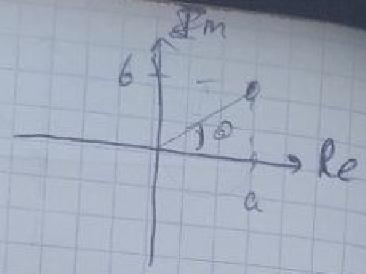
Exercize 1.



- $z = \rho e^{j\theta}$

$$\begin{cases} a = \rho \cos\theta \\ b = \rho \sin\theta \end{cases}$$

$$z = \rho[\cos\theta + j\sin\theta]$$

- $z = a + jb$

Pythagorean theorem

$$\begin{cases} \rho = \sqrt{a^2+b^2} \\ tg\,\theta = \frac{b}{a} \Rightarrow \theta = arctg\left(\frac{b}{a}\right) \end{cases}$$

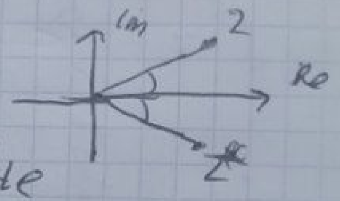
Exercize 2.

$z = a + jb = \rho e^{j\theta}$

$w = c + jd = \sigma e^{j\varphi}$

- $z + w = (a+jb) + (c+jd) = (a+c) + j(b+d)$
- $z - w = a + jb - c - jd = (a-c) + j(b-d)$
- $z/w = \frac{\rho e^{j\theta}}{\sigma e^{j\varphi}} = \frac{\rho}{\sigma} e^{j(\theta-\varphi)}$

- $|z| = \sqrt{a^2+b^2} = \rho$



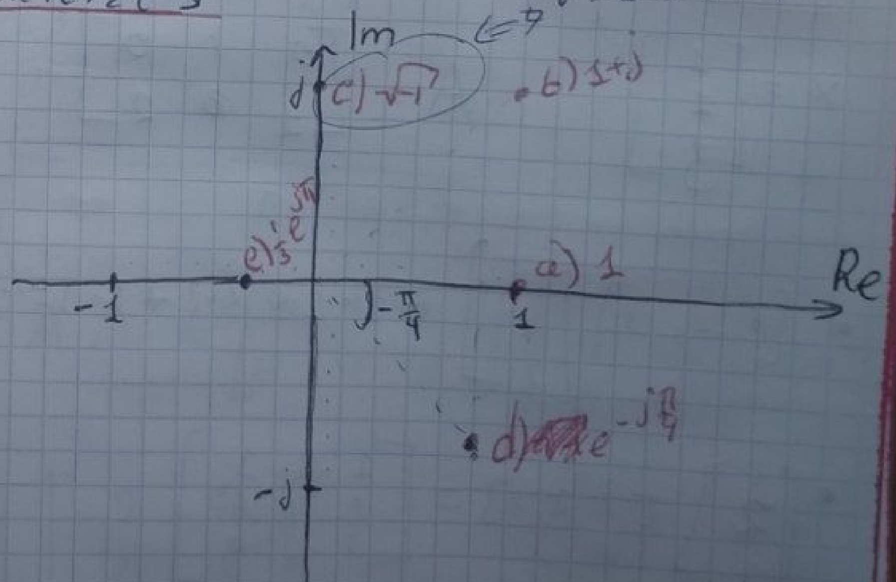
- $z^* = a - jb$

complex conjugate

- $z^5 = (\rho e^{j\theta})^5 = \rho e^{5j\theta}$
- $\sqrt{z} = z^{1/2} = (\rho e^{j\theta})^{1/2} = \rho e^{\frac{j\theta}{2}}$

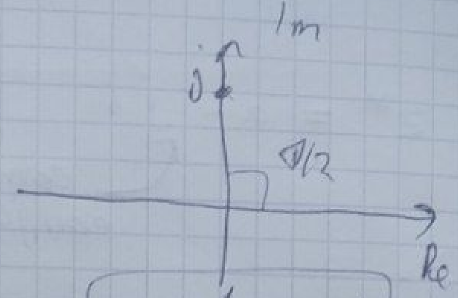
Exercize 3

$i^2 = -1 \Rightarrow i = \sqrt{-1}$



Exercize 4

$j^j = ?$



$j = e^{j\frac{\pi}{2}}$

$$\boxed{(e^a)^b = e^{ab}}$$

$$j^j = \left(e^{j\frac{\pi}{2}}\right)^j = e^{j^2\frac{\pi}{2}} = e^{-\pi/2}$$

Exercize 5

$$x = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \qquad A = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 2 \end{bmatrix}$$

