**FIN 503: Advanced derivatives** **(Due: 24/11/20)**

# Assignment #10

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Ex 1

$$dR = k(\theta - r(t))dt + \sigma_r \sqrt{R}\, dW_t$$

Find the PDE for a Derivative V

Any Derivative that Depends on t, R has respect to the following PDE in order to be arbitrage free

$$\frac{1}{2}\frac{d^2V(t,R)}{dR^2}\sigma^2 + \frac{dV(t,R)}{dt} - RV = -\mu^Q(R,t)\frac{dV}{dR}$$

where r has dynamics $dR = \mu^Q(R,t)dt + \sigma dW$

So in our case with CIR Dynamics!

the PDE is

$$\frac{1}{2}\frac{d^2V(t,R)}{dR}\sigma_R^2 R + \frac{dV(t,R)}{dt} - RV = -\frac{dV(t,R)}{dR}k(\theta - R_t)$$

If V is a ZCB(T) with expiration T the boundary condition for this PDE is

If $V = P(t,T)$ $P(T,T) = 1$

Solve using $V = A(T-t)\, e^{-B(T-t)R_t}$

Sub in PDE: and omitting the dependence of A and B

$$\begin{cases} \frac{1}{2}B^2 V \sigma_r^2 R + \left(-A' e^{-BR_t} + B'_R A\, e^{-BR_t}\right) - RV = BVk(\theta - R_t) \\ V(0) = 1 \end{cases}$$

We used the following result in the above derivation:

$V = A(\tau)\, e^{-B(\tau)R_t}$ ; $\frac{dV(\tau)}{dt} = -\frac{dV}{d\tau}$ ; $P(T,T) = V(0)$

And we can rewrite

$$\frac{dV}{dt} = -A' e^{-BR_t} + B'RA\, e^{-BR_t} = V\left(B'R_t - \frac{A'}{A}\right)$$

where $A' = \frac{dA}{d\tau}$ $B' = \frac{dB}{d\tau}$ $\tau = T-t$

So we obtain the following PDE:

$$\begin{cases} V\left(B' R_t - \frac{A'}{A}\right) - B V K(\theta - R_t) + \frac{1}{2} B^2 V \sigma_R^2 R_t = R_t V \\ V(0) = 1 = A(0) e^{-B(0) R_0} \end{cases}$$

rearranging and using the fact that $V \geq 0$ can be semplified:

$$= \begin{cases} R_t\left(B' + BK + \frac{1}{2} B^2 \sigma_R^2 - 1\right) - \frac{A'}{A} - BK\theta = 0 \\ 1 = A(0) e^{-B(0) R_0} \end{cases}$$

from the boundary condition is straightforward: $A(0)=1$ and $B(0) = \varnothing$

By separating the variables multiplied by R and not we get the following system of ODE:

$$\begin{cases} B' + BK + 1/2\, B^2 \sigma_R^2 - 1 = 0 & B(0) = 0 \\ -\frac{A'}{A} - BK\theta = 0 & A(0) = 1 \end{cases}$$

$B' + BK + \frac{1}{2} B^2 \sigma_R^2 - 1 = 0 \quad B(0) = 0$ riccati equation

$$\frac{dB}{d\tau} + BK + \frac{1}{2} B^2 \sigma_R^2 - 1 = 0$$

set $\sigma_R^2 = \alpha$

$$\frac{dB}{d\tau} = -\frac{1}{2} B^2 \sigma_R^2 - BK + 1$$

set $C(\tau) = -\frac{1}{2} \alpha B \qquad C' = -\frac{1}{2} \alpha B'$

so we get in our ODE:

$$C' = C^2 - KC - \frac{1}{2}\alpha$$

set $C = -\frac{D'}{D} \rightarrow C' = -\left(\frac{D'' \cdot D - D' D'}{D^2}\right)$

$$C' = -\frac{D''}{D} + \frac{D'^2}{D^2} = -\frac{D''}{D} + C^2$$

So we get $\frac{D''}{D} = C^2 - C' = KC + \frac{1}{2}\partial$

$$\frac{D''}{D} = -K\frac{D'}{D} + \frac{1}{2}\partial \quad \text{OR}$$

$$D'' = -KD' + \frac{1}{2}\partial D$$ THIS IS the STANDARD Homogeneous equation

$$D'' + KD' - \frac{1}{2}\partial D = 0$$

Characteristic equation!

