Purpose: The purpose of this assignment is to help you determine whether your mathematics background is strong enough for you to succeed in this course. If you have serious difficulty with these problems, even after getting a hint at office hours, you should consider dropping the course.

Problem 1 Differentiation

a. Maximize the function

$$f(x) = xe^{-(x-1)^2}$$

for $x > 0$.

b. Consider a random variable X with cumulative distribution function

$$F(x) = \Pr(X < x) = \begin{cases} 0 & x < 0 \\ \dfrac{1 - e^{-\lambda x}}{1 - e^{-\lambda a}} & 0 \leq x \leq a \\ 1 & x > a \end{cases}$$

where $a > 0$ is a constant. Invent a probability density $f(x)$ that corresponds to $F(x)$. What is $f(a + 17)$?

Problem 2 Integration

a. Let X be an Exponential random variable with density $f(x) = e^{-x}$. Find $\Pr(X > 1)$.

b. Let

$$f(x) = \frac{1}{C}e^{-a|x-b|}$$

be a probability density defined for $x \in (-\infty, +\infty)$ where $a, b > 0$ are fixed parameters. Find the normalizing constant C.

Problem 3 Linear algebra

a. Suppose X is a $n \times n$ matrix. Show that the matrix $Y = X'X$ is symmetric.

b. Show that $x = \left(\sqrt{197}, \sqrt{197}\right)'$ is an eigenvector of the matrix

$$A = \begin{pmatrix} 4 & -5 \\ 2 & -3 \end{pmatrix}.$$

What is the corresponding eigenvalue?

Problem 4 Counting

a. Show that

$$x(1+x)^{n-1} = \sum_{k=1}^{n} \binom{n}{k} \frac{k}{n} x^k$$

for $x \neq 0$ and n a positive integer.

b. Consider an urn containing r red balls and b blue balls. I draw one ball at random. If it is red, I place it back in the urn along with seven other red ones. If it is blue, I remove it from the urn forever. What is the probability of drawing a blue ball on the second draw?

Problem 5 Series

a. Show that

$$\sum_{k=0}^{\infty} \sum_{j=0}^{\infty} (-1)^j \frac{\lambda^{k+j}}{k!j!} = 1$$

for any $\lambda > 0$.

b. Show that

$$\sum_{k=0}^{\infty} \frac{k}{2^{k+1}} = 1.$$

Problem 6 Limits

a. Evaluate

$$\lim_{n \to \infty} \frac{n - 10^{897}}{n}.$$

b. Consider the function

$$f(x) = \begin{cases} 0 & x < 0 \\ 1 & x \geq 0. \end{cases}$$

Evaluate

$$\lim_{n \to \infty} f(-1/n) \quad \text{and} \quad \lim_{n \to \infty} f(1/n).$$