- $\begin{bmatrix} 1 & 2 & 3 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} = 1\cdot 1 + 2\cdot 2 + 3\cdot 3 = 14$

- $\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \begin{bmatrix} 1 & 2 & 3 \end{bmatrix} = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 6 \\ 3 & 6 & 9 \end{bmatrix}$

- $|x| = \begin{vmatrix} 1 & 0 & 0 \\ 2 & 0 & 0 \\ 3 & 0 & 0 \end{vmatrix} = 0$

- $Ax = \underbrace{\begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 2 \end{bmatrix}}_{2\times 3} \underbrace{\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}}_{3\times 1} =$

$$= \begin{bmatrix} 2 \\ 7 \end{bmatrix}$$

- $\underbrace{\underbrace{\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}}_{[3\times 1]} \underbrace{\begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 2 \end{bmatrix}}_{[2\times 3]}}_{\text{Operation Not Aplicable}} = \begin{bmatrix} 1 & 1 & 2 \\ 21 & 22 & 23 \\ 31 & 32 & 33 \end{bmatrix}$

- $AA^T = \underbrace{\begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 2 \end{bmatrix}}_{2\times 3} \underbrace{\begin{bmatrix} 0 & 1 \\ 1 & 0 \\ 0 & 2 \end{bmatrix}}_{3\times 2} =$

$$= \begin{bmatrix} 1 & 0 \\ 0 & 5 \end{bmatrix}$$

Exercize 6

- $\begin{bmatrix} a & 0 \\ 0 & 6 \end{bmatrix} = A$

$\vec{u}$ - eigenvector -

$A\vec{u} = \lambda \vec{u}$

eigenvalue

let $\vec{u} = \begin{pmatrix} x \\ y \end{pmatrix}$

$\begin{bmatrix} a & 0 \\ 0 & 6 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \lambda \begin{bmatrix} x \\ y \end{bmatrix}$

$\begin{bmatrix} ax \\ 6y \end{bmatrix} = \begin{bmatrix} \lambda x \\ \lambda y \end{bmatrix}$

Both matrices are equal if corresponding elements are equal.

$\lambda = a$
$\lambda = 6$

- $\begin{bmatrix} 0 & a \\ 6 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \lambda \begin{bmatrix} x \\ y \end{bmatrix}$

2x2 2x1

$\begin{bmatrix} ay \\ 6x \end{bmatrix} = \begin{bmatrix} \lambda x \\ \lambda y \end{bmatrix}$

$\begin{cases} ay = \lambda x \\ 6x = \lambda y \end{cases} \Rightarrow y = \frac{\lambda x}{a}$

$\cancel{x} = \frac{\lambda}{6} y = \frac{\lambda \lambda \cancel{x}}{6a}$

$\Downarrow$

$\lambda^2 = a6 \Rightarrow \lambda = \pm\sqrt{a6}$

- $\begin{bmatrix} 2 & 0 & 0 \\ 1 & 8 & 8 \\ -3 & -2 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} \lambda x \\ \lambda y \\ \lambda z \end{bmatrix}$

Matrix characteristic equation:

$\det(A - \lambda I) = 0$

$\begin{bmatrix} 2-\lambda & 0 & 0 \\ 1 & 8-\lambda & 8 \\ -3 & -2 & 0-\lambda \end{bmatrix} = 0$

$(2-\lambda)(\lambda^2 - 8\lambda + 16) = 0$

$(2-\lambda)(\lambda - 4)^2 = 0$

$\lambda = 2$
$\lambda = 4$

Exercize 7

$\lambda = 2$

$$\underset{3\times 3}{\begin{bmatrix} 2 & 0 & 0 \\ 1 & 8 & 8 \\ -3 & -2 & 0 \end{bmatrix}} \underset{3\times 1}{\begin{pmatrix} x \\ y \\ z \end{pmatrix}} = \begin{pmatrix} \lambda x \\ \lambda y \\ \lambda z \end{pmatrix}$$

$$\cancel{\begin{bmatrix} 2x \\ x+8y+8z \\ -3x-2y \end{bmatrix} = \begin{pmatrix} \lambda x \\ \lambda y \\ \lambda z \end{pmatrix}}$$