$$R^2 + KR - \frac{1}{2}\partial = 0 \qquad R_{1,2} = -K \pm \sqrt{K^2 + 2\partial}$$

$$R_{1,2} = -\frac{1}{2}K \pm \frac{1}{2}\sqrt{K^2 + 2\partial} \equiv \beta \pm \frac{1}{2}\psi = \beta + \lambda$$

$$\beta \equiv -\frac{1}{2}K \;; \quad \psi \equiv \sqrt{K^2 + 2\partial} \qquad \lambda = \frac{1}{2}\psi$$

General solution:

$$D(\tau) = C_1 e^{(\beta+\lambda)(\tau)} + C_2 e^{(\beta-\lambda)\tau}$$

$$D'(\tau) = C_1(\beta+\lambda) e^{(\beta+\lambda)\tau} + C_2(\beta-\lambda) e^{(\beta-\lambda)\tau}$$

Substituting BACK:

$$B(\tau) = -\frac{2C(\tau)}{\partial} = \frac{2D'(\tau)}{\partial D}$$

Terminal condition: $B(T,T) = 0 \Rightarrow D'(0) = 0$

so $C_1 = -C_2 \frac{\beta-\lambda}{\beta+\lambda}$

Thus

$$D(\tau) = \frac{C_2}{\beta+\lambda} e^{\beta\tau}\left[(\beta+\lambda)e^{-\lambda\tau} - (\beta-\lambda)e^{\lambda\tau}\right]$$

AND

$$D'(\tau) = (\beta-\lambda)C_2 e^{\beta\tau}(e^{-\lambda\tau} - e^{\lambda\tau})$$

And we get

$$B(\tau) = \frac{2\, D'(\tau)}{\partial\, D(\tau)}$$

$$= \frac{2(\beta^2 - \lambda^2)\left(e^{-\lambda\tau} - e^{\lambda\tau}\right)}{\partial\left((\beta+\lambda)e^{-\lambda\tau} - (\beta-\lambda)e^{\lambda\tau}\right)}$$

$$\beta^2 - \lambda^2 = \frac{1}{4}K^2 - \frac{1}{4}\left(K^2 + 2\partial\right) = -\frac{\partial}{2}$$

$$= \frac{2(\partial/2)\left(e^{\lambda\tau} - e^{-\lambda\tau}\right)}{\partial\left((\beta+\lambda)e^{-\lambda\tau} - (\beta-\lambda)e^{\lambda\tau}\right)}$$

$$= \frac{\left(e^{2\lambda\tau} - 1\right)}{(\beta+\lambda) - (\beta-\lambda)e^{2\lambda\tau}}$$

$$\beta = -\frac{1}{2}K \qquad \lambda = \frac{1}{2}\psi$$

$$= \frac{\left(e^{\psi_1\tau} - 1\right)}{\left(-\frac{1}{2}K - \psi\cdot\frac{1}{2}\right) + \psi + \left(\frac{1}{2}\psi + \frac{1}{2}K\right)e^{\psi_1\tau}}$$

$$= \frac{2\left(e^{\psi_1\tau} - 1\right)}{(\psi + K)\left(e^{\psi\tau} - 1\right) + 2\psi} = B(\tau)$$

where $\psi = \sqrt{K^2 + 2\sigma_R^2}$

for A! By substituting $B(\tau)$ in the ODE

$$\int_0^{\tau} \frac{dA(s)}{A(s)} = -K\theta \int_0^{\tau} B(s)\, ds$$

$$\int_0^{\tau} \frac{dA(s)}{A(s)} = -K\theta \int_0^{\tau} \frac{2}{\sigma} \frac{D'(s)}{D(s)} ds$$

$$= -K\theta \int_0^{\tau} \frac{2}{\sigma} \frac{dD(s)}{D(s)}$$

$$\ln(A(\tau)) = -K\theta \frac{2}{\sigma^2} \ln(D(\tau))$$

$$D(\tau) = \frac{C2}{\beta+\lambda} e^{\beta\tau}\left((\beta+\lambda)e^{-\lambda\tau} - (\beta-\lambda)e^{\lambda\tau}\right)$$

