Digital Signal Processing

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Contents

1 Software Installation

1.1 software installation done

2 Digital Filter

2.1 Download the sound file
Solution: Sound File can be found in the link below

```
wget https://github.com/anitadash/EE3900/
    blob/main/Soundfiles/Sound_Noise.wav
```

2.2 You will find a spectrogram at https://academo.org/demos/spectrum-analyzer. Upload the sound file that you downloaded in Problem 2.1 in the spectrogram and play. Observe the spectrogram. What do you find?
Solution: After observing the spectogram, we find that there are a lot of yellow lines between 440 Hz to 5.2 KHz. They represent the synthesizer key tones. The key strokes are audible along with the background noise.

2.3 Write the python code for removal of out of band noise and execute the code.
Solution: The python code for removal of out of band noise can be found in the link below.

```
wget https://github.com/anitadash/EE3900/
    blob/main/Codes/2_3.py
```

2.4 The output of the python script in Problem 2.3 is the audio file Sound_With_ReducedNoise.wav. Play the file in the spectrogram in Problem 2.2. What do you observe?
Solution: The output for the above python code can be found in the link below

```
wget https://github.com/anitadash/EE3900/
    blob/main/Soundfiles/
    Sound_With_ReducedNoise.wav
```

After playing the output in the spectogram we observe that the key strokes as well as background noise is subdued in the audio. The signal is blank for frequencies above 5.1 kHz.

3 Difference Equation

3.1 Let

$$x(n) = \left\{ \underset{\uparrow}{1}, 2, 3, 4, 2, 1 \right\} \tag{3.1}$$

Sketch $x(n)$
Solution: The Python code for sketching the graph of $x(n)$ can be found in the link below. The code yields the graph shown in figure 3.1

```
wget https://github.com/anitadash/EE3900/
    blob/main/Codes/3_1.py
```

3.2 Let

$$y(n) + \frac{1}{2}y(n-1) = x(n) + x(n-2),$$
$$y(n) = 0, n < 0 \tag{3.2}$$

Sketch $y(n)$.
Solution: The Python code for sketching the graph of $y(n)$ can be found in the link below. The code yields the graph shown in figure 3.2

```
wget https://github.com/anitadash/EE3900/
    blob/main/Codes/3_2.py
```

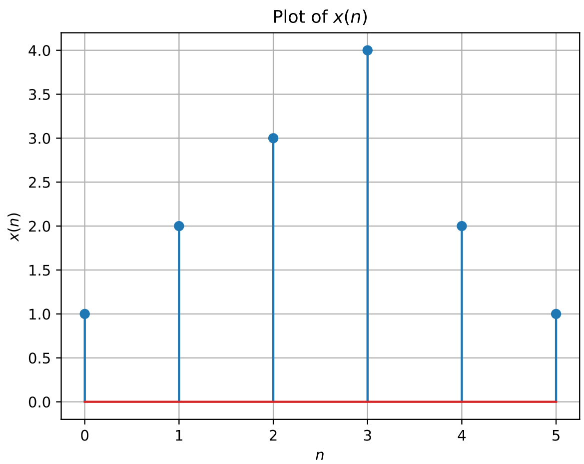

Fig. 3.1

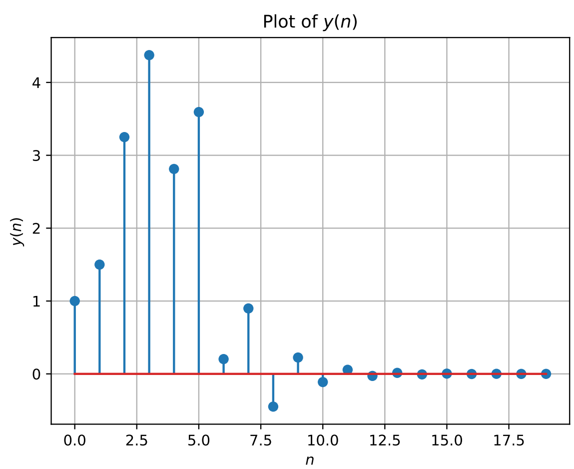

Fig. 3.2

3.3 Repeat the above exercise using a C code.
Solution: The C-code for the above two exercises can be found in the link below. The code yields the graph shown in figure 3.3

```
wget https://github.com/anitadash/EE3900/
    blob/main/Codes/3_3.c
```

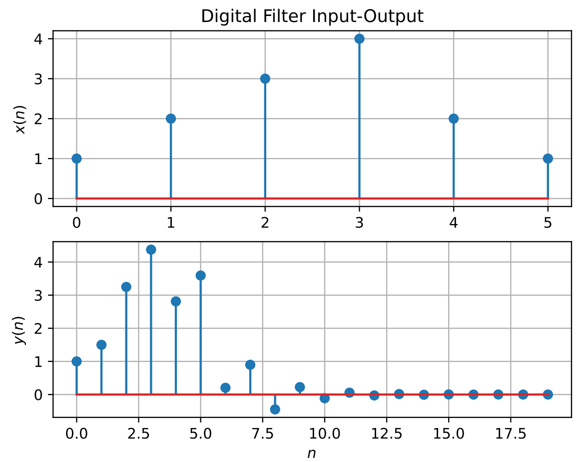

Fig. 3.3

4 Z-transform

4.1 The Z-transform of $x(n)$ is defined as

$$X(z) = \mathcal{Z}\{x(n)\} = \sum_{n=-\infty}^{\infty} x(n) z^{-n} \tag{4.1}$$

Show that

$$\mathcal{Z}\{x(n-1)\} = z^{-1} X(z) \tag{4.2}$$

and find

$$\mathcal{Z}\{x(n-k)\} \tag{4.3}$$

Solution: From (4.1),

$$\mathcal{Z}\{x(n-1)\} = \sum_{n=-\infty}^{\infty} x(n-1) z^{-n} \tag{4.4}$$

$$= \sum_{n=-\infty}^{\infty} x(n) z^{-n-1} = z^{-1} \sum_{n=-\infty}^{\infty} x(n) z^{-n} \tag{4.5}$$

Similarly,

$$\mathcal{Z}\{x(n-k)\} = \sum_{n=-\infty}^{\infty} x(n-k) z^{-n} \tag{4.6}$$

$$= \sum_{n=-\infty}^{\infty} x(n) z^{-n-k} = z^{-k} \sum_{n=-\infty}^{\infty} x(n) z^{-n} \tag{4.7}$$

Therefore,

$$\mathcal{Z}\{x(n-k)\} = z^{-k} X(z) \tag{4.8}$$

4.2 Obtain $X(z)$ for $x(n)$ defined in problem 3.1.
Solution: We Know that,

$$X(z) = \sum_{n=-\infty}^{\infty} x(n) z^{-n} \tag{4.9}$$

$$\implies X(z) = \sum_{n=0}^{5} x(n) z^{-n} \tag{4.10}$$

Therefore,

$$X(z) = 1 + \frac{2}{z} + \frac{3}{z^2} + \frac{4}{z^3} + \frac{2}{z^4} + \frac{1}{z^5} \tag{4.11}$$

4.3 Find

$$H(z) = \frac{Y(z)}{X(z)} \tag{4.12}$$

from (3.2) assuming that the Z-transform is a linear operation.
Solution: from (3.2) assuming that the Z-transform is a linear operation.
Applying (4.8) in (3.2) we get,

$$Y(z) + \frac{1}{2}z^{-1}Y(z) = X(z) + z^{-2}X(z) \tag{4.13}$$

$$\implies \frac{Y(z)}{X(z)} = \frac{1 + z^{-2}}{1 + \frac{1}{2}z^{-1}} \tag{4.14}$$

