and similarly for $\mathbb{Q}^S$: $\quad \sigma_t^{\mathbb{Q}^S} = -\sigma B(t,S)$

where $\sigma$ is the deterministic volatility of the Vasicek SDE and $B(t,T)$ comes from the closed-form solution of the zero-coupon price

3. We note that under $\mathbb{Q}^T$:

$$\frac{dP(t,S)}{P(t,S)} = r_t dt + \sigma_t^{\mathbb{Q}^S} dW_t^{\mathbb{Q}} = r_t dt + \sigma_t^{\mathbb{Q}^S}\left[dW_t^{\mathbb{Q}^T} + \sigma_t^{\mathbb{Q}^T} dt\right]$$
$$= \left[r_t + \sigma_t^{\mathbb{Q}^S}\sigma_t^{\mathbb{Q}^T}\right]dt + \sigma_t^{\mathbb{Q}^S} dW_t^{\mathbb{Q}^T}$$

and under $\mathbb{Q}^S$:

$$\frac{dP(t,S)}{P(t,S)} = r_t dt + \sigma_t^{\mathbb{Q}^S} dW_t^{\mathbb{Q}} = r_t dt + \sigma_t^{\mathbb{Q}^S}\left[dW_t^{\mathbb{Q}^S} + \sigma_t^{\mathbb{Q}^S} dt\right]$$
$$= \left[r_t + \sigma_t^{\mathbb{Q}^S}\sigma_t^{\mathbb{Q}^S}\right]dt + \sigma_t^{\mathbb{Q}^S} dW_t^{\mathbb{Q}^S}$$

4. Let's observe that $\mathbb{E}_t^{\mathbb{Q}^S}\left[\mathbb{1}_{\{P(T,S)\geq K\}}\right] = \mathbb{Q}^S\left(P(T,S)\geq K\right) = \mathbb{Q}^S\left(\frac{P(T,T)}{P(T,S)} \leq \frac{1}{K}\right)$

Therefore, we have:

$$\ln\left(\frac{P(T,T)}{P(T,S)}\right) = \ln\left(\frac{P(0,T)}{P(0,S)}\right) + \ln \mathcal{E}\left(\sigma_t^{\mathbb{Q}^S}\cdot W^{\mathbb{Q}^S}\right)$$
$$= \ln\left(\frac{P(0,T)}{P(0,S)}\right) + \int_0^T \sigma_s^{\mathbb{Q}^S} dW_s^{\mathbb{Q}^S} - \frac{1}{2}\int_0^T (\sigma_s^{\mathbb{Q}^S})^2 ds$$

We further observe that:

- $\mathbb{E}^{\mathbb{Q}^S}\left[\ln\left(\frac{P(T,T)}{P(T,S)}\right)\right] = \ln\left(\frac{P(0,T)}{P(0,S)}\right) - \frac{1}{2}\int_0^T(\sigma_s^{\mathbb{Q}^S})^2 ds = \mu(T)$
- $\mathrm{Var}^{\mathbb{Q}^S}\left[\ln\left(\frac{P(T,T)}{P(T,S)}\right)\right] = \mathbb{E}^{\mathbb{Q}^S}\left[\int_0^T(\sigma_s^{\mathbb{Q}^S})^2 ds\right] = \int_0^T(\sigma_s^{\mathbb{Q}^S})^2 ds = \sigma^2(T)$

where we used the fact that the expectation of the integral in $dW_s^{\mathbb{Q}^S}$ is zero and Itô's isometry

$$\Rightarrow \ln\left(\frac{P(T,T)}{P(T,S)}\right) \sim N\left(\mu(T), \sigma^2(T)\right)$$

This implies that:

$$\mathbb{Q}^S\left(\frac{P(T,T)}{P(T,S)} < \frac{1}{K}\right) = \mathbb{Q}^S\left(\ln\left(\frac{P(T,T)}{P(T,S}\right) < \ln\left(\frac{1}{K}\right)\right) = \mathbb{Q}^S\left(Z < \frac{\ln(1/K) - \mu(T)}{\sigma(T)}\right)$$
$$= \Phi\left(\frac{\ln\left(\frac{P(0,S)}{K\,P(0,T)}\right) + \frac{1}{2}\int_0^T(\sigma_s^{\mathbb{Q}^S})^2 ds}{\sqrt{\int_0^T \|(\sigma_s^{\mathbb{Q}^S})^2\| ds}}\right) = \Phi(d_1)$$

Similarly, we have $\mathbb{Q}^T(P(T,S)>K) = \mathbb{Q}^T\left(\frac{P(T,S)}{P(T,T)} > K\right)$ and thus:

$$\ln\left(\frac{P(T,S)}{P(T,T)}\right) \sim N\left(\ln\frac{P(0,S)}{P(0,T)} - \frac{1}{2}\int_0^T(\sigma_s^{\mathbb{Q}^T})^2 ds\,,\ \int_0^T(\sigma_s^{\mathbb{Q}^T})^2 ds\right)$$

$$\ln\left(\frac{P(T,S)}{P(T,T)}\right) \sim N\left(\ln\frac{P(0,S)}{P(0,T)} - \frac{1}{2}\int_0 (\sigma_s \quad)\, ds \,,\, \int_0 (\sigma_s \quad)\, ds\right)$$

$$\Rightarrow Q^T\left(\frac{P(T,S)}{P(T,T)} > K\right) = \Phi\left(\frac{\ln\frac{P(0,S)}{K P(0,T)} - \frac{1}{2}\int_0^T (\sigma_s^{Q^S})^2 ds}{\sqrt{\int_0^T \|(\sigma_s^{Q^S})^2\| ds}}\right) = \Phi(d_2)$$

