Assignment 10

1. Consider the SDE $dr_t = \kappa(\vartheta - r_t)\,dt + \sigma_r\sqrt{r_t}\,dW_t$ where W_t is a $\mathbb{Q}$-Brownian motion

Furthermore, consider $V_t = \mathbb{E}_t^{\mathbb{Q}}\left[e^{-\int_t^T r_s\,ds}\right]$ the value at t of a zero-coupon bond with maturity T

If we apply Ito's lemma to $e^{-\int_0^t r_s\,ds}\,V_t$ and we impose the drift to be zero (i.e. the discounted value is a $\mathbb{Q}$-martingale), we have:

$$-rV + \frac{\partial V}{\partial t} + \kappa(\vartheta - r)\frac{\partial V}{\partial r} + \frac{1}{2}\sigma_r^2 r\frac{\partial^2 V}{\partial r^2} = 0$$

with boundary condition $V(T,r) = 1$

To find a solution to the PDE, assume $V = A(T-t)e^{-B(T-t)r_t} = A(\tau)e^{-B(\tau)r_t}$ with $\tau = T-t$ and let's differentiate:

- $\frac{\partial V}{\partial t} = -\frac{\partial V}{\partial \tau} = -A'(\tau)e^{-B(\tau)r} + A(\tau)B'(\tau)\,r\,e^{-B(\tau)} = -\frac{A'}{A}V + B' r V$
- $\frac{\partial V}{\partial r} = -A(\tau)B(\tau)e^{-B(\tau)r} = -BV$
- $\frac{\partial^2 V}{\partial r^2} = A(\tau)B^2(\tau)e^{-B(\tau)r} = B^2 V$

If we plug the terms into the PDE:

$$-rV - \frac{A'}{A}V + B'rV - \kappa(\vartheta - r)BV + \frac{1}{2}\sigma_r^2 r B^2 V = 0$$

$$\Rightarrow -Ar - A' + AB'r - \kappa\vartheta AB + \kappa ABr + \frac{1}{2}\sigma_r^2 AB^2 r = 0$$

and grouping together the terms with r and the terms without r:

$$-A + AB' + \kappa AB + \frac{1}{2}\sigma_r^2 AB^2 = 0 \quad\Rightarrow\quad \begin{cases} -1 + B' + \kappa B + \sigma_r^2/2\, B^2 = 0 \\ B(0) = 0 \end{cases}$$

$$-A' - \kappa\vartheta AB = 0 \quad\Rightarrow\quad \begin{cases} -A' - \kappa\vartheta AB = 0 \\ A(0) = 1 \end{cases}$$

Let's solve as first the first-order non linear ODE: $B' + \kappa B + \frac{1}{2}\sigma_r^2 B^2 = 1$

$$\Rightarrow \frac{dB}{d\tau} = 1 - \kappa B(\tau) - \frac{1}{2}\sigma_r^2 B^2(\tau) \quad\Rightarrow\quad \frac{dB/d\tau}{1 - \kappa B(\tau) - \sigma_r^2/2\, B^2(\tau)} = 1$$

and thus $\int \frac{1}{1-\kappa B(\tau) - \sigma_r^2/2\, B^2(\tau)}\frac{dB}{\cancel{d\tau}}\cancel{d\tau} = \int \frac{1}{1-\kappa B - \sigma_r^2/2\, B^2}\,dB = \int d\tau = \tau + c_1$

Hence, we have to solve the integral:

$$\int \frac{1}{1-\kappa B - \sigma_r^2/2\, B^2}\,dB = -\frac{2}{\sigma_r^2}\int \frac{1}{B^2 + \frac{2\kappa}{\sigma_r^2}B - \frac{2}{\sigma_r^2}}\,dB = -\frac{2}{\sigma_r^2}\int \frac{1}{\left(B + \frac{\kappa}{\sigma_r^2}\right)^2 - \frac{2}{\sigma_r^2} - \frac{\kappa^2}{\sigma_r^4}}\,dB$$

$$= -\frac{2}{\sigma_r^2}\int \frac{dB}{\left(B + \frac{\kappa}{\sigma_r^2} + \psi\right)\left(B + \frac{\kappa}{\sigma_r^2} - \psi\right)} \quad \text{with } \psi = \sqrt{\frac{\kappa^2}{\sigma_r^4} + \frac{2}{\sigma_r^2}}$$

$$= -\frac{2}{\sigma_r^2}\left[\frac{1}{2\psi}\int \frac{dB}{B + \frac{\kappa}{\sigma_r^2} - \psi} - \frac{1}{2\psi}\int \frac{dB}{B + \frac{\kappa}{\sigma_r^2} + \psi}\right]$$

$$= -\frac{2}{\sigma_r^2}\left[\frac{1}{2\psi}\int \frac{dB}{B+\frac{\kappa}{\sigma_r^2}-\psi} - \frac{1}{2\psi}\int \frac{dB}{B+\frac{\kappa}{\sigma_r^2}+\psi}\right]$$