since $\ln(A(0)) = \ln(1) = 0$

$$\ln(D(0)) = 0 \rightarrow D(0) = 1$$

$$C2 = \frac{\beta+\lambda}{(\beta+\lambda)-(\beta-\lambda)}$$

$$D(\tau) = \frac{e^{\beta\tau}\left[(\beta+\lambda)e^{-\lambda\tau} - (\beta-\lambda)e^{\lambda\tau}\right]}{(\beta+\lambda)-(\beta-\lambda)}$$

$$= \frac{e^{\beta\tau}\left[(\beta+\lambda) - (\beta-\lambda)e^{2\lambda\tau}\right]}{2\lambda e^{\lambda\tau}}$$

$$= \frac{(\beta+\lambda) - (\beta-\lambda)e^{2\lambda\tau}}{2\lambda e^{(\lambda-\beta)\tau}}$$

As already demonstrated in the steps before

$$(\beta+\lambda) - (\beta-\lambda)e^{2\lambda\tau} = \frac{(\psi+K)(e^{\psi\tau}-1)+2\psi}{2}$$

and

$$2\lambda e^{(\lambda-\beta)\tau} = \psi e^{(\psi+K)\tau/2}$$

so

$$D(\tau) = \left((\psi+K)(e^{\psi\tau}-1)+2\psi\right) \cdot \frac{1}{2\psi \exp((\psi+K)\tau/2)}$$

SUB BACK

$$\ln(A(\tau)) = \ln\left( D(\tau)^{-\frac{2k\theta}{\sigma^2_r}} \right)$$

$$A(\tau) = D(\tau)^{-2k\theta/\sigma^2_r}$$

$$= \left( \frac{2\psi e^{(\psi+k)\tau/2}}{(\psi+k)(e^{\psi\tau}-1)+2\psi} \right)^{\frac{2k\theta}{\sigma^2_r}}$$

PROBLEM 2

Show how to obtain the analytical formula for the call option on a zero-coupon bond in the Vasicek model.

expiration option: $T$

expiration underlying bond: $T+\tau$

- Show that the expected discounted payoff of the option can be written as

$$P(t,T+\tau)\,\mathbb{E}_t^{Q^{T+\tau}}\left[I_{P(T,T+\tau)\geq K}\right]-K\,P(t,T)\,\mathbb{E}^{Q^T}\left[I_{P(T,T+\tau)\geq K}\right] \quad (*)$$

Solution:

We want to find $\pi(t)=\mathbb{E}^{Q}\left[e^{-\int_t^T r(s)ds}\,(P(T,T+\tau)-K)^+\,|\mathcal{F}_t\right]$ and show it is equal to $*$.

$$ZBC(t,T,T+\tau;K)=\mathbb{E}^{Q}\Big[\underbrace{e^{-\int_t^T r(s)ds}}_{\frac{B(t)}{B(T)}}\,(P(T,T+\tau)-K)\,\mathbb{1}_{P(T,T+\tau)>K}|\mathcal{F}_t\Big] \qquad \leftarrow B(t)=e^{\int_0^t r(s)ds}$$

$$=\mathbb{E}^{Q}\left[\frac{B(t)}{B(T)}P(T,T+\tau)\,\mathbb{1}_{P(T,T+\tau)>K}|\mathcal{F}_t\right]-K\cdot\mathbb{E}^{Q}\left[\frac{B(t)}{B(T)}\mathbb{1}_{P(T,T+\tau)>K}|\mathcal{F}_t\right] \ominus$$

Let us introduce the new measure $Q^T\sim Q$: $\quad \frac{dQ^T}{dQ}=\frac{e^{-\int_0^T r(s)ds}}{P(0,T)}=\frac{1}{B(T)P(0,T)}$

In the same way: $\frac{dQ^{T+\tau}}{dQ}=\frac{1}{B(T)P(0,T+\tau)}$

Therefore $Z_t^T=\frac{P(t,T)}{B(t)P(0,T)}$ and $Z_t^{T+\tau}=\frac{P(t,T+\tau)}{B(t)P(0,T+\tau)}$

