Problem set 8

Due on November 10, 2020, at 11:15am

Exercise 1

Assume the risk-neutral dynamics for the stock price $S(t)$ is given by the risk-neutral process

$$\frac{dS_t}{S_t} = (r - q + \lambda\gamma)dt + \sigma dW_t - \gamma dN_t \tag{1}$$

where γ is the constant jump size and λ is the jump intensity.

- Derive the process for $log(S(t))$.
- Compute
$$E^Q\left[log\left(\frac{S(T)}{S(0)}\right)\right] \tag{2}$$
- Compute the quadratic variation of log(S) over $[0, T]$:
$$\int_0^T (d\ log(S(t)))^2 \tag{3}$$
- Show that, due to the jump process, the expression
$$2\left(rT - E^Q\left[log\left(\frac{S(T)}{S(0)}\right)\right]\right) \tag{4}$$
does not perfectly replicate the quadratic variation (3).

Exercise 2

Consider a payoff $g(S_T)$. Given the Breeden-Litzenberger formula for the pdf of the stock price S_T (conditional on the stock price S_t at time t)

$$p(S_T, T; S_t, t) = \left.\frac{\partial^2 \tilde{C}(S_t, K, T-t)}{\partial K^2}\right|_{K=S_T} = \left.\frac{\partial^2 \tilde{P}(S_t, K, T-t)}{\partial K^2}\right|_{K=S_T} \tag{5}$$

where $\tilde{C}$ and $\tilde{P}$ are undiscounted option prices, we can write

$$E[g(S_T)|S_t] = \int_0^{S^*} dK \frac{\partial^2 \tilde{P}(S_t, K, T-t)}{\partial K^2} g(K) + \int_{S^*}^{\infty} dK \frac{\partial^2 \tilde{C}(S_t, K, T-t)}{\partial K^2} g(K) \tag{6}$$

Show that, by integrating by parts and using put-call parity, the latter expression is equivalent to the formula obtained in class

$$E[g(S_T)] = (g(S^*) - g'(S^*)S^*) + g'(S^*)S_T + \int_0^{S^*} dK \tilde{P}(K) g''(K) + \int_{S^*}^{\infty} dK \tilde{C}(K) g''(K) \tag{7}$$

Exercise 3

the stock price $S(t)$ follows a pure diffusion process

$$\frac{dS_t}{S_t} = (r - q)dt + \sigma dW_t \tag{8}$$

You want to replicate a variance swap, but only a discrete set of options (calls and puts) with strikes $\{K_0, K_1, ...K_n\}$ is available. (You can still use forwards, bonds and the stock in your replication). How would you build a portfolio that approximately replicates the payoff of the variance swap using these options?

Imagine that you are short the variance swap and long this portfolio. For which values of the final stock price is the payoff of the variance swap higher than the payoff of this portfolio, and for which values of S_T is it lower?