4.4 Find the Z transform of

$$\delta(n) = \begin{cases} 1 & n = 0 \\ 0 & \text{otherwise} \end{cases} \tag{4.15}$$

and show that the Z-transform of

$$u(n) = \begin{cases} 1 & n \geq 0 \\ 0 & \text{otherwise} \end{cases} \tag{4.16}$$

is

$$U(z) = \frac{1}{1 - z^{-1}}, \quad |z| > 1 \tag{4.17}$$

Solution: From (4.1), we can see that,

$$\mathcal{Z}\{\delta(n)\} = \delta(0)z^0 \implies \mathcal{Z}\{\delta(n)\} = 1 \tag{4.18}$$

To show that the Z-transform of $u(n)$ is

$$U(z) = \frac{1}{1 - z^{-1}}, \quad |z| > 1 \tag{4.19}$$

From (4.1), we can see that,

$$\mathcal{Z}\{u(n)\} = \sum_{n=0}^{\infty} u(n)z^{-n} \tag{4.20}$$

$$= \sum_{n=0}^{\infty} z^{-n} \tag{4.21}$$

The above summation is an infinite geometric progression with common ratio z^{-1}, and for an infinite geometric progression to converge the common ratio must be strictly lesser than 1.

Therefore

$$U(z) = \frac{1}{1 - z^{-1}}, \quad |z| > 1 \tag{4.22}$$

4.5 Show that,

$$a^n u(n) \overset{\mathcal{Z}}{\rightleftharpoons} \frac{1}{1 - az^{-1}} \quad |z| > |a| \tag{4.23}$$

Solution: From (4.16) we can see that,

$$\mathcal{Z}\{a^n u(n)\} = \sum_{n=0}^{\infty} a^n z^{-n} \tag{4.24}$$

The above summation is an infinite geometric progression with common ratio az^{-1}, and for an infinite geometric progression to converge the common ratio must be strictly lesser than 1. (Which is true as given $|z| > |a|$)
Therefore the Z-tranform of $a^n un$ is,

$$a^n u(n) \overset{\mathcal{Z}}{\rightleftharpoons} \frac{1}{1 - az^{-1}} \quad |z| > |a| \tag{4.25}$$

4.6 Let

$$H\left(e^{j\omega}\right) = H\left(z = e^{j\omega}\right). \tag{4.26}$$

Plot $|H\left(e^{j\omega}\right)|$. Is it periodic? If so, find the period. $H(e^{j\omega})$ is known as the *Discret Time Fourier Transform* (DTFT) of $h(n)$.
Solution: The Python code for sketching the graph of $H\left(e^{j\omega}\right)$ can be found in the link below. The above code yields figure 4.6.

```
wget https://github.com/anitadash/EE3900/
    blob/main/Codes/4_5.py
```

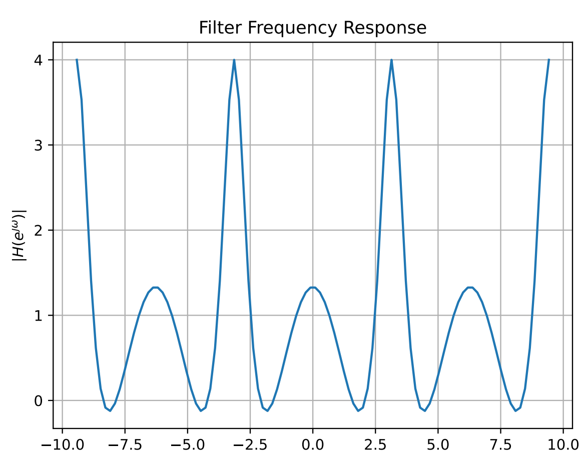

Fig. 4.6

It is periodic with a period of 2π.
Proof:

$$H\left(e^{j(\omega+2\pi)}\right) = H\left(e^{j\omega}e^{j2\pi}\right) \tag{4.27}$$

$$= H\left(e^{j\omega}(\cos(2\pi) + j\sin(2\pi))\right) \tag{4.28}$$

$$= H\left(e^{j\omega}\right) \tag{4.29}$$

2π is a period, we need to verify whether if it is the fundamental period.and thus need to check the periodicity for $T = \pi$

$$H\left(e^{j(\omega+\pi)}\right) = -H\left(e^{j\omega}\right) \tag{4.30}$$

and thus 2π is the period of the function.

4.7 Express $h(n)$ in terms of $H\left(e^{j\omega}\right)$.
Solution: $h(n)$ can be expressed as:

$$h(n) = \frac{1}{2\pi}\int_0^{2\pi} H\left(e^{\jmath\omega}\right) e^{\jmath n\omega} d\omega \tag{4.31}$$

Proof: We know that

$$H\left(e^{\jmath\omega}\right) = \sum_{k=0}^{\infty} h(k) z^{-\jmath k\omega} \tag{4.32}$$

Substituting the above equation in (4.31)

$$h(n) = \frac{1}{2\pi}\int_0^{2\pi}\left(\sum_{k=0}^{\infty} h(k) z^{-\jmath k\omega}\right) e^{\jmath n\omega} d\omega \tag{4.33}$$

$$h(n) = \frac{1}{2\pi}\sum_{k=0}^{\infty} h(k)\int_0^{2\pi} z^{\jmath(n-k)\omega} d\omega \tag{4.34}$$

The above integral is 1 when $k = n$ and 0 otherwise. Thus,

$$h(n) = \frac{1}{2\pi}\int_0^{2\pi} H\left(e^{\jmath\omega}\right) e^{\jmath n\omega} d\omega \tag{4.35}$$

Hence Shown.

5 Impulse Response

5.1 Using long division, find

$$h(n), \quad n < 5 \tag{5.1}$$

for H(z) in (4.14).
Solution: Given,

$$H(z) = \frac{1 + z^{-2}}{1 + \frac{1}{2}z^{-1}} \tag{5.2}$$

$$H(z) = \frac{2z^2 + 2}{2z^2 + z} \tag{5.3}$$

By Long Division, $H(z) =$

$$\begin{array}{r} 1 - \frac{1}{2z} + \frac{5}{4z^2} - \frac{5}{8z^3}\dots \\ \hline 2z^2 + z \,\big)\, 2z^2 + 2 \\ -\ (2z^2 + z) \\ \hline -z + 2 \\ -\ (-z - \frac{1}{2}) \\ \hline \frac{5}{2} \\ -\ (\frac{5}{2} + \frac{5}{4z}) \\ \hline \frac{-5}{4z} \\ -\ (\frac{-5}{4z} + \frac{-5}{8z^2}) \\ \hline \frac{-5}{8z^2} \\ \vdots \end{array} \tag{5.4}$$

From the above we can see that

$$H(z) = 1 - \frac{1}{2z} + \frac{5}{4z^2} - \frac{5}{8z^3} + \dots 5\left(\frac{-1}{2z}\right)^n (n > 1) \tag{5.5}$$

given n $<$ 5 , Therefore

$$H(z) = \sum_{n=-\infty}^{4} h(n) z^{-n} \tag{5.6}$$

Therefore, comparing the above equation with (5.5) we can say that,

$$h(n) = \left\{1, \frac{-1}{2}, \frac{5}{4}, \frac{-5}{8}, \frac{5}{16}\right\} \tag{5.7}$$