where we used the fact that $\Phi(z > d) = \Phi(z < -d)$

Finally, we have: $\Pi = P(t,S)\,\Phi(d_1) - K\,P(t,T)\,\Phi(d_2)$

where $d_2 = d_1 - \sqrt{\int_0^T \|(\sigma_s^{Q^S})^2\| ds}$

3 Consider $dS_t = S_t\left[r\,dt + \sigma\, dW_t^{Q}\right]$ under $Q$ and $r \in \mathbb{R}$

1. $C_t = E_t^{Q}\left[e^{-r(T-t)}(S_T - K)^+\right]$

If we rearrange the expectation using the indicator function and noticing that $e^{-r(T-t)}$ is always measurable

$$= e^{-r(T-t)} E_t^{Q}\left[S_T\, \mathbb{1}_{\{S_T \geq K\}}\right] - K e^{-r(T-t)} E_t^{Q}\left[\mathbb{1}_{\{S_T \geq K\}}\right]$$

If we define $\frac{dQ^R}{dQ} = \frac{S_T\, B(t)}{S_t\, B(T)}$ with $B(u) = e^{-ru}$

$\Rightarrow E_t^{Q}\left[\frac{S_T B(t)}{S_t B(T)} \frac{S_t B(T)}{B(t)} \mathbb{1}_{\{S_T \geq K\}}\right]$ given that $S_t$ is measurable with respect to $\mathcal{F}_t$ and that $B(u)$ is deterministic

$\Rightarrow$ by Bayes lemma $E_t^{Q^R}\left[\frac{S_t B(T)}{B(t)} \mathbb{1}_{\{S_T \geq K\}}\right] = e^{r(T-t)} S_t\, E_t^{Q^R}\left[\mathbb{1}_{\{S_T \geq K\}}\right]$

In conclusion: $C_t = e^{-r(T-t)} e^{r(T-t)} S_t\, E_t^{Q^R}[\dots] - K e^{-r(T-t)} E_t^{Q}[\dots]$

$$= S_t\, E_t^{Q^R}\left[\mathbb{1}_{\{S_T \geq K\}}\right] - K e^{-r(T-t)} E_t^{Q}\left[\mathbb{1}_{\{S_T \geq K\}}\right]$$

2. By Girsanov we have: $dW_t^{Q^R} = dW_t^{Q} - \sigma\, dt$

This results follows immediately from the fact that $\frac{dQ^R}{dQ} = \mathcal{E}_t(-\sigma \cdot W^Q)$, i.e. it is a stochastic exponential

3. We note that $\frac{dS_t}{S_t} = r\,dt + \sigma\left[dW_t^{Q^R} + \sigma\, dt\right] = (r + \sigma^2)\,dt + \sigma\, dW_t^{Q^R}$

Hence, $S_t$ follows a GBM under $Q^R$ with drift $r + \sigma^2$ and volatility $\sigma$

If we apply Itô's lemma to $\ln S_t$ under $Q^R$:

$$d\ln S_t = \left[\frac{1}{S_t}(r + \sigma^2) S_t - \frac{1}{2}\frac{1}{S_t^2}\sigma^2 S_t^2\right] dt + \frac{1}{S_t}\sigma S_t\, dW_t^{Q^R}$$

$$= \left(r + \frac{1}{2}\sigma^2\right) dt + \sigma\, dW_t^{Q^R}$$

Integrating in $(t,T)$: $\ln S_T = \ln S_t + \left(r + \frac{1}{2}\sigma^2\right)(T-t) + \sigma\left(W_T^{Q^R} - W_t^{Q^R}\right)$

$\Rightarrow S_T = S_t \exp\left\{\left(r + \frac{1}{2}\sigma^2\right)(T-t) + \sigma\varepsilon\right\}$ with $\varepsilon = W_T^{Q^R} - W_t^{Q^R}$

$$\Rightarrow \quad S_T = S_t \exp\left\{ \left(r + \frac{1}{2}\sigma^2\right)(T-t) + \sigma \varepsilon \right\} \quad \text{with } \varepsilon = W_T^{\mathbb{Q}^R} - W_t^{\mathbb{Q}^R}$$

In conclusion, we have:

$$\mathbb{E}_t^{\mathbb{Q}^R}\left[ \mathbb{1}_{\{S_T \geq K\}} \right] = \mathbb{Q}^R\left[ S_T \geq K \right] = \mathbb{Q}^R\left[ e^{(r+\sigma^2/2)(T-t) + \sigma\varepsilon} \geq \frac{K}{S_t} \right]$$

$$= \mathbb{Q}^R\left[ \left(r + \frac{1}{2}\sigma^2\right)(T-t) + \sigma\varepsilon \geq \ln\left(\frac{K}{S_t}\right) \right]$$

since $\varepsilon \sim N(0, T-t) \Rightarrow \frac{\varepsilon}{\sqrt{T-t}} = Z \sim N(0,1)$

$$= \mathbb{Q}^R\left[ Z \geq \frac{1}{\sigma\sqrt{T-t}} \left( \ln\left(\frac{K}{S_t}\right) - \left(r + \frac{1}{2}\sigma^2\right)(T-t) \right) \right]$$

$$= \mathbb{Q}^R\left[ Z \leq \frac{1}{\sigma\sqrt{T-t}} \left( \ln\left(\frac{S_t}{K}\right) + \left(r + \frac{1}{2}\sigma^2\right)(T-t) \right) \right]$$

$$= \Phi(d_1)$$