$Z_t^T$ is a $Q$ martingale $\left(Z_t^T=\mathbb{E}_t^{Q}[Z_T^T]\right)$ satisfying $Z_0=1$

Note $\mathbb{E}^{Q}[\ldots|\mathcal{F}_t]=\mathbb{E}_t^{Q}[\ldots]$ for semplicity

$$\ominus\ \mathbb{E}_t^{Q^{T+\tau}}\left[P(T,T+\tau)\,\mathbb{1}_{P(T,T+\tau)>K}\,\frac{B(t)}{B(T)}\,\frac{Z_t^{T+\tau}}{Z_T^{T+\tau}}\right]-K\,\mathbb{E}_t^{Q^T}\left[\mathbb{1}_{P(T+\tau)>K}\,\frac{B(t)}{B(T)}\,\frac{Z_t^T}{Z_T^T}\right]=$$

$$=\mathbb{E}_t^{Q^{T+\tau}}\left[\cancel{P(T,T+\tau)}\,\mathbb{1}_{P(T,T+\tau)>K}\,\frac{P(t,T+\tau)\,\cancel{P(0,T+\tau)}}{\cancel{P(T,T+\tau)}\,\cancel{P(0,T+\tau)}}\right]-K\,\mathbb{E}_t^{Q^T}\left[\mathbb{1}_{P(T,T+\tau)>K}\,\frac{P(t,T)\,\cancel{P(0,T)}}{P(T,T)\,\cancel{P(0,T)}}\right]=$$

Zero coupon bond: $P(T,T)=1$

$$=\underbrace{P(t,T+\tau)}_{\mathcal{F}_t\text{-measurable}}\,\mathbb{E}_t^{Q^{T+\tau}}\left[\mathbb{1}_{P(T,T+\tau)>K}\right]-K\cdot\underbrace{P(t,T)}_{\mathcal{F}_t\text{-measurable}}\cdot\mathbb{E}_t^{Q^T}\left[\mathbb{1}_{P(T,T+\tau)>K}\right]$$

- What is the relationship between the BM increment $dW^{Q}$ in the $Q$-measure and the BM increment in the two probability measures appearing in $*$?

Solution

By the Girsanov theorem, $dW_t^{Q^T}=dW_t^{Q}-\sigma_p^T(t)\,dt$

What is $\sigma_P^T(t)$ and where does it come from?

Let us introduce the dynamic of $r$: $\quad dr = \kappa(\theta - r(t))dt + \sigma dW_t$

In the Vasicek model, a bond price $P$ satisfies the PDE: $\frac{1}{2} P_{rr}\sigma^2 + P_r \kappa(\theta - r) + P_t = rP$

As in the lecture, we guess a solution in the form $P(t,T;r) = A(T-t)\, e^{-B(T-t)r}$
where $A$ and $B$ solve

$$\begin{cases} \frac{1}{2} B^2\sigma^2 - \kappa\theta B - \frac{A'}{A} = 0 \\ \kappa B + B' - 1 = 0 \end{cases} \qquad A(0)=1 \quad B(0)=0$$

Once we have solution of $A$ and $B$, it is easy to get $\dfrac{dP(t,T)}{P(t)} = r_t\, dt + \sigma_P^T(t)\, dW_t^{\mathbb{Q}}$

where $\sigma_P^T = -\sigma B(t,T)$

We anticipated that $Z^T$ is a $\mathbb{Q}$ martingale. In particular:

$$Z_t^T = \mathbb{E}_t^{\mathbb{Q}}[Z_T^T] = \frac{e^{-\int_0^t r_s ds} A(T-t)\, e^{-r_t B(T-t)}}{P(0,T)}$$

$$dZ_t^T = Z_t^T \sigma_P^T(t)\, dW_t^{\mathbb{Q}} \qquad Z_t^T = \exp\left(\frac{1}{2}\int_0^t \sigma_P^T(s)^2 ds + \int_0^t \sigma_P^T dW_s\right).$$

In the same way, $\begin{cases} dW_t^{\mathbb{Q}^{T+\tau}} = dW_t^{\mathbb{Q}} - \sigma_P^{T+\tau}(t)dt \\ \sigma_P^{T+\tau} = -\sigma B(t, T+\tau) \end{cases}$

- What is the process followed by the underlying process $P(t, T+\tau)$ under $\mathbb{Q}^T$ and $\mathbb{Q}^{T+\tau}$?