5.2 Find an expression for $h(n)$ using $H(z)$, given that

$$h(n) \overset{\mathcal{Z}}{\rightleftharpoons} H(z) \tag{5.8}$$

and there is a one to one relationship between $h(n)$ and $H(z)$. $h(n)$ is known as the *impulse response* of the system defined by (3.2).
Solution: Given,

$$h(n) \overset{\mathcal{Z}}{\rightleftharpoons} H(z) \tag{5.9}$$

We know that,

$$H(z) = \frac{1 + z^{-2}}{1 + \frac{1}{2}z^{-1}} \tag{5.10}$$

$$= \frac{1}{1 + \frac{1}{2}z^{-1}} + \frac{z^{-2}}{1 + \frac{1}{2}z^{-1}} \tag{5.11}$$

From (4.23) and (4.8), we can say that

$$h(n) = \left(-\frac{1}{2}\right)^n u(n) + \left(-\frac{1}{2}\right)^{n-2} u(n-2) \tag{5.12}$$

5.3 Sketch $h(n)$. Is it bounded? Justify theoretically.
Solution: The Python code for sketching the graph of $h(n)$ can be found in the link below. The above code yields the graph shown in figure 5.3

```
wget https://github.com/anitadash/EE3900/
    blob/main/Codes/5_2.py
```

We know that,

$$0 \leq u(n) \leq 1 \quad \forall n \geq 0 \tag{5.13}$$

$$0 \leq \left(\frac{-1}{2}\right)^n u(n) \leq \left(\frac{-1}{2}\right)^n \leq 1 \quad \forall n \geq 0 \tag{5.14}$$

Similarly,

$$0 \leq \left(\frac{-1}{2}\right)^{n-2} u(n-2) \leq \left(\frac{-1}{2}\right)^{n-2} \leq 1 \quad \forall n \geq 0 \tag{5.15}$$

Adding (5.14) and (5.15), We get

$$0 \leq h(n) \leq 2 \quad \forall n \geq 0 \tag{5.16}$$

Thus, $h(n)$ is bounded.

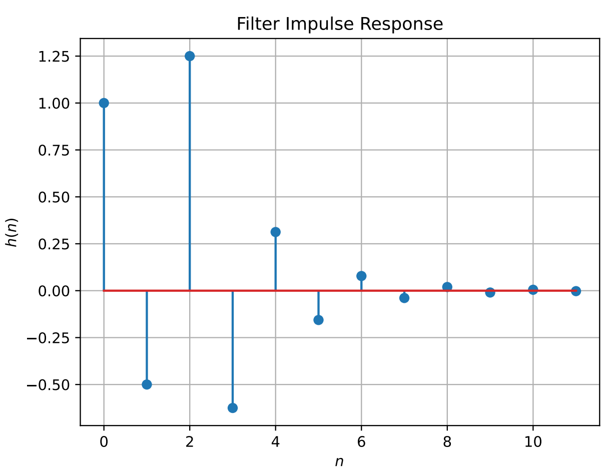

Fig. 5.3

5.4 Convergent? Justify using the ratio test.
Solution: from (5.5) We can see that for $n > 1$

$$h(n) = 5\left(\frac{-1}{2}\right)^n \tag{5.17}$$

Ratio test states that:
If $a(n)$ is a sequence such that

$$\lim_{n\to+\infty} |\frac{a_{n+1}}{a_n}| = L \tag{5.18}$$

and if $L < 1$, then $a(n)$ converges. Thus,

$$\lim_{n\to+\infty} |\frac{h_{n+1}}{h_n}| = \frac{5\left(\frac{1}{2}\right)^{n+1}}{5\left(\frac{1}{2}\right)^n} = \frac{1}{2} \tag{5.19}$$

Hence, $h(n)$ is convergent.

5.5 The system with $h(n)$ is defined to be stable if

$$\sum_{n=-\infty}^{\infty} h(n) < \infty \tag{5.20}$$

Is the system defined by (3.2) stable for the impulse response in (5.9)?
Solution: We know that,

$$h(n) = \left(-\frac{1}{2}\right)^n u(n) + \left(-\frac{1}{2}\right)^{n-2} u(n-2) \tag{5.21}$$

$$\sum_{n=-\infty}^{\infty} h(n) = \sum_{n=0}^{\infty}\left(-\frac{1}{2}\right)^n + \sum_{n=2}^{\infty}\left(-\frac{1}{2}\right)^{n-2} \tag{5.22}$$

$$= \left(\frac{1}{1+\frac{1}{2}}\right) + \left(\frac{1}{1+\frac{1}{2}}\right) \tag{5.23}$$

$$= \frac{4}{3} \tag{5.24}$$

Therefore,

$$\sum_{n=-\infty}^{\infty} h(n) < \infty \tag{5.25}$$

Hence the system defined by (3.2) is stable for the impulse response in (5.9)

5.6 Verify the above result using a python code.
Solution: The python code to the question can be found in the link below

```
wget https://github.com/anitadash/EE3900/
    blob/main/Codes/5_6.py
```

5.7 Compute and sketch $h(n)$ using

$$h(n) + \frac{1}{2}h(n-1) = \delta(n) + \delta(n-2), \tag{5.26}$$

This is the definition of $h(n)$.
Solution: The Python code for sketching the graph of $h(n)$ from (5.26) can be found in the link below. The code yields the graph shown in figure 5.7

```
wget https://github.com/anitadash/EE3900/
    blob/main/Codes/5_4.py
```

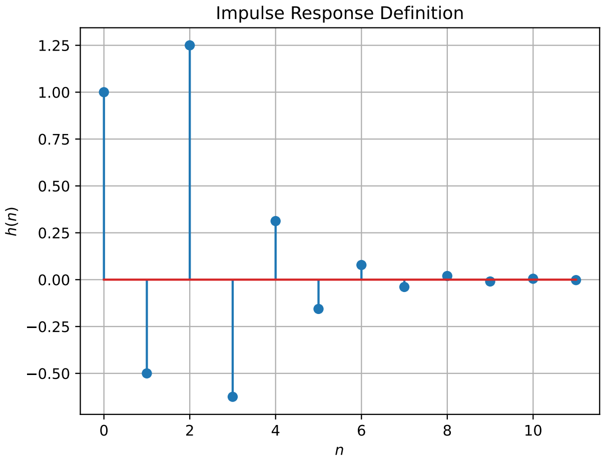

Fig. 5.7

5.8 Compute

$$y(n) = x(n) * h(n) = \sum_{n=-\infty}^{\infty} x(k)h(n-k) \quad (5.27)$$

Comment. The operation in (5.27) is known as *convolution*.

Solution: The Python code for the question can be found in the link below. The code yields the graph shown in figure 5.8. $y(n)$ is the same as the $y(n)$ from (3.2)

```
wget https://github.com/anitadash/EE3900/
    blob/main/Codes/5_8.py
```

