Purpose: Explore advanced topics in optimization, regression, random number generation, and the bootstrap.

Assignment: Each problem involves mathematical and computational parts. For the computational parts, please turn in the simplest/shortest algorithm you can write that accomplishes the task. You may discuss the problems with other students, but all work must be your own. If you consult a book or online resource, cite it.

Problem 1: Rejection sampling

Do Robert & Casella Exercise 2.18.

Problem 2: A uniform random number generator

Consider the linear congruential generator

$$X_{i+1} = 65539 X_i \bmod 2^{31}$$

Dividing by X_i by its maximum value 2^{31} returns a pseudorandom number that appears to follow the uniform distribution on (0,1). Let $X_0 = 1$ and generate numbers as follows:

```
seed = 1.0
myrnd = function() {
  seed <<- ((2^16 + 3) * seed) %% (2^31)
  return(seed/(2^31))
}
u = sapply(1:3000, function(i)myrnd())
```

Let $Y_i = X_i/2^{31}$ for $i = 1, \ldots, n$.

a. Plot the values in `u` in order. Do they look uniform?

b. Plot a histogram of the values. Does the distribution look uniform?

c. Perform a Kolmogorov-Smirnov test of goodness of fit to the uniform distribution using the `ks.test` function. Do you reject the hypothesis of uniform distribution?

d. Install the `rgl` package and make a 3D scatterplot as follows:

```
library(rgl)
y = matrix(u, nrow=1000, ncol=3, byrow=TRUE)
plot3d(y[,1], y[,2], y[,3], axes=TRUE, xlab="", ylab="", zlab="")
```

Rotate the plot. Turn in an image of the plot displaying weird patterns.

e. **Extra credit**: Explain formally what is going wrong.

Problem 3: The inverse CDF method for generating random variables

Use the inverse CDF method to derive and implement random number generators for the following distributions. Do not use any random number generator except `runif`. For each distribution, turn in your derivation, algorithm, and a histogram of 1000 deviates with the true density or mass function overlaid.

a. Poisson with mean $\lambda > 0$. [Hint: Do not compute the CDF of the Poisson distribution. What is the relationship between exponential and Poisson random variables?]

b. Weibull with shape $\alpha > 0$ and rate $\beta > 0$.

c. The triangle distribution

$$f(x) = \begin{cases} 4x & 0 \leq x \leq \frac{1}{2} \\ 4(1-x) & \frac{1}{2} < x \leq 1 \\ 0 & \text{otherwise} \end{cases}$$

[Hint: do this analytically!]