Hyperbolic Delaunay Geometric Alignment - Supplementary Material

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1 Other Models of Hyperbolic Geometry

Hyperbolic geometry is a non-Euclidean geometry that has a constant negative Gaussian curvature. Spaces following this geometry are said to be hyperbolic. It is possible to construct several models of hyperbolic geometry, for example the Lorentz hyperboloid and the Poincaré ball model, which are commonly deployed in machine learning to represent hierarchical data. All these models are related by isometric diffeomorphisms.

As emphasized in [2], the Poincaré ball possesses remarkable visualization capabilities and is intuitively interpretable. On the other hand, the Lorentz hyperboloid model is more convenient for Riemannian optimization and avoids numerical instabilities since its distance function does not involve fractions. We present briefly these two models below. The Poincaré and the Klein-Beltrami model can be seen as map projections of the Lorentz model and were both introduced by Beltrami [1].

1.1 Lorentz hyperboloid.

To begin with, the Lorentzian product – or, equivalently, the negative Minkowski bilinear form – between two vectors $z = (z_0, ..., z_n)$, $z' = (z'_0, ..., z'_n) \in \mathbb{R}^{n+1}$ is defined as $\langle z, z' \rangle_{\mathcal{L}} = -z_0 z'_0 + \sum_{i=1}^{n} z_i z'_i$. The Lorentz hyperboloid model $\mathbb{L}^n$ is the set of points $z \in \mathbb{R}^{n+1}$ with $z_0 > 0$ such that its Lorentzian product with itself is equal to -1, i.e.;

$$\mathbb{L}^n = \left\{ z \in \mathbb{R}^{n+1} \mid \langle z, z \rangle_{\mathcal{L}} = -1, \quad z_0 > 0 \right\}. \tag{1}$$

The space $\mathbb{L}^n$ is a Riemannian manifold once equipped with the metric tensor given in the standard coordinate system of $\mathbb{R}^{n+1}$ by the diagonal matrix with entries $-1, 1, ..., 1$. The geodesic distance between z and z' in $\mathbb{L}^n$ is given by

$$d_{\mathbb{L}^n}(z, z') = \operatorname{arccosh} \langle z, z' \rangle_{\mathcal{L}}. \tag{2}$$

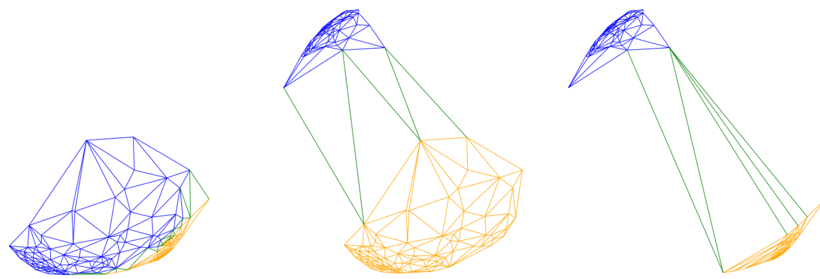
1.2 Poincaré ball.

The n-dimensional Poincaré ball model of the hyperbolic space is defined as the Riemannian manifold consisting of the open Euclidean ball $\mathbb{P}^n = \left\{z \in \mathbb{R}^n \mid \|z\|^2 < 1\right\}$, equipped with the metric tensor given in the standard coordinate system of $\mathbb{R}^n$ by the diagonal matrix with entries equal to $\frac{4}{(1-\|z\|^2)^2}$, $z \in \mathbb{P}^n$. The geodesic distance between z and z' in $\mathbb{P}^n$ is given by

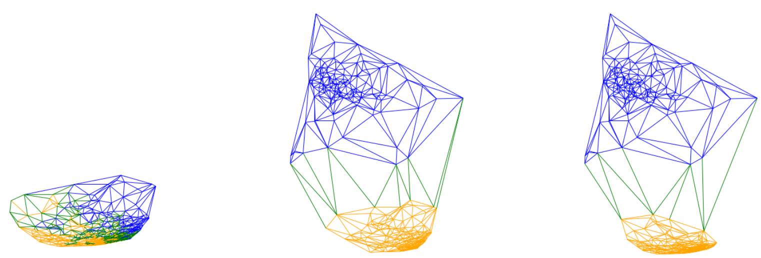
$$d_{\mathbb{P}^n}(z, z') = \operatorname{arccosh}\left(1 + \frac{2\|z - z'\|^2}{(1 - \|z\|^2)(1 - \|z'\|^2)}\right). \tag{3}$$

The geodesics in the Poincaré ball model are portions of circular arcs orthogonal to the boundary of the open ball.

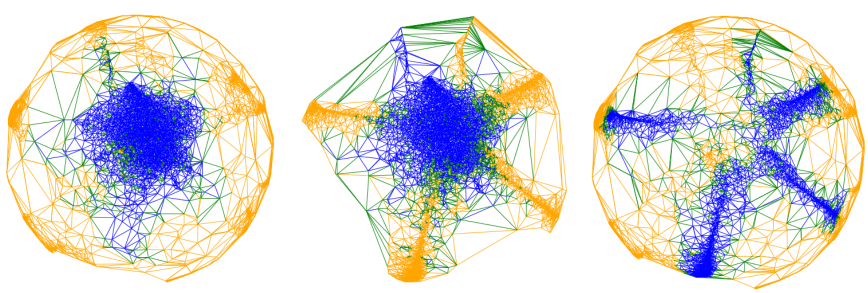
2 Additional Real-Life Visualizations



(a) (b) (c)



(a) (b) (c)



(a) (b) (c)

Fig. 1: HyperDGA visualizations of homogeneous edges (blue and orange) and heterogeneous edges (green) for Experiment 3 in the Klein-Beltrami model. Top: *Olsson*. (a) Mono vs. Gran, (b) HSPC-1 vs. Mono, and (c) HSPC-1 vs. Gran. Center: *Paul*. (a) 14Mo vs. 13Baso, (b) 7MEP vs. 13Baso, and (c) 7MEP vs. 14Mo. Bottom: *Planaria*. (a) neoblasts vs. differentiated, (b) neoblasts vs. progenitors, and (c) progenitors vs. differentiated.

References

1. Beltrami, E.: Teoria fondamentale degli spazii di curvatura costante. Annali di Matematica Pura ed Applicata (1867-1897) **2**, 232–255 (1868)
2. Nickel, M., Kiela, D.: Learning continuous hierarchies in the Lorentz model of hyperbolic geometry. In: Dy, J., Krause, A. (eds.) Proceedings of the 35th International Conference on Machine Learning. Proceedings of Machine Learning Research, vol. 80, pp. 3779–3788. PMLR (10–15 Jul 2018)