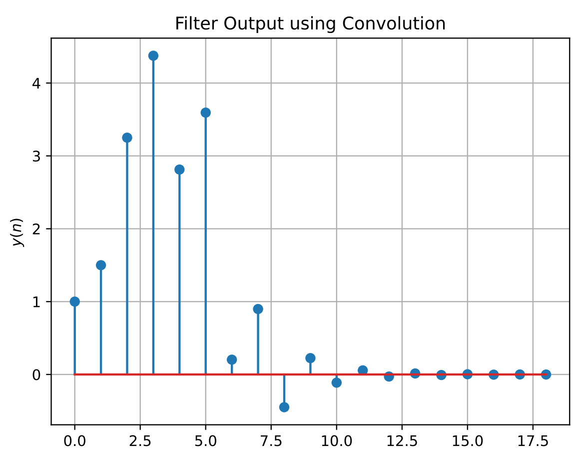

Fig. 5.8

5.9 Express the above convolution using a Teoplitz matrix.

Solution: following is the Teoplitz matrix T

$$\begin{array}{cccccc}
1 & 0 & 0 & 0 & 0 & 0 \\
-5 & 1 & 0 & 0 & 0 & 0 \\
1.25 & -5 & 1 & 0 & 0 & 0 \\
-0.62 & 1.25 & -5 & 1 & 0 & 0 \\
0.31 & -0.62 & 1.25 & -5 & 1 & 0 \\
-0.16 & 0.31 & -0.62 & 1.25 & -5 & 1 \\
0.08 & -0.16 & 0.31 & -0.62 & 1.25 & -5 \\
-0.04 & 0.08 & -0.16 & 0.31 & -0.62 & 1.25 \\
0.02 & -0.04 & 0.08 & -0.16 & 0.31 & 0.62 \\
-9.76e^{-3} & 0.02 & -0.04 & 0.08 & -0.16 & 0.31 \\
4.88e^{-3} & -9.76e^{-3} & 0.02 & -0.04 & 0.08 & -0.16 \\
-2.44e^{-3} & 4.88e^{-3} & -9.76e^{-3} & 0.02 & -0.04 & 0.08 \\
1.22e^{-3} & -2.44e^{-3} & 4.88e^{-3} & -9.76e^{-3} & 0.02 & -0.04 \\
-6.1e^{-4} & 1.22e^{-3} & -2.44e^{-3} & 4.88e^{-3} & -9.76e^{-3} & 0.02 \\
0 & -6.1e^{-4} & 1.22e^{-3} & -2.44e^{-3} & 4.88e^{-3} & -9.76e^{-3} \\
0 & 0 & -6.1e^{-4} & 1.22x10^{-3} & -2.44e^{-3} & 4.88e^{-3} \\
0 & 0 & 0 & -6.1e^{-4} & 1.22e^{-3} & -2.44e^{-3} \\
0 & 0 & 0 & 0 & -6.1e^{-4} & 1.22e^{-3} \\
0 & 0 & 0 & 0 & 0 & -6.1e^{-4}
\end{array}$$

Therefore $T.X$ gives,

$$y = \begin{bmatrix} 1 \\ 1.5 \\ 3.25 \\ 4.375 \\ 2.81 \\ 3.59 \\ 0.203 \\ 0.89 \\ -0.449 \\ 0.224 \\ -0.11 \\ 5.61e^{-2} \\ -2.80e^{-2} \\ 1.40e^{-2} \\ -7.01e^{-3} \\ 3.50e^{-3} \\ -1.75e^{-3} \\ 8.77e^{-4} \\ -4.38e^{-4} \\ 0 \end{bmatrix} \quad (5.28)$$

5.10 Show that

$$y(n) = \sum_{n=-\infty}^{\infty} x(n-k)h(k) \quad (5.29)$$

Solution: We know that,

$$Y(z) = H(z).X(z) \quad (5.30)$$

Therefore,

$$\sum_{n=-\infty}^{\infty} y(n)z^{-n} = \sum_{n=-\infty}^{\infty} h(n)z^{-n} \sum_{n=-\infty}^{\infty} x(n)z^{-n} \quad (5.31)$$

$$= \sum_{n=-\infty}^{\infty} \sum_{k=0}^{n} h(k)x(n-k)z^{-n} \quad (5.32)$$

Therefore, By comparison we can see that,

$$y(n) = \sum_{k=0}^{n} h(k)x(n-k) \quad (5.33)$$

6 DFT

6.1 Compute

$$X(k) \triangleq \sum_{n=0}^{N-1} x(n)e^{-\jmath 2\pi kn/N}, \quad k = 0, 1, \dots, N-1 \quad (6.1)$$

and $H(k)$ using $h(n)$.
Solution: We know that,

$$x(n) = \{1, 2, 3, 4, 2, 1\} \quad (6.2)$$

Adding and subtracting $e^{-j2\pi k6/6}$ we get,

$$X(k) = \sum_{n=0}^{6} x(n)e^{-\jmath 2\pi kn/N} - 1 \quad (6.3)$$

$$\frac{2\pi kn}{6} + \frac{2\pi kn}{6} \quad (6.4)$$

6.2 Compute

$$Y(k) = X(k)H(k) \quad (6.5)$$

Solution: We know that,

$$X(k) = \sum_{n=0}^{N-1} x(n)e^{-\jmath 2\pi kn/N} \quad (6.6)$$

$$H(k) = \sum_{n=0}^{N-1} h(n)e^{-\jmath 2\pi kn/N} \quad (6.7)$$

$$X(k)H(k) = \sum_{n=0}^{N-1} (x(n) * h(n))\, e^{-\jmath 2\pi kn/N} \quad (6.8)$$

$$\implies \sum_{n=0}^{N-1} y(n)e^{-\jmath 2\pi kn/N} \quad (6.9)$$

therefore,

$$Y(k) = X(k)H(k) \quad (6.10)$$

6.3 Compute

$$y(n) = \frac{1}{N} \sum_{k=0}^{N-1} Y(k) \cdot e^{\jmath 2\pi kn/N}, \quad n = 0, 1, \dots, N-1 \quad (6.11)$$

Solution: The Python code for the computation can be found below . The code yields the graph shown in figure 6.3

```
wget https://github.com/anitadash/EE3900/
    blob/main/Codes/6_3.py
```

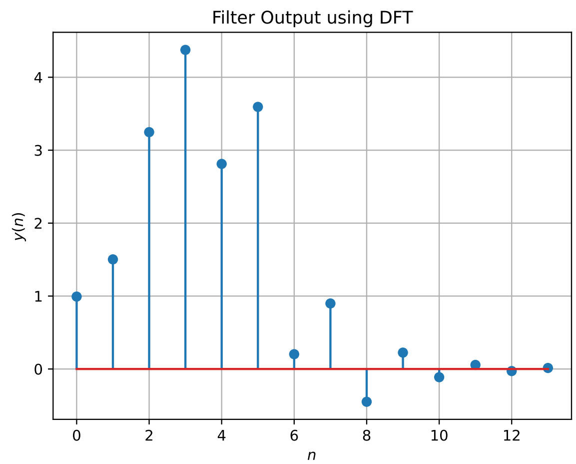

Fig. 6.3

6.4 Repeat the previous exercise by computing $X(k), H(k)$ and $y(n)$ through FFT and IFFT.
Solution: The Python code for the computation can be found below.

```
wget https://github.com/anitadash/EE3900/
    blob/main/Codes/6_4.py
```