$\cancel{\det(A-\lambda I)=0}$

$2 - \lambda$

$$\begin{bmatrix} 2x \\ x+8y+8z \\ -3x-2y \end{bmatrix} = \begin{pmatrix} 2x \\ 2y \\ 2z \end{pmatrix}$$

$$\begin{cases} 2x = 2x \\ x+8y+8z = 2y \\ -3x-2y = 2z \end{cases} \qquad \begin{cases} 2x=2x \\ x+6y+8z=0 \mid \cdot(-3) \\ 3x+2y+2z=0 \end{cases}$$

$$+\begin{cases} -3x-18y-24z=0 \\ 3x+2y+2z=0 \end{cases}$$

$-16y-22z=0$

$-8y = 11z$

$y = -\frac{11}{8}z$

$x - 6\cdot\frac{11}{8}z + 8z = 0$

$x = \frac{33}{4}z - \frac{32}{4}z$

$x = \frac{1}{4}z$

$$\begin{cases} y = -\frac{11}{8}z \\ x = \frac{1}{4}z \end{cases}$$

let $z=4$

$x=1$

$y=-\frac{11}{2}$

let $z=8$

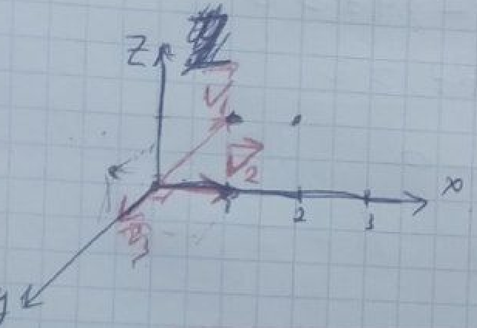
$y=-11$

$x=2$

Exercize 8

Easy to answer =)

Exercize 9



$\vec{v}_1 \in XZ$

$\vec{v}_2 \in XZ$

$\vec{v}_1 = [1, 0, 1]$

$\vec{v}_2 = [1, 0, 0]$

• distance $\boxed{d = \sqrt{(x_2-x_1)^2+(y_2-y_1)^2+(z_2-z_1)^2}}$

$d = \sqrt{(1-1)^2 + 0^2 + 1^2} = 1$

• angle

magnitude of $\vec{a}$ magnitude of $\vec{b}$

$\vec{a} \cdot \vec{b} = |\vec{a}| \cdot |\vec{b}| \cos\angle(\vec{a};\vec{b})$ – dot product (скалярное произведение

$\vec{a} = [a_1, a_2, \dots, a_n] = \sum a_i \vec{e}_i$ ← standard basis

$\vec{b} = [b_1, b_2, \dots, b_n] = \sum_i b_i \vec{e}_i$

$\vec{a}\vec{b} = \sum a_i b_i$

$\vec{e}_i \cdot \vec{e}_i = 1$

$\vec{e}_i \cdot \vec{e}_j = 0$

$\cos\alpha = \frac{\vec{a}\cdot\vec{b}}{\|\vec{a}\|\cdot\|\vec{b}\|}$

$\alpha = \arccos \frac{1\cdot 1 + 0\cdot 0 + 1\cdot 0}{\sqrt{1^2+0^2+1^2}\cdot\sqrt{1^2+0^2+0^2}} \Rightarrow$

$\alpha = \arccos\left(\frac{1}{\sqrt{2}}\right)$

$\alpha = \arccos\left(\frac{\sqrt{2}}{2}\right) = 45$

• $\boxed{\text{find } \vec{v}_3[x_3; y_3; z_3]}$ $\vec{v}_3 \perp \vec{v}_1$ & $\vec{v}_3 \perp \vec{v}_2$

$\cancel{\begin{cases}\vec{v}_3 \cdot \vec{v}_1 = 0 \\ \vec{v}_3 \cdot \vec{v}_2 = 0\end{cases}}$ $\cancel{(\cos 90° = 0)}$ All we need is cross product

$\cancel{\begin{cases} x_3x_1 + y_3y_1 + z_3z_1 = 0 \\ x_3x_2 + y_3y_2 + z_3z_2 = 0 \end{cases}}$

$\cancel{x_3 + z_3 = 0}$

$\cancel{x_3 = 0}$

$\frac{[\vec{v}_1 \times \vec{v}_2]}{\|\vec{v}_1 \times \vec{v}_2\|}$

$\|\vec{v}_1 \times \vec{v}_2\| = \sqrt{0^2 + (-1)^2 + 0^2} = 1$

$\begin{vmatrix} i & j & k \\ 1 & 0 & 1 \\ 1 & 0 & 0 \end{vmatrix} = \underbrace{0\cdot\vec{i} + (-1)\vec{j} + 0\cdot\vec{k}}$