$$= -\frac{1}{\psi\sigma_r^2}\left[\ln\left(B+\frac{\kappa}{\sigma_r^2}-\psi\right) - \ln\left(B+\frac{\kappa}{\sigma_r^2}+\psi\right)\right] + c_2$$

$$\Rightarrow -\psi\sigma_r^2\tau = \ln\left(B+\frac{\kappa}{\sigma_r^2}-\psi\right) - \ln\left(B+\frac{\kappa}{\sigma_r^2}+\psi\right) + c \quad \text{with } c \in \mathbb{R}$$

$$\Rightarrow -\psi\sigma_r^2\tau = \ln\left(\frac{B+\kappa/\sigma_r^2-\psi}{B+\kappa/\sigma_r^2+\psi}\right) + c$$

Using the initial condition we have: $B(0) = 0 \Rightarrow c = -\ln\left(\frac{\kappa/\sigma_r^2-\psi}{\kappa/\sigma_r^2+\psi}\right) = \ln\left(\frac{\kappa/\sigma_r^2+\psi}{\kappa/\sigma_r^2-\psi}\right)$

Thus we have $-\psi\sigma_r^2\tau = \ln\left(\frac{B+\kappa/\sigma_r^2-\psi}{B+\kappa/\sigma_r^2+\psi}\cdot\frac{\kappa/\sigma_r^2+\psi}{\kappa/\sigma_r^2-\psi}\right)$

Isolating $B(T-t)$ from the previous expression: $e^{-\psi\sigma_r^2\tau}\,\frac{\kappa/\sigma_r^2-\psi}{\kappa/\sigma_r^2+\psi} = \frac{B+\kappa/\sigma_r^2-\psi}{B+\kappa/\sigma_r^2+\psi}$

$$\Rightarrow B+\frac{\kappa}{\sigma_r^2}-\psi = \frac{\kappa/\sigma_r^2-\psi}{\kappa/\sigma_r^2+\psi}e^{-\psi\sigma_r^2\tau}\left[B+\frac{\kappa}{\sigma_r^2}+\psi\right] = \left(\frac{\kappa}{\sigma_r^2}-\psi\right)e^{-\psi\sigma_r^2\tau} + B\,\frac{\kappa/\sigma_r^2-\psi}{\kappa/\sigma_r^2+\psi}e^{-\psi\sigma_r^2\tau}$$

$$\Rightarrow B\left[1-\frac{\kappa/\sigma_r^2-\psi}{\kappa/\sigma_r^2+\psi}e^{-\psi\sigma_r^2\tau}\right] = \left[\frac{\kappa}{\sigma_r^2}-\psi\right]\left[e^{-\psi\sigma_r^2\tau}-1\right] = \frac{\kappa-\psi\sigma_r^2}{\sigma_r^2}\left[e^{-\psi\sigma_r^2\tau}-1\right]$$

Let's note that $\psi = \sqrt{\frac{\kappa}{\sigma_r^4}+\frac{2}{\sigma_r^2}} = \frac{1}{\sigma_r^2}\sqrt{\kappa+2\sigma_r^2} \Rightarrow \psi = \frac{\gamma}{\sigma_r^2}$ or $\gamma = \sigma_r^2\psi$

$$\Rightarrow B\left[1-\frac{\kappa-\gamma}{\kappa+\gamma}e^{-\gamma\tau}\right] = \frac{\kappa-\gamma}{\sigma_r^2}\left[e^{-\gamma\tau}-1\right]$$

$$\Rightarrow B\left[e^{\gamma\tau}-\frac{\kappa-\gamma}{\kappa+\gamma}\right] = \frac{\kappa-\gamma}{\sigma_r^2}\left[1-e^{\gamma\tau}\right]$$

$$\Rightarrow B(\tau) = \frac{(e^{\gamma\tau}-1)(\kappa-\gamma)}{\sigma_r^2\left(\frac{\kappa-\gamma}{\kappa+\gamma}-e^{\gamma\tau}\right)} = \frac{e^{\gamma\tau}-1}{\sigma^2\left(\frac{1}{\kappa+\gamma}+\frac{e^{\gamma\tau}}{\kappa-\gamma}\right)} = 2\,\frac{e^{\gamma\tau}-1}{(e^{\gamma\tau}-1)(\gamma+\kappa)+2\gamma}$$

Finally, moving on to the other ODE:

$$\frac{A'}{A} = -\kappa\vartheta B \Rightarrow \int\frac{dA}{A} = -\kappa\vartheta\int B(\tau)\,d\tau \Rightarrow A(\tau) = e^{-\kappa\vartheta B^*(\tau)+\delta} \quad \text{with } \delta\in\mathbb{R}$$

Using the initial condition: $1 = e^{\delta}$ and thus $\delta = 0 \Rightarrow A(\tau) = e^{-\kappa\vartheta B^*(\tau)}$

