Purpose: Explore Newton's method in one dimension.

Assignment: Some problems involve both mathematical and computational parts. For the computational parts, please turn in the simplest/shortest algorithm you can write that accomplishes the task. You may discuss the problems with other students, but all work must be your own. If you consult a book or online resource, cite it.

Problem 1: Factoring polynomials

Adapted from Lange "Numerical Analysis for Statisticians" (2010) problem 5.17. Suppose a n-degree polynomial $p(x)$ has known roots $r_1, \ldots, r_m$, where $m < n$. Maehly's algorithm is a method for finding one more root, r_{m+1}. Consider the new polynomial

$$q(x) = \frac{p(x)}{(x - r_1) \cdots (x - r_m)}.$$

a. Intuitively, what is the relationship of the new polynomial $q(x)$ to the original polynomial $p(x)$? [Hint: what does the fundamental theorem of algebra say about the roots of $p(x)$?]

b. Show that

$$\frac{\mathrm{d}}{\mathrm{d}x}(x - r_1) \cdots (x - r_m) = (x - r_1) \cdots (x - r_m) \sum_{i=1}^{m} \frac{1}{x - r_i}.$$

c. Show that

$$\frac{\mathrm{d}q}{\mathrm{d}x} = \frac{p'(x)}{(x - r_1) \cdots (x - r_m)} - q(x) \sum_{i=1}^{m} \frac{1}{x - r_i}.$$

d. Show that Newton's method update for r_{m+1} is

$$x_{\text{new}} = x - \frac{p(x)}{p'(x) - p(x) \sum_{i=1}^{m} (x - r_i)^{-1}}.$$

e. Implement and test your algorithm with the polynomial

$$p(x) = x^4 - 12x^3 + 47x^2 - 60x.$$

Report the roots and the implied factorization of $p(x)$. (Hint: $p(x)$ has four real roots and $x = 0$ is one of them. Find three more.) Compare your results to those obtained by applying a generic rootfinding routine like (`uniroot` in `R`) repeatedly to $p(x)$.

Problem 2: Newton's method convergence

The function

$$f(x) = \frac{e^x}{1+e^x} - \frac{1}{2}$$

has exactly one root, located at $x = 0$. Suppose we try to find this root using Newton's method starting at $x_0 > 0$. This function is smooth, continuous, and not very "wiggly" (plot it!). We might imagine that rootfinding for this function will be easy.

a. Show that x_0 must satisfy

$$4x_0 - e^{x_0} + e^{-x_0} > 0$$

in order for the Newton's method iterates to converge to the root. [Hint: you may use the fact that in this problem, a sufficient condition for convergence is that x_1 is closer to zero than x_0.]

b. Write an algorithm to determine the largest value of $x_0 > 0$ that results in convergence of Newton's method. Describe (numerically) the interval in which Newton's method converges to the root. Turn in your code.

c. Confirm that your answers to (a) and (b) above are correct by starting Newton's method at various starting values x_0 and observing whether the iterates converge to a value less than 10^{-8} within 1000 iterations. Plot the results in a way that illustrates the interval in which Newton's method converges. Turn in your code.