$\vec{v}_3 = [0; -1; 0]$

- $\vec{v}_1 = [1, 0, 1]$
  $\vec{v}_2 = [1, 0, 0]$
  $\vec{v}_3 = [0\ -1\ 0]$

  If all of them are linear independent, It means they make up basis of 3D space.

$$\Downarrow$$

$$\begin{vmatrix} 1 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & -1 & 0 \end{vmatrix} \neq 0 =$$

$= 0 + 0 + (-1) \neq 0$

$\vec{v}_1, \vec{v}_2, \vec{v}_3$ make up 3D basis.

$v_3 \perp v_2$

$v_3 \perp v_1$

$\angle(\vec{v}_1, \vec{v}_2) = 45$

$$\vec{v}_\perp = [\vec{v}_3 \times \vec{v}_2] = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 0 & -1 & 0 \\ 1 & 0 & 0 \end{vmatrix} = 0\vec{i} + 0\vec{j} + 1\vec{k}$$

$\vec{v}_\perp = [0, 0, 1]$, $\vec{v}_2$ and $\vec{v}_3$ — are orthonormal basis

- $\vec{v}_4 = [2, 0, 3]$

  Is $(\vec{v}_1, \vec{v}_2, \vec{v}_4)$ basis?

$$\begin{vmatrix} 1 & 1 & 2 \\ 0 & 0 & 0 \\ 1 & 0 & 3 \end{vmatrix} = 0 \Rightarrow$$ It is a complanarity of vectors.

They don't make up basis.

Exercize 10.

- $$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{bmatrix} = A$$
$$+ \begin{bmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \end{bmatrix}$$

$$= \begin{bmatrix} a_{11} + b_{11}, & a_{12} + b_{12}, & a_{13} + b_{13} \\ a_{21} + b_{21}, & a_{22} + b_{22}, & a_{23} + b_{23} \end{bmatrix}$$

$$\alpha A = \begin{bmatrix} \alpha a_{11} & \alpha a_{12} & \alpha a_{13} \\ \alpha a_{21} & \alpha a_{22} & \alpha a_{23} \end{bmatrix}$$

- $a_{11}X^3 + a_{12}X^2 + a_{13}X^1 + a_{14}X^0 = 0 \Rightarrow A$

  $b_{11}X^3 + b_{12}X^2 + b_{13}X^1 + b_{14}X^0 = 0 \Rightarrow B$

$A + B = (a_{11}+b_{11})X^3 + (a_{12}+b_{12})X^2 + (a_{13}+b_{13})X^1 + (a_{14}+b_{14})X$

$\alpha A = \alpha a_{11}X^3 + \alpha a_{12}X^2 + \alpha a_{13}X^1 + \alpha a_{14}X^0 = 0$

Exercize 11

Let $\vec{P} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$

$\vec{X} = \begin{bmatrix} x_0 \\ x_1 \end{bmatrix}$

← matrix multiplication

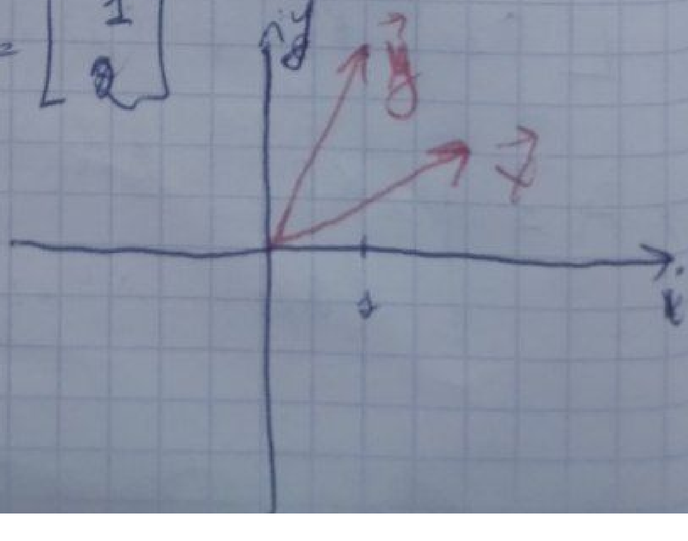
$\vec{y} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} x_0 \\ x_1 \end{bmatrix} =$

