time, O(1) space
  }
  return result;
}

vector<int> vec = {1, 2, 3, 4, 5};
vector<int> result = increment(vec, 42); // O(n) time and O(n) space
```

Case study

Count unique integers in an unsorted vector, i.e.

`int count_unique(const vector<int>& vec)`

e.g.

`count_unique([3, 2, 3, 4, 3])` → `3` (unique: 3, 2, 4)

Naïve solution

```cpp
int count_unique(const vector<int>& vec) {
  vector<int> unique_vec;
  for (const auto& v1 : vec) {              // a) O(?)
    bool found = false;
    for (const auto& v2 : unique_vec) {  // b) O(?)
      if (v1 == v2) {
        found = true;
        break;
      }
    }
    if (!found) {
      unique_vec.push_back(v1);            // c) O(?)
    }
  }
  return unique_vec.size();               // d) O(?)
}

vector<int> vec = {3, 2, 3, 4, 3};
int result = count_unique(vec);            // e) O(?)
```

Naïve solution

```
int count_unique(const vector<int>& vec) {
  vector<int> unique_vec;
  for (const auto& v1 : vec) {              // a) O(n)
    bool found = false;
    for (const auto& v2 : unique_vec) {  // b) O(n) *
      if (v1 == v2) {
        found = true;
        break;
      }
    }
    if (!found) {
      unique_vec.push_back(v1);              // c) O(1) **
    }
  }
  return unique_vec.size();                // d) O(1)
}

vector<int> vec = {3, 2, 3, 4, 3};
int result = count_unique(vec);              // e) O(n * (n + 1) + 1) = O(n^2)
```

The naïve solution runs $O(n^2)$ time and uses $O(n)$ space.

Ideas for a better solution?

Using a sorted vector

```cpp
int count_unique(const vector<int>& vec) {
  if (vec.empty()) return 0;
  vector<int> sorted_vec = vec;                        // O(n)
  sort(sorted_vec.begin(), sorted_vec.end());          // O(n * log(n))
  int unique_count = 1;
  for (size_t i = 1; i < sorted_vec.size(); i++) { // O(n)
    if (sorted_vec[i] != sorted_vec[i - 1]) {          // O(1 + 1)
      unique_count++;
    }
  }
  return unique_count;
}

vector<int> vec = {3, 2, 3, 4, 3};
int result = count_unique(vec); // O(n + n * log(n) + n * 2) = O(n * log(n))
```

Using `std::set` (ordered set)

```cpp
int count_unique(const vector<int>& vec) {
  std::set<int> unique_set;
  for (const auto& v : vec) { // O(n)
    unique_set.insert(v);     // O(log(n))
  }
  return unique_set.size();   // O(1)
}

vector<int> vec = {3, 2, 3, 4, 3};
int result = count_unique(vec); // O(n * log(n))
```

Using `std::unordered_set`

```
int count_unique(const vector<int>& vec) {
  std::unordered_set<int> unique_set;
  for (const auto& v : vec) { // O(n)
    unique_set.insert(v);     // O(1) (amortized)
  }
  return unique_set.size();   // O(1)
}

vector<int> vec = {3, 2, 3, 4, 3};
int result = count_unique(vec); // O(n * 1 + 1) = O(n)
```

Space using `std::unordered_set`

```
int count_unique(const vector<int>& vec) {
  std::unordered_set<int> unique_set;
  for (const auto& v : vec) { // O(n)
    unique_set.insert(v);     // O(1) space (amortized)
  }
  return unique_set.size();
}

vector<int> vec = {3, 2, 3, 4, 3};
int result = count_unique(vec); // O(n) space
```

What is the time and space complexity of this recursive solution?

... but first, how does it work?