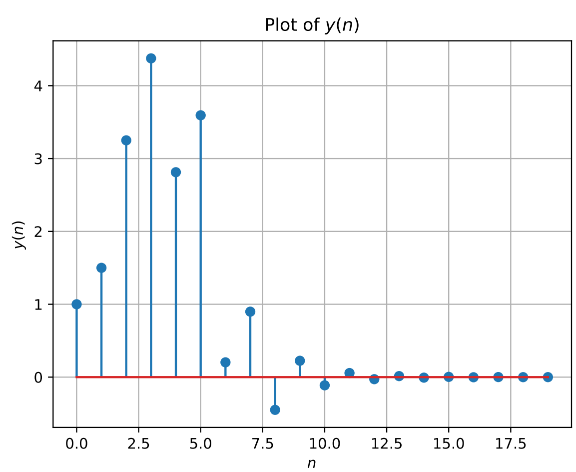

Fig. 6.4

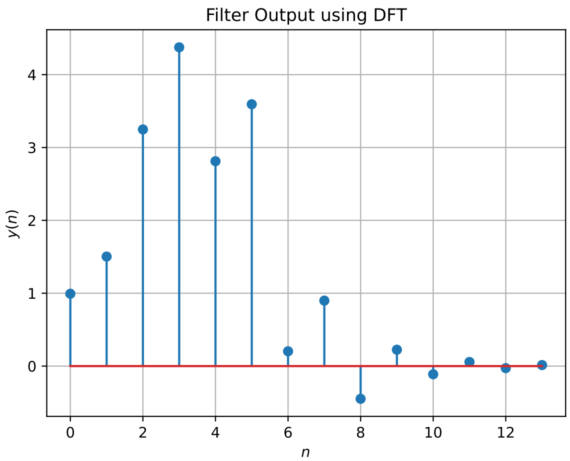

Fig. 6.4

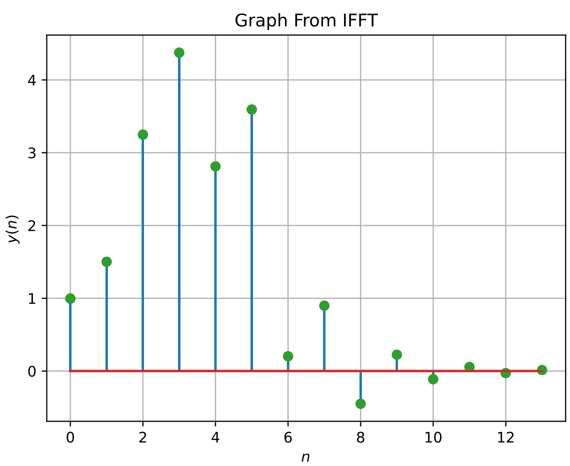

Fig. 6.4

7 FFT

7.1 The DFT of $x(n)$ is given by

$$X(k) \triangleq \sum_{n=0}^{N-1} x(n)e^{-j2\pi kn/N}, \quad k = 0, 1, \ldots, N-1 \tag{7.1}$$

7.2 Let

$$W_N = e^{-j2\pi/N} \tag{7.2}$$

Then the N-point *DFT matrix* is defined as

$$\mathbf{F}_N = \left[W_N^{mn}\right], \quad 0 \le m, n \le N-1 \tag{7.3}$$

where W_N^{mn} are the elements of $\mathbf{F}_N$.

7.3 Let

$$\mathbf{I}_4 = \begin{pmatrix} \mathbf{e}_4^1 & \mathbf{e}_4^2 & \mathbf{e}_4^3 & \mathbf{e}_4^4 \end{pmatrix} \tag{7.4}$$

be the 4×4 identity matrix. Then the 4 point *DFT permutation matrix* is defined as

$$\mathbf{P}_4 = \begin{pmatrix} \mathbf{e}_4^1 & \mathbf{e}_4^3 & \mathbf{e}_4^2 & \mathbf{e}_4^4 \end{pmatrix} \tag{7.5}$$

7.4 The 4 point *DFT diagonal matrix* is defined as

$$\mathbf{D}_4 = diag\begin{pmatrix} W_8^0 & W_8^1 & W_8^2 & W_8^3 \end{pmatrix} \tag{7.6}$$

7.5 Show that

$$W_N^2 = W_{N/2} \tag{7.7}$$

Solution: We know that,

$$W_N^2 = \left(e^{\frac{-j2\pi}{N}}\right)^2 = e^{\frac{-j2\pi}{N/2}} = W_{N/2} \tag{7.8}$$

therefore,

$$W_N^2 = W_{N/2} \tag{7.9}$$

7.6 Show that

$$\mathbf{F}_4 = \begin{bmatrix} \mathbf{I}_2 & \mathbf{D}_2 \\ \mathbf{I}_2 & -\mathbf{D}_2 \end{bmatrix} \begin{bmatrix} \mathbf{F}_2 & 0 \\ 0 & \mathbf{F}_2 \end{bmatrix} \mathbf{P}_4 \tag{7.10}$$

Solution: We know that for $n \in \mathbb{N}$,

$$W_4^{4n} = 1 \tag{7.11}$$
$$W_4^{4n+2} = -1 \tag{7.12}$$

Thus,

$$\mathbf{D}_2\mathbf{F}_2 = \begin{bmatrix} W_4^0 & 0 \\ 0 & W_4^1 \end{bmatrix} \begin{bmatrix} W_2^0 & W_2^0 \\ W_2^0 & W_2^1 \end{bmatrix} \tag{7.13}$$
$$= \begin{bmatrix} W_4^0 & 0 \\ 0 & W_4^1 \end{bmatrix} \begin{bmatrix} W_4^0 & W_4^0 \\ W_4^0 & W_4^2 \end{bmatrix} \tag{7.14}$$
$$= \begin{bmatrix} W_4^0 & W_4^0 \\ W_4^1 & W_4^3 \end{bmatrix} \tag{7.15}$$
$$\implies -\mathbf{D}_2\mathbf{F}_2 = \begin{bmatrix} W_4^2 & W_4^6 \\ W_4^3 & W_4^9 \end{bmatrix} \tag{7.16}$$

and

$$\mathbf{F}_2 = \begin{pmatrix} W_2^0 & W_2^0 \\ W_2^0 & W_2^1 \end{pmatrix} \tag{7.17}$$
$$= \begin{pmatrix} W_4^0 & W_4^0 \\ W_4^0 & W_4^2 \end{pmatrix} \tag{7.18}$$

Hence,

$$\mathbf{W}_4 = \begin{pmatrix} W_4^0 & W_4^0 & W_4^0 & W_4^0 \\ W_4^0 & W_4^2 & W_4^1 & W_4^3 \\ W_4^0 & W_4^4 & W_4^2 & W_4^6 \\ W_4^0 & W_4^6 & W_4^3 & W_4^9 \end{pmatrix} \tag{7.19}$$

$$= \begin{bmatrix} \mathbf{I}_2\mathbf{F}_2 & \mathbf{D}_2 F_2 \\ \mathbf{I}_2\mathbf{F}_2 & -\mathbf{D}_2 F_2 \end{bmatrix} \tag{7.20}$$

$$= \begin{bmatrix} \mathbf{I}_2 & \mathbf{D}_2 \\ \mathbf{I}_2 & \mathbf{D}_2 \end{bmatrix} \begin{bmatrix} \mathbf{F}_2 & 0 \\ 0 & \mathbf{F}_2 \end{bmatrix} \tag{7.21}$$

Multiplying (7.21) by $\mathbf{P}_4$ on both sides, and noting that $\mathbf{W}_4\mathbf{P}_4 = \mathbf{F}_4$ gives us (7.10).

7.7 Show that

$$\mathbf{F}_N = \begin{bmatrix} \mathbf{I}_{N/2} & \mathbf{D}_{N/2} \\ \mathbf{I}_{N/2} & -\mathbf{D}_{N/2} \end{bmatrix} \begin{bmatrix} \mathbf{F}_{N/2} & 0 \\ 0 & \mathbf{F}_{N/2} \end{bmatrix} \mathbf{P}_N \tag{7.22}$$

Solution: Observe that for even N and letting $\mathbf{f}_N^i$ denote the i^{th} column of $\mathbf{F}_N$, from (7.15) and (7.16),

$$\begin{pmatrix} \mathbf{D}_{N/2}\mathbf{F}_{N/2} \\ -\mathbf{D}_{N/2}\mathbf{F}_{N/2} \end{pmatrix} = \begin{pmatrix} \mathbf{f}_N^2 & \mathbf{f}_N^4 & \dots & \mathbf{f}_N^N \end{pmatrix} \tag{7.23}$$

and

$$\begin{pmatrix} \mathbf{I}_{N/2}\mathbf{F}_{N/2} \\ \mathbf{I}_{N/2}\mathbf{F}_{N/2} \end{pmatrix} = \begin{pmatrix} \mathbf{f}_N^1 & \mathbf{f}_N^3 & \dots & \mathbf{f}_N^{N-1} \end{pmatrix} \tag{7.24}$$

