Problem set 10

Due on November 23, 2020, at 11:15am

Exercise 1

Write the PDE for a derivative V when the short rate follows the CIR model

$$dr = k(\theta - r(t))dt + \sigma_r \sqrt{r} dW_t \tag{1}$$

and specify the boundary conditions if V is a zero-coupon bond with expiration T. To find a formula for the bond price, try a solution

$$V = A(T-t)e^{-B(T-t)r_t} \tag{2}$$

- Find the system of ODEs satisfied by A and B.
- Find A and B by solving this ODE system.

Exercise 2

Show how to obtain the analytical formula for the call option on a zero-coupon bond in the Vasicek model, following the method described on the slides. The expiration of the option is T and the expiration of the underlying bond is $T+\tau$.

- Show that the expected discounted payout of the option can be written as

$$P(t,T+\tau)\mathrm{E}_t^{Q^{T+\tau}}[\mathcal{I}_{P(T,T+\tau)\geq K}] - KP(t,T)\mathrm{E}^{Q^T}[\mathcal{I}_{P(T,T+\tau)\geq K}] \tag{3}$$

- What is the relationship between the Brownian motion increment dW^Q in the Q-measure and the Brownian motion increment in the two probability measures appearing in (3)?
- What is the process followed by the underlying process $P(t,T+\tau)$ under these two measures?
- Now derive a formula for $\mathrm{E}_t^{Q^{T+\tau}}[\mathcal{I}_{P(T,T+\tau)\geq K}]$ and $\mathrm{E}^{Q^T}[\mathcal{I}_{P(T,T+\tau)\geq K}]$ and finally write the final formula for the value of the option.

Exercise 3

Consider a simple call option in the Black-Scholes model. The process for the stock price under the Q-measure is

$$\frac{dS}{S} = rdt + \sigma dW^Q \tag{4}$$

with constant r

- Show that the value of the option can be written as

$$C_t = S_t E^R[\mathcal{I}_{S_T>K}|\mathcal{F}_t] - e^{-r(T-t)} K E^Q[\mathcal{I}_{S_T>K}|\mathcal{F}_t] \tag{5}$$

  and specify what is the Radon-Nykodym derivative associated with the new measure R.

- What is the relationship between dW^Q and dW^R?

- What is the process followed by the stock price under the measure R? Show that $E^R[\mathcal{I}_{S_T>K}|\mathcal{F}_t]$ is the familiar $N(d_1)$.