```cpp
int count_unique(const vector<int>& vec) {
  count_unique(vec, 0, vec.size());
}

int count_unique(const vector<int>& arr, int index, int size) {
  if (index >= size) return 0;
  bool has_seen_value = is_in_subarray(arr, index - 1, arr[index]);
  return (has_seen_value ? 0 : 1) + count_unique(arr, index + 1, size);
}

bool is_in_subarray(const vector<int>& arr, int size, int elem) {
  if (size < 0) return false;
  return arr[size] == elem || is_in_subarray(arr, size - 1, elem);
}
```

Example call stack for `count_unique({5, 6, 7, 6});`

```
// int count_unique(_, int index, int size)
count_unique(0, 4)
    // bool is_in_subarray(_, int size, int elem)
    is_in_subarray(-1, 5)

    count_unique(1, 4)
        is_in_subarray(0, 6)
            is_in_subarray(-1, 6)

        count_unique(2, 4)
            is_in_subarray(1, 7)
                is_in_subarray(0, 7)
                    is_in_subarray(-1, 7)

            count_unique(3, 4)
                is_in_subarray(2, 6)
                    is_in_subarray(1, 6)
                        is_in_subarray(0, 6)
                            is_in_subarray(-1, 6)

                count_unique(4, 4)
                  ...
```

What is the time and space complexity here?

```
Time:
count_unique:     O(n) calls
  is_in_subarray: O(n) calls
               => O(n * n) calls
                = O(n^2) time

Space:
count_unique:     O(n) stack frame allocations
  is_in_subarray: O(n) stack frame allocations
               => O(n * n) stack frame allocations
                = O(n^2) space
```

Recursive algorithms are *often* $O(n^B)$ time and O(n) space, where `B` is the number of recursive calls in each step.

Data Structure	Time Complexity								Space Complexity
	Average				Worst				Worst
	Access	Search	Insertion	Deletion	Access	Search	Insertion	Deletion	
Array	Θ(1)	Θ(n)	Θ(n)	Θ(n)	O(1)	O(n)	O(n)	O(n)	O(n)
Stack	Θ(n)	Θ(n)	Θ(1)	Θ(1)	O(n)	O(n)	O(1)	O(1)	O(n)
Queue	Θ(n)	Θ(n)	Θ(1)	Θ(1)	O(n)	O(n)	O(1)	O(1)	O(n)
Singly-Linked List	Θ(n)	Θ(n)	Θ(1)	Θ(1)	O(n)	O(n)	O(1)	O(1)	O(n)
Doubly-Linked List	Θ(n)	Θ(n)	Θ(1)	Θ(1)	O(n)	O(n)	O(1)	O(1)	O(n)
Skip List	Θ(log(n))	Θ(log(n))	Θ(log(n))	Θ(log(n))	O(n)	O(n)	O(n)	O(n)	O(n log(n))
Hash Table	N/A	Θ(1)	Θ(1)	Θ(1)	N/A	O(n)	O(n)	O(n)	O(n)
Binary Search Tree	Θ(log(n))	Θ(log(n))	Θ(log(n))	Θ(log(n))	O(n)	O(n)	O(n)	O(n)	O(n)
Cartesian Tree	N/A	Θ(log(n))	Θ(log(n))	Θ(log(n))	N/A	O(n)	O(n)	O(n)	O(n)
B-Tree	Θ(log(n))	Θ(log(n))	Θ(log(n))	Θ(log(n))	O(log(n))	O(log(n))	O(log(n))	O(log(n))	O(n)
Red-Black Tree	Θ(log(n))	Θ(log(n))	Θ(log(n))	Θ(log(n))	O(log(n))	O(log(n))	O(log(n))	O(log(n))	O(n)
Splay Tree	N/A	Θ(log(n))	Θ(log(n))	Θ(log(n))	N/A	O(log(n))	O(log(n))	O(log(n))	O(n)
AVL Tree	Θ(log(n))	Θ(log(n))	Θ(log(n))	Θ(log(n))	O(log(n))	O(log(n))	O(log(n))	O(log(n))	O(n)
KD Tree	Θ(log(n))	Θ(log(n))	Θ(log(n))	Θ(log(n))	O(n)	O(n)	O(n)	O(n)	O(n)

Memory