Thus,

$$\begin{bmatrix} \mathbf{I}_2\mathbf{F}_2 & \mathbf{D}_2\mathbf{F}_2 \\ \mathbf{I}_2\mathbf{F}_2 & -\mathbf{D}_2\mathbf{F}_2 \end{bmatrix} = \begin{bmatrix} \mathbf{I}_{N/2} & \mathbf{D}_{N/2} \\ \mathbf{I}_{N/2} & -\mathbf{D}_{N/2} \end{bmatrix} \begin{bmatrix} \mathbf{F}_{N/2} & 0 \\ 0 & \mathbf{F}_{N/2} \end{bmatrix}$$
$$= \begin{pmatrix} \mathbf{f}_N^1 & \dots & \mathbf{f}_N^{N-1} & \mathbf{f}_N^2 & \dots & \mathbf{f}_N^N \end{pmatrix} \tag{7.25}$$

and so,

$$\begin{bmatrix} \mathbf{I}_{N/2} & \mathbf{D}_{N/2} \\ \mathbf{I}_{N/2} & -\mathbf{D}_{N/2} \end{bmatrix} \begin{bmatrix} \mathbf{F}_{N/2} & 0 \\ 0 & \mathbf{F}_{N/2} \end{bmatrix} \mathbf{P}_N$$
$$= \begin{pmatrix} \mathbf{f}_N^1 & \mathbf{f}_N^2 & \dots & \mathbf{f}_N^N \end{pmatrix} = \mathbf{F}_N \tag{7.26}$$

7.8 Find

$$\mathbf{P}_4\mathbf{x} \tag{7.27}$$

Solution: We have,

$$\mathbf{P}_4\mathbf{x} = \begin{pmatrix} \mathbf{e}_4^1 & \mathbf{e}_4^3 & \mathbf{e}_4^2 & \mathbf{e}_4^4 \end{pmatrix} \begin{pmatrix} x(0) \\ x(1) \\ x(2) \\ x(3) \end{pmatrix} = \begin{pmatrix} x(0) \\ x(2) \\ x(1) \\ x(3) \end{pmatrix} \tag{7.28}$$

7.9 Show that

$$\mathbf{X} = \mathbf{F}_N\mathbf{x} \tag{7.29}$$

where $\mathbf{x}, \mathbf{X}$ are the vector representations of $x(n), X(k)$ respectively.

Solution: Writing the terms of X,

$$X(0) = x(0) + x(1) + \dots + x(N-1) \tag{7.30}$$

$$X(1) = x(0) + x(1)e^{-\frac{\jmath 2\pi}{N}} + \dots + \\ + x(N-1)e^{-\frac{\jmath 2(N-1)\pi}{N}} \tag{7.31}$$

$$\vdots$$

$$X(N-1) = x(0) + x(1)e^{-\frac{\jmath 2(N-1)\pi}{N}} + \dots + \\ + x(N-1)e^{-\frac{\jmath 2(N-1)(N-1)\pi}{N}} \tag{7.32}$$

Clearly, the term in the m^{th} row and n^{th} column is given by ($0 \le m \le N-1$ and $0 \le n \le N-1$)

$$T_{mn} = x(n)e^{-\frac{\jmath 2mn\pi}{N}} \tag{7.33}$$

and so, we can represent each of these terms as a matrix product

$$\mathbf{X} = \mathbf{F}_N\mathbf{x} \tag{7.34}$$

where $\mathbf{F}_N = \left[e^{-\frac{-\jmath 2mn\pi}{N}}\right]_{mn}$ for $0 \le m \le N-1$ and $0 \le n \le N-1$.

7.10 Derive the following Step-by-step visualisation of 8-point FFTs into 4-point FFTs and so on

$$\begin{bmatrix} X(0) \\ X(1) \\ X(2) \\ X(3) \end{bmatrix} = \begin{bmatrix} X_1(0) \\ X_1(1) \\ X_1(2) \\ X_1(3) \end{bmatrix} + \begin{bmatrix} W_8^0 & 0 & 0 & 0 \\ 0 & W_8^1 & 0 & 0 \\ 0 & 0 & W_8^2 & 0 \\ 0 & 0 & 0 & W_8^3 \end{bmatrix} \begin{bmatrix} X_2(0) \\ X_2(1) \\ X_2(2) \\ X_2(3) \end{bmatrix} \tag{7.35}$$

$$\begin{bmatrix} X(4) \\ X(5) \\ X(6) \\ X(7) \end{bmatrix} = \begin{bmatrix} X_1(0) \\ X_1(1) \\ X_1(2) \\ X_1(3) \end{bmatrix} - \begin{bmatrix} W_8^0 & 0 & 0 & 0 \\ 0 & W_8^1 & 0 & 0 \\ 0 & 0 & W_8^2 & 0 \\ 0 & 0 & 0 & W_8^3 \end{bmatrix} \begin{bmatrix} X_2(0) \\ X_2(1) \\ X_2(2) \\ X_2(3) \end{bmatrix} \tag{7.36}$$

4-point FFTs into 2-point FFTs

$$\begin{bmatrix} X_1(0) \\ X_1(1) \end{bmatrix} = \begin{bmatrix} X_3(0) \\ X_3(1) \end{bmatrix} + \begin{bmatrix} W_4^0 & 0 \\ 0 & W_4^1 \end{bmatrix} \begin{bmatrix} X_4(0) \\ X_4(1) \end{bmatrix} \tag{7.37}$$

$$\begin{bmatrix} X_1(2) \\ X_1(3) \end{bmatrix} = \begin{bmatrix} X_3(0) \\ X_3(1) \end{bmatrix} - \begin{bmatrix} W_4^0 & 0 \\ 0 & W_4^1 \end{bmatrix} \begin{bmatrix} X_4(0) \\ X_4(1) \end{bmatrix} \tag{7.38}$$

$$\begin{bmatrix} X_2(0) \\ X_2(1) \end{bmatrix} = \begin{bmatrix} X_5(0) \\ X_5(1) \end{bmatrix} + \begin{bmatrix} W_4^0 & 0 \\ 0 & W_4^1 \end{bmatrix} \begin{bmatrix} X_6(0) \\ X_6(1) \end{bmatrix} \tag{7.39}$$

$$\begin{bmatrix} X_2(2) \\ X_2(3) \end{bmatrix} = \begin{bmatrix} X_5(0) \\ X_5(1) \end{bmatrix} - \begin{bmatrix} W_4^0 & 0 \\ 0 & W_4^1 \end{bmatrix} \begin{bmatrix} X_6(0) \\ X_6(1) \end{bmatrix} \tag{7.40}$$

$$P_8 \begin{bmatrix} x(0) \\ x(1) \\ x(2) \\ x(3) \\ x(4) \\ x(5) \\ x(6) \\ x(7) \end{bmatrix} = \begin{bmatrix} x(0) \\ x(2) \\ x(4) \\ x(6) \\ x(1) \\ x(3) \\ x(5) \\ x(7) \end{bmatrix} \tag{7.41}$$

$$P_4 \begin{bmatrix} x(0) \\ x(2) \\ x(4) \\ x(6) \end{bmatrix} = \begin{bmatrix} x(0) \\ x(4) \\ x(2) \\ x(6) \end{bmatrix} \tag{7.42}$$

$$P_4 \begin{bmatrix} x(1) \\ x(3) \\ x(5) \\ x(7) \end{bmatrix} = \begin{bmatrix} x(1) \\ x(5) \\ x(3) \\ x(7) \end{bmatrix} \tag{7.43}$$