In conclusion, we have: $B^*(\tau) = \int B(\tau)\,d\tau = 2\int\frac{e^{\gamma\tau}-1}{(e^{\gamma\tau}-1)(\gamma+\kappa)+2\gamma}\,d\tau$

$$\Rightarrow A(\tau) = \left[\frac{2\gamma\, e^{(\gamma+\kappa)\tau/2}}{(\gamma+\kappa)(e^{\gamma\tau}-1)+2\gamma}\right]^{\frac{2\kappa\vartheta}{\sigma_r^2}}$$

② For simplicity, suppose $S = T+\tau$. We have that:

1. Suppose Π is the price at t of the call with maturity T written on the zero-coupon bond with maturity S. Under $\mathbb{Q}$, the price is:

$$\Pi = \mathbb{E}_t^{\mathbb{Q}}\left[e^{-\int_t^T r_s ds}\left(P(T,S)-K\right)^+\right]$$

maturity S. Under $\mathbb{Q}$, the price is:

$$\Pi = \mathbb{E}_t^{\mathbb{Q}}\left[e^{-\int_t^T r_s ds} \left(P(T,S) - K \right)^+ \right]$$

$$\overset{!}{=} \underbrace{\mathbb{E}_t^{\mathbb{Q}}\left[e^{-\int_t^T r_s ds} P(T,S) \mathbb{1}_{\{P(T,S) \geq K\}} \right]}_{I_1} - K \underbrace{\mathbb{E}_t^{\mathbb{Q}}\left[e^{-\int_t^T r_s ds} \mathbb{1}_{\{P(T,S) \geq K\}} \right]}_{I_2}$$

with $\mathbb{1}_{\{\cdot\}}$ the indicator function

We have that:

- $I_2 = \mathbb{E}_t^{\mathbb{Q}}\left[\frac{B(t)}{B(T)} \frac{P(0,T)}{P(0,T)} \mathbb{1}_{\{P(T,S) \geq K\}} \right]$

  recalling that $\frac{d\mathbb{Q}^T}{d\mathbb{Q}} = \frac{1}{P(0,T)B(T)}$ with $B(\cdot)$ the money market account

  by application of Bayes lemma

  $= \mathbb{E}_t^{\mathbb{Q}^T}\left[B(t)P(0,T) \mathbb{1}_{\{P(T,S) \geq K\}} \right]$

  and given that $B(t)P(0,T) = P(t,T)$ and is measurable

  $= P(t,T)\, \mathbb{E}_t^{\mathbb{Q}^T}\left[\mathbb{1}_{\{P(T,S) \geq K\}} \right]$

- $I_1 = \mathbb{E}_t^{\mathbb{Q}}\left[\frac{B(t)}{B(T)} \frac{P(0,T)}{P(0,T)} P(T,S) \mathbb{1}_{\{P(T,S) \geq K\}} \right]$

  proceeding as before

  $= \mathbb{E}_t^{\mathbb{Q}^T}\left[B(t)P(0,T)P(T,S) \mathbb{1}_{\{P(T,S) \geq K\}} \right]$

  recalling that $\frac{d\mathbb{Q}^S}{d\mathbb{Q}^T} = \frac{d\mathbb{Q}^S}{d\mathbb{Q}} \frac{d\mathbb{Q}}{d\mathbb{Q}^T} = \frac{P(0,T)B(T)}{P(0,S)B(S)}$ and that $P(T,S) = \frac{B(T)}{B(S)}$

  and that $P(0,S)$ is measurable

  $= \mathbb{E}_t^{\mathbb{Q}^T}\left[B(t)P(0,S) \frac{P(0,T)B(T)}{P(0,S)B(S)} \mathbb{1}_{\{P(T,S) \geq K\}} \right]$

  by Bayes lemma and noticing that $B(t)P(0,S) = P(t,S)$ is measurable

  $= P(t,S)\, \mathbb{E}_t^{\mathbb{Q}^S}\left[\mathbb{1}_{\{P(T,S) \geq K\}} \right]$

2. The relationship is simply obtained by Girsanov theorem:

- $dW_t^{\mathbb{Q}^T} = dW_t^{\mathbb{Q}} - \sigma_t^{\mathbb{Q}^T} dt$
- $dW_t^{\mathbb{Q}^S} = dW_t^{\mathbb{Q}} - \sigma_t^{\mathbb{Q}^S} dt$

Specifically, with Vasicek dynamics of the short rate, we know that there exists a closed form solution and thus:

$$dP(t,T) = P(t,T)\left[r_t dt + \sigma_t^{\mathbb{Q}^T} dW_t^{\mathbb{Q}} \right] \implies \sigma_t^{\mathbb{Q}^T} = -\sigma B(t,T)$$

and similarly for $\mathbb{Q}^S$: $\sigma_t^{\mathbb{Q}^S} = -\sigma B(t,S)$