$\cancel{2\times 2}$ $\cancel{2\times 1}$ 2×1

$\cancel{= x_0(\cos\theta - \sin\theta)}$

$y = \begin{bmatrix} x_0\cos\theta - x_1\sin\theta \\ x_0\sin\theta + x_1\cos\theta \end{bmatrix}$; let $x = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$, $\theta = \pi/2$

$y = \begin{bmatrix} 2\cdot 0 - 1\cdot 1 \\ 2\cdot 1 + 1\cdot 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$



Exercize 12

$\boxed{\begin{aligned} D &= b^2 - 4ac \\ x_{1,2} &= \frac{-b \pm \sqrt{D}}{2a} \end{aligned}}$

- $2x^2 - 5x + 2 = 0$

$D = 25 - 16 = 9$

$$\begin{cases} x_1 = \frac{5+3}{4} = 2 \\ x_2 = \frac{5-3}{4} = 0,5 \end{cases}$$

- $x^3 + 1 = 0 \Rightarrow x^3 = -1$

$x = -1$

$\cancel{x = \sqrt[3]{-1}}$

$\boxed{a^3 + b^3 = (a+b)(a^2 - ab + b^2)}$

$(x+1)(x^2 - x + 1) = 0$

$$\begin{cases} x + 1 = 0 \Rightarrow x = -1 \\ x^2 - x + 1 = 0 \Rightarrow D = -3 \Rightarrow x = \frac{1 \pm \sqrt{-3}}{2} \end{cases}$$

$\boxed{\begin{aligned} i^2 &= -1 \\ i &= \sqrt{-1} \end{aligned}}$

$$\begin{cases} x_1 = -1 \\ x_{2,3} = \frac{1}{2} \pm \frac{i\sqrt{3}}{2} \end{cases}$$

- $x^4 + 3x^2 + 2 = 0$

let $x^2 = t \Rightarrow x = \pm\sqrt{t}$

$t^2 + 3t + 2 = 0 \Rightarrow D = b^2 - 4ac = 1$

$$\begin{cases} t_1 = \frac{-3-1}{2} = -2 \\ t_2 = \frac{-3+1}{2} = -1 \end{cases}$$

$\Updownarrow$

$$\begin{cases} x_1 = +\sqrt{-2} = i\sqrt{2} \\ x_2 = -i\sqrt{2} \\ x_3 = +\sqrt{-1} = i \\ x_4 = -i \end{cases}$$

Exercize 13

- $\sum_{n=0}^{17} n = ?$ It looks like arithmetic progression with $d=1$

$a_0 = 0 \quad a_1 = 1$

$a_n = 17$

$$\boxed{S = \frac{(a_1 + a_n)n}{2}}$$ - arithmetic series

$$\sum_{n=0}^{17} n = \frac{(1+17)17}{2} = 17 \cdot 9 = 153$$

- $\sum_{n=0}^{N} a^n = \underbrace{1 + a + a^2 + \dots a^N}_{\text{looks like geometric series}}$

$q = a$

$$\boxed{S_n = \frac{b_1(q^n - 1)}{q - 1}}$$

$$\sum_{n=0}^{N} a^n = \frac{a^n - 1}{a - 1}$$

- $\sum_{n=0}^{\infty} \frac{x^n}{n!} = e^x$

We can try to take derivative

$$\frac{d}{dx}\sum_{n=0}^{\infty} \frac{x^n}{n!} = \sum_{n=0}^{\infty} \frac{d}{dx}\frac{x^n}{n!} = \sum_{n=0}^{\infty} \frac{n x^{n-1}}{n!} =$$

$$= \sum_{n=1}^{\infty} \frac{x^{n-1}}{(n-1)!} = \sum_{n=0}^{\infty} \frac{x^n}{n!}$$

the same

- $\sum_{n=1}^{\infty} \frac{1}{n} = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} \dots = \infty$

$$\int_0^{\infty} \frac{1}{x}dx = \lim_{n\to\infty} \ln x \Big|_0^{\infty} = \infty - 0 = \infty$$

Exercise 14

uncertaintly

- $\lim\limits_{x\to\infty} \left(\frac{\log x}{x}\right) \overset{\frac{\infty}{\infty}}{=}$

$$\boxed{\lim\limits_{x\to h} \frac{f(x)}{g(x)} \overset{\frac{0}{0}(\frac{\infty}{\infty})}{=} \lim\limits_{x\to h} \frac{d/dx\,(f(x))}{d/dx\,(g(x))}}$$

$$\lim\limits_{x\to\infty} \frac{\frac{d}{dx}\log x}{\frac{d}{dx}x} \overset{L'H}{=} \lim\limits_{x\to\infty} \frac{\frac{1}{x}}{1} = \lim\limits_{x\to\infty} \frac{1}{x}$$