Therefore,

$$\begin{bmatrix} X_3(0) \\ X_3(1) \end{bmatrix} = F_2 \begin{bmatrix} x(0) \\ x(4) \end{bmatrix} \tag{7.44}$$

$$\begin{bmatrix} X_4(0) \\ X_4(1) \end{bmatrix} = F_2 \begin{bmatrix} x(2) \\ x(6) \end{bmatrix} \tag{7.45}$$

$$\begin{bmatrix} X_5(0) \\ X_5(1) \end{bmatrix} = F_2 \begin{bmatrix} x(1) \\ x(5) \end{bmatrix} \tag{7.46}$$

$$\begin{bmatrix} X_6(0) \\ X_6(1) \end{bmatrix} = F_2 \begin{bmatrix} x(3) \\ x(7) \end{bmatrix} \tag{7.47}$$

Solution: We write out the values of performing an 8-point FFT on $\mathbf{x}$ as follows.

$$X(k) = \sum_{n=0}^{7} x(n) e^{-\frac{\jmath 2kn\pi}{8}} \tag{7.48}$$

$$= \sum_{n=0}^{3} \left(x(2n) e^{-\frac{\jmath 2kn\pi}{4}} + e^{-\frac{\jmath 2k\pi}{8}} x(2n+1) e^{-\frac{\jmath 2kn\pi}{4}} \right) \tag{7.49}$$

$$= X_1(k) + e^{-\frac{\jmath 2k\pi}{4}} X_2(k) \tag{7.50}$$

where $\mathbf{X}_1$ is the 4-point FFT of the even-numbered terms and $\mathbf{X}_2$ is the 4-point FFT of the odd numbered terms. Noticing that for $k \geq 4$,

$$X_1(k) = X_1(k-4) \tag{7.51}$$

$$e^{-\frac{\jmath 2k\pi}{8}} = -e^{-\frac{\jmath 2(k-4)\pi}{8}} \tag{7.52}$$

we can now write out $X(k)$ in matrix form as in (**??**) and (**??**). We also need to solve the two 4-point FFT terms so formed.

$$X_1(k) = \sum_{n=0}^{3} x_1(n) e^{-\frac{\jmath 2kn\pi}{8}} \tag{7.53}$$

$$= \sum_{n=0}^{1} \left(x_1(2n) e^{-\frac{\jmath 2kn\pi}{4}} + e^{-\frac{\jmath 2k\pi}{8}} x_2(2n+1) e^{-\frac{\jmath 2kn\pi}{4}} \right) \tag{7.54}$$

$$= X_3(k) + e^{-\frac{\jmath 2k\pi}{4}} X_4(k) \tag{7.55}$$

using $x_1(n) = x(2n)$ and $x_2(n) = x(2n+1)$. Thus we can write the 2-point FFTs

$$\begin{bmatrix} X_3(0) \\ X_3(1) \end{bmatrix} = F_2 \begin{bmatrix} x(0) \\ x(4) \end{bmatrix} \tag{7.56}$$

$$\begin{bmatrix} X_4(0) \\ X_4(1) \end{bmatrix} = F_2 \begin{bmatrix} x(2) \\ x(6) \end{bmatrix} \tag{7.57}$$

Using a similar idea for the terms X_2,

$$\begin{bmatrix} X_5(0) \\ X_5(1) \end{bmatrix} = F_2 \begin{bmatrix} x(1) \\ x(5) \end{bmatrix} \tag{7.58}$$

$$\begin{bmatrix} X_6(0) \\ X_6(1) \end{bmatrix} = F_2 \begin{bmatrix} x(3) \\ x(7) \end{bmatrix} \tag{7.59}$$

But observe that from (7.28),

$$\mathbf{P}_8 \mathbf{x} = \begin{pmatrix} \mathbf{x}_1 \\ \mathbf{x}_2 \end{pmatrix} \tag{7.60}$$

$$\mathbf{P}_4 \mathbf{x}_1 = \begin{pmatrix} \mathbf{x}_3 \\ \mathbf{x}_4 \end{pmatrix} \tag{7.61}$$

$$\mathbf{P}_4 \mathbf{x}_2 = \begin{pmatrix} \mathbf{x}_5 \\ \mathbf{x}_6 \end{pmatrix} \tag{7.62}$$

where we define $x_3(k) = x(4k)$, $x_4(k) = x(4k+2)$, $x_5(k) = x(4k+1)$, and $x_6(k) = x(4k+3)$ for $k = 0, 1$.

7.11 For

$$\mathbf{x} = \begin{pmatrix} 1 \\ 2 \\ 3 \\ 4 \\ 2 \\ 1 \end{pmatrix} \tag{7.63}$$

compte the DFT using (7.29)

Solution: The Python code for the above question can be found below

```
wget https://github.com/anitadash/EE3900/
    blob/main/Codes/7_11.py
```

7.12 Repeat the above exercise using the FFT after zero padding **x**.

7.13 Write a C program to compute the 8-point FFT.
Solution: The Python code for the above two questions can be found below

```
wget https://github.com/anitadash/EE3900/
    blob/main/Codes/7_13.c
```

Folowing are the codes for finding the runtime complexity of the algorithms so far.

```
wget https://github.com/anitadash/EE3900/
    blob/main/Codes/time_cmpx.c
wget https://github.com/anitadash/EE3900/
    blob/main/Codes/TC_Convolution.py
wget https://github.com/anitadash/EE3900/
    blob/main/Codes/TC_FFT.py
```

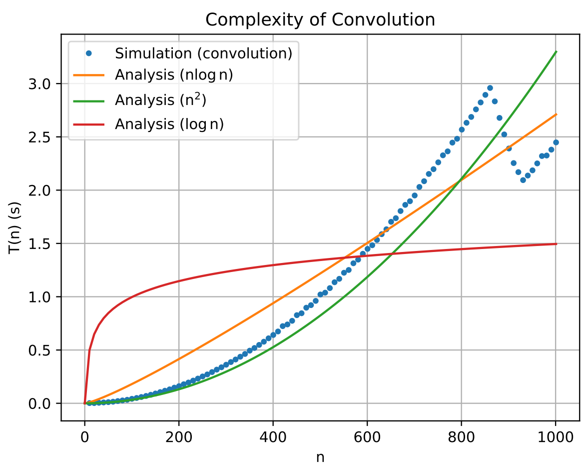

Fig. 7.13

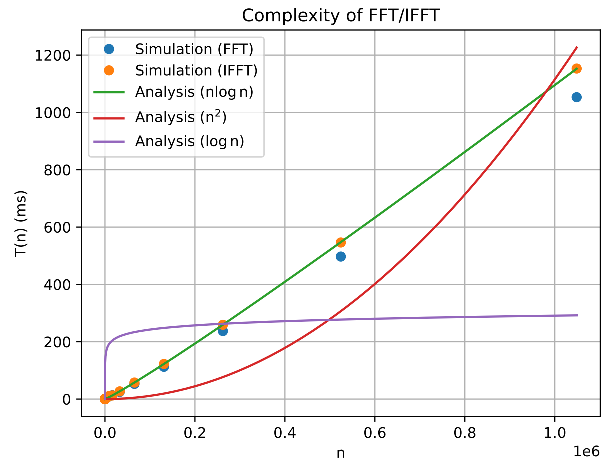

Fig. 7.13

8 Exercises

Answer the following questions by looking at the python code in Problem 2.3.

