Problem set 2

Due date: Tuesday, September 29^{th}, 11:15 am. Submit your Matlab code, plots and written report.

Exercise 1

The objective of the exercise is to work on a simple option model in which the underlying equity process is not lognormal and derive an *implied volatility surface.*
Inspired by Merton's model, we assume that

- A firm's assets V follow a lognormal stochastic process

$$\frac{dV_t}{V_t} = \mu dt + \sigma dW_t \tag{1}$$

  The value of the assets at the initial time t is $V_t = 100$. The assets yearly volatility is $\sigma = 0.4$.

- The firm has debt $D = 50$ due in 10 years.

- The (constant) riskless interest rate is $r = 5\%$.

Under this conditions, the equity of the firm is a residual claim: in 10 years, equity will be worth $max(0, V_T - D)$. Hence equity is a call option on the firm's assets, with strike $D = 50$, expiration $T = 10$. **Compute the current value of the firm's equity**.

Now consider a call option on the firm's equity, with expiration $\tau < T$ and strike K. This can be regarded as a compound option (call on call) on the firm's assets V.

- Use the analytic formula for the compound option (call on call) to find the value of a call option on the firm's equity with strike K and maturity $\tau < T$.

- Write a small Matlab program to obtain the numerical value of the option for different strikes ($K = 60\%, 80\%, 100\%, 120\%$ of the current equity value) and maturities ($\tau = 2, 5, 7, 9$ years).

- Derive the Black-Scholes implied volatility of the firm's equity corresponding to all these strikes and expirations.

- For every maturity τ, plot the implied volatility as a function of strike.

Background: compound option

For this exercise you will have to use the formula to price a *compound option*, an option to buy at T_1 an option with expiration $T_2 > T_1$ for a predetermined price. Consider a (non-dividend paying) stock following a stochastic process

$$\frac{dS}{S} = \mu dt + \sigma dW \tag{2}$$

A call with strike K_1 and expiration T_1 on a call with strike K_2 and expiration $T_2 > T_1$ is worth

$$C(t,S) = SN_2\left(a_1, b_1, \sqrt{\frac{(T_1-t)}{(T_2-t)}}\right) - K_2 e^{-r(T_2-t)} N_2\left(a_2, b_2, \sqrt{\frac{(T_1-t)}{(T_2-t)}}\right) - e^{-r(T_1-t)} K_1 N(a_2) \tag{3}$$

where $N_2(x_1, x_2, \rho)$ is the bivariate cumulative distribution function of two standard normal variables with correlation ρ, $N(x)$ is the univariate standard normal cumulative distribution function,

$$a_1 = \frac{ln(S/S^*) + (r + \sigma^2/2)(T_1 - t)}{\sigma\sqrt{T_1 - t}} \tag{4}$$

$$a_2 = a_1 - \sigma\sqrt{T_1 - t} \tag{5}$$

$$b_1 = \frac{ln(S/K_2) + (r + \sigma^2/2)(T_2 - t)}{\sigma\sqrt{T_2 - t}} \tag{6}$$

$$b_2 = b_2 - \sigma\sqrt{T_2 - t} \tag{7}$$

S^* is a stock price for which the price of the underlying option will be equal to K_1 at T_1, so the lowest stock price for which the compound option will be exercised.

In Matlab, for the bivariate normal distribution $N_2(a, b, \rho)$ use the command $mvncdf(x, \mu, \sigma)$ with $x = [a, b]$, $\mu = [0, 0]$, $\sigma = [1, \rho; \rho, 1]$

Exercise 2

Consider a non-dividend-paying stock with current value $S = 100$. In the risk-neutral measure the stock follows a jump-diffusion process

$$\frac{dS}{S} = (r - \lambda^Q \gamma)dt + \sigma dW_t + \gamma dN_t \tag{8}$$

Take $\gamma = -0.08$ (i.e. the stock price is subject to downward 8% jumps), $\lambda^Q = 0.2$ (i.e. on average the stock experiences a jump every 5 years), $\sigma = 0.2$, $r = 0.04$. Compute the value of a call option with expirations $T = 0.02,\ 0.08,\ 0.25,\ 0.5$ (i.e. approximately 1 week, 1 month, 3 months, 6 months). and strikes $K/S = 0.8,\ 0.9,\ 1, 1.1$. For every expiration, plot the Black-Scholes implied volatility of the option as a function of strike.