Purpose: Explore advanced topics in optimization, regression, random number generation, and the bootstrap.

Assignment: Each problem involves mathematical and computational parts. For the computational parts, please turn in the simplest/shortest algorithm you can write that accomplishes the task. You may discuss the problems with other students, but all work must be your own. If you consult a book or online resource, cite it.

Problem 1: A problem with the parametric bootstrap

Suppose $X_1, \ldots, X_n$ are iid Uniform$(0, \theta)$, where $\theta > 0$ and you wish to estimate θ.

a. Show that the maximum likelihood estimate is $\hat{\theta} = \max\{x_1, \ldots, x_n\}$.

b. Suppose you estimate θ in this manner and simulate a parametric bootstrap sample. That is, you simulate

$$X_1^* \ldots X_n^* \sim \text{Uniform}\left(0, \hat{\theta}\right)$$

and find $\hat{\theta}^*$. Argue that $\Pr\left(\hat{\theta}^* = \hat{\theta}\right) = 0$.

c. Find $\Pr\left(\theta > \hat{\theta}\right)$ and $\Pr\left(\hat{\theta}^* > \hat{\theta}\right)$. What do you conclude about the parametric bootstrap confidence interval for θ from the above probabilities?

Problem 2: A problem with the nonparametric bootstrap

Suppose $X_1, \ldots, X_n$ are iid Uniform$(0, \theta)$, where $\theta > 0$ and you wish to estimate θ. As above, the MLE is $\hat{\theta} = \max\{X_1, \ldots, X_n\}$. Let $\hat{F}$ be the empirical distribution function of the original samples $X_1, \ldots, X_n$ and let $X_1^*, \ldots X_n^* \sim \hat{F}$. Note that under the true distribution function F, $\Pr(\hat{\theta} = \theta) = 0$.

a. However, the nonparametric bootstrap produces a different result. Show that

$$\Pr(\hat{\theta}^* = \hat{\theta}) = 1 - \left(1 - \frac{1}{n}\right)^n.$$

b. Let $n \to \infty$ in the above expression to show that

$$\Pr(\hat{\theta}^* = \hat{\theta}) \to 0.6321206\ldots$$

What does this mean?