8.1 The command

```
output_signal = signal.lfilter(b, a,
    input_signal)
```

in Problem 2.3 is executed through the following difference equation

$$\sum_{m=0}^{M} a(m)\, y(n-m) = \sum_{k=0}^{N} b(k)\, x(n-k) \quad (8.1)$$

where the input signal is $x(n)$ and the output signal is $y(n)$ with initial values all 0. Replace **signal.filtfilt** with your own routine and verify.
Solution: The following code executes the above question.

```
wget https://github.com/anitadash/EE3900/
    blob/main/Codes/8_1.py
```

8.2 Repeat all the exercises in the previous sections for the above a and b.
Solution: For the given values, the difference equation is

$$\begin{aligned} &y(n) - (2.52)\, y(n-1) + (2.56)\, y(n-2) \\ &- (1.21)\, y(n-3) + (0.22)\, y(n-4) \\ &= \left(3.45 \times 10^{-3}\right) x(n) + \left(1.38 \times 10^{-2}\right) x(n-1) \\ &+ \left(2.07 \times 10^{-2}\right) x(n-2) + \left(1.38 \times 10^{-2}\right) x(n-3) \\ &+ \left(3.45 \times 10^{-3}\right) x(n-4) \end{aligned} \quad (8.2)$$

From (8.1), we see that the transfer function can be written as follows

$$H(z) = \frac{\sum_{k=0}^{N} b(k) z^{-k}}{\sum_{k=0}^{M} a(k) z^{-k}} \quad (8.3)$$

$$= \sum_i \frac{r(i)}{1 - p(i) z^{-1}} + \sum_j k(j) z^{-j} \quad (8.4)$$

where $r(i)$, $p(i)$, are called residues and poles respectively of the partial fraction expansion of $H(z)$. $k(i)$ are the coefficients of the direct polynomial terms that might be left over. We can now take the inverse z-transform of (8.4) and get using (4.23),

$$h(n) = \sum_i r(i)[p(i)]^n u(n) + \sum_j k(j)\delta(n-j) \tag{8.5}$$

Substituting the values,

$$\begin{aligned} h(n) = [&(-0.24 - 0.71\jmath)(0.56 + 0.14\jmath)^n \\ &+ (-0.24 + 0.71\jmath)(0.56 - 0.14\jmath)^n \\ &+ (-0.25 + 0.12\jmath)(0.70 + 0.41\jmath)^n \\ &+ (-0.25 - 0.12\jmath)(0.70 - 0.41\jmath)^n]u(n) \\ &+ \left(1.6 \times 10^{-2}\right)\delta(n) \end{aligned} \tag{8.6}$$

$$\begin{aligned} \implies h(n) = &(1.5)(0.58)^n \cos(n\alpha_1 + \beta_1) \\ &+ (0.55)(0.81)^n \cos(n\alpha_2 + \beta_2) \\ &+ \left(1.6 \times 10^{-2}\right)\delta(n) \end{aligned} \tag{8.7}$$

where

$$\tan\alpha_1 = 0.25 \tag{8.8}$$

$$\tan\beta_1 = 2.96 \tag{8.9}$$

$$\tan\alpha_2 = 0.59 \tag{8.10}$$

$$\tan\beta_2 = -0.48 \tag{8.11}$$

The values $r(i)$, $p(i)$, $k(i)$ and thus below is the code for the impulse response function

```
wget https://github.com/anitadash/EE3900/
    blob/main/Codes/8_2_1.py
```

code for the filter frequency response.

```
wget https://github.com/anitadash/EE3900/
    blob/main/Codes/8_2_2.py
```

Observe that for a series $t_n = r^n$, $\frac{t_{n+1}}{t_n} = r$. By the ratio test, t_n converges if $|r| < 1$. We observe that for all i, $|p(i)| < 1$ and so, as $h(n)$ is the sum of many convergent series, we see that $h(n)$ converges and is bounded. From (4.1),

$$\sum_{n=0}^{\infty} h(n) = H(1) = \frac{\sum_{k=0}^{N} b(k)}{\sum_{k=0}^{M} a(k)} = 1 < \infty \tag{8.12}$$

Therefore, the system is stable. From Fig. (**??**), $h(n)$ is negligible after $n \geq 64$, and we can apply a 64-bit FFT to get y(n). The following code uses the DFT matrix to generate $y(n)$ in Fig. (8.2)

```
wget https://github.com/anitadash/EE3900/
    blob/main/Codes/8_2_3.py
```

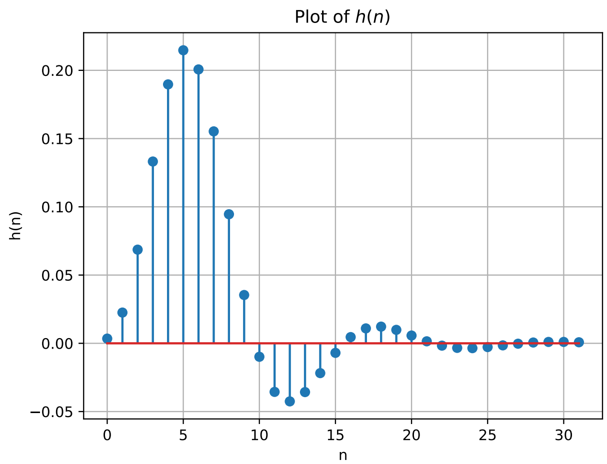

Fig. 8.2

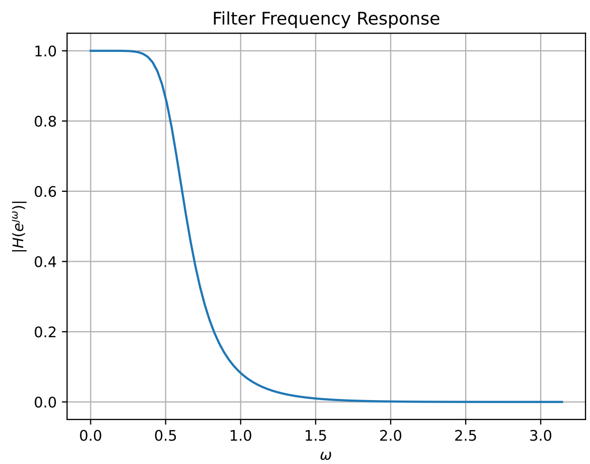

Fig. 8.2

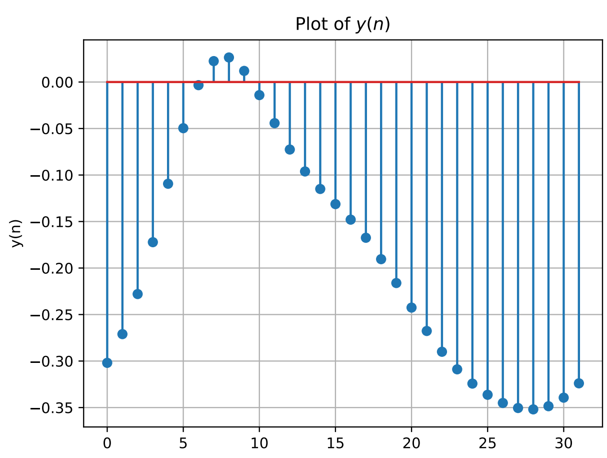

Fig. 8.2

8.3 What is the sampling frequency of the input signal?
Solution: Sampling frequency(fs)=44.1kHZ.
8.4 What is type, order and cutoff-frequency of the above butterworth filter
Solution: The given butterworth filter is low pass with order=2 and cutoff-frequency=4kHz.
8.5 Modifying the code with different input parameters and to get the best possible output.
Solution: A better filtering was found on setting the order of the filter to be 7.