- $\lim\limits_{x\to 0} \frac{\sin x}{x}$ – The first wonderful limit

$\frac{0}{0}$ - uncertainty

$$\boxed{\lim\limits_{x\to 0} \frac{\sin x}{x} = 1}$$

- $\lim\limits_{x\to\infty} \left[1+\frac{1}{x}\right)^x$ – The second wonderful limit

$$\boxed{\lim\limits_{x\to\infty} \left(1+\frac{1}{x}\right)^x = e}$$

Exercise 15

- $\int_{-\pi}^{\pi} \sin x \, dx = (-\cos x) + C \Big|_{-\pi}^{\pi} =$

$= (-\cos\pi) - (-\cos(-\pi)) + C =$

$1 - 1 + C = 0$

- $\int e^{\alpha x} dx = \frac{e^{\alpha x}}{\alpha} + C$

- $\int x \cos(nx)\, dx =$

$$\boxed{\int u\,dv = uv - \int v\,du}$$ - integrate by parts

$u = x \qquad dv = \cos(nx)$

$du = dx \qquad v = \frac{\sin(nx)}{n}$

$= \frac{x\cdot\sin(nx)}{n} - \int \frac{\sin(nx)}{n} dx =$

$= \frac{x\sin(nx)}{n} - \int \frac{\sin(nx)\, d(nx)}{n^2} =$

$= \frac{1}{n^2}\left(n\sin(nx) + \cos(nx)\right) + C$

Exercize 16

$a \in \{-1, 2\}$

$P[a=-1] = 0.3$

$p[a=2] = 0.7$

- mean (mathematical expectation)

$\boxed{E[X] = \sum x_i p_i}$ - expectation of variable X withe finite outcomes

↑ if $p_1 = p_2 = \ldots = p_i \Rightarrow E(x) =$ standard average

$E(a) = -1 \cdot 0.3 + 2 \cdot 0,7 = 1,1$

- variance - squared deviation from the mean. (Дисперсия рассеяние) =

$\boxed{Var(X) = E[(X-\mu)^2]}$, where $\mu = E[X]$

$\boxed{Var(\cancel{X}) = \sum_{i=1}^{n} p_i (x_i - \mu)^2} = D$

$\boxed{D = E(x^2) - (E(x))^2}$

$Var(a) = 0,3 \cdot (-1 - 1,1)^2 + 0,7 (2 - 1,1)^2 =$

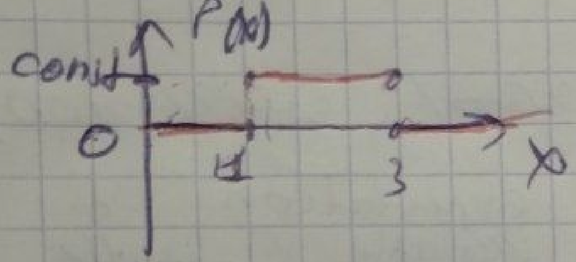
$= 1,452 + 0,567 = 2.019.$

$\boxed{\sigma = \sqrt{D}}$ — Mean Square Deviation (How far value from mean?)

Exercize 17

$X \in [1,3]$

$p(x)$ - uniformly distributed



$p(x) = const$

$$\boxed{E(v) = \int_a^b x\, p(x)\, dx}$$ - mean of continous value.

- $E(x) = \int_1^3 \frac{x\,dx}{3-1} = \frac{x^2}{4}\Big|_1^3 = \frac{9}{4} - \frac{1}{4} = \frac{8}{4} = 2$

- $E[(x-2)^2] = \int_1^3 \frac{(x-2)^2}{2}\, dx \overset{x-2=u}{=}$

$\frac{1}{2} \int_1^3 u^2\, du = \frac{1}{2} \frac{u^3}{3}\Big|_1^3 = \frac{(x-2)^3}{6}\Big|_1^3 =$

$= 1/6 + 1/6 = 2/6 = 1/3$

- $E[\frac{1}{x}] = \frac{1}{2} \int_1^3 \frac{dx}{x} = \frac{1}{2}\left[\ln 3 - \overset{0}{\ln 1}\right]$