<u>Solution</u>

We have already introduced that $\dfrac{dP(t,T+\tau)}{P(t)} = r_t\, dt + \sigma_P^{T+\tau}(t)\, dW_t^{\mathbb{Q}}$

Therefore, for $\mathbb{Q}^T$: $\dfrac{dP(t,T+\tau)}{P(t)} = r\,dt + \sigma_P^{T+\tau}(t)\left(dW_t^{\mathbb{Q}^T} + \sigma_P^T(t)\,dt\right)$

$$= \left(r + \sigma_P^{T+\tau}(t)\sigma_P^T(t)\right)dt + \sigma_P^{T+\tau}(t)\, dW_t^{\mathbb{Q}^T}$$

While for $\mathbb{Q}^{T+\tau}$: $\dfrac{dP(t,T+\tau)}{P(t)} = r\,dt + \sigma_P^{T+\tau}(t)\left(dW_t^{\mathbb{Q}^{T+\tau}} + \sigma_P^{T+\tau}(t)\,dt\right)$

$$= \left(r + \sigma_P^{T+\tau}(t)\sigma_P^{T+\tau}(t)\right)dt + \sigma_P^{T+\tau}(t)\, dW_t^{\mathbb{Q}^{T+\tau}}$$

Hence, for $U = T$ or $T+\tau$:

$$\frac{dP(t,T+\tau)}{P(t)} = \left(r + \sigma_P^{T+\tau}(t)\sigma_P^U(t)\right)dt + \sigma_P^{T+\tau} dW_t^{\mathbb{Q}^U}$$

Therefore $P(t,T,T+\tau)$ follows a Geometric BM:

$$P(t,T+\tau) = P(0,T+\tau)\, e^{\int_0^t r + \sigma_P^T\sigma_P^U - \frac{(\sigma_P^{T+\tau})^2}{2} ds + \int_0^t \sigma_P^{T+\tau} dW_s^{\mathbb{Q}^U}}$$

The exponent of this formula is gaussian with mean $= \int_0^t r + \sigma_P^T\sigma_P^U - \frac{(\sigma_P^{T+\tau})^2}{2} ds$
and variance $\int_0^t |\sigma_P^{T+\tau}|^2 ds$

- Now derive the formula for $\mathbb{E}_t^{\mathbb{Q}^{T+\tau}}[I_{P(T,T+\tau)\geq K}]$ and $\mathbb{E}_t^{\mathbb{Q}^T}[I_{P(T,T+\tau)>K}]$ and write the final formula.

<u>Solution</u>

$$\mathbb{E}_t^{\mathbb{Q}^{T+\tau}}[I_{P(T,T+\tau)\geq K}] = \mathbb{Q}^{T+\tau}(\underbrace{P(T,T+\tau)}>K) =$$

Using the solution of the previous point, we can rewrite it as

$$P(T,T+\tau) = P(t,T+\tau)\, e^{\mu_{T+\tau}+\nu z} \quad \text{where } z\sim N(0,1)$$

$$\mu_v = \int_t^T r + \sigma_p^T \sigma_p^v - \frac{(\sigma_p^{T+\tau})^2}{2} ds$$

$$\nu^2 = \int_t^T |\sigma_p^{T+\tau}|^2 ds$$

$$= \mathbb{Q}^{T+\tau}(P(t,T+\tau)\, e^{\mu_{T+\tau}+\nu z} > K) =$$

$$= \mathbb{Q}^{T+\tau}\left(z > \frac{\log(K/P(t,T+\tau)) - \mu_{T+\tau}}{\nu}\right) = \mathbb{Q}^{T+\tau}\left(z > -\underbrace{\frac{\log(P(t,T+\tau)/K) + \mu_{T+\tau}}{\nu}}_{d_{T+\tau}}\right)$$

$$= 1 - \Phi(-d_{T+\tau}) = \Phi(d_{T+\tau})$$

In the same way $\mathbb{E}_t^{\mathbb{Q}^T}[I_{P(T,T+\tau)\geq K}] = \Phi(d_T)$

$$d_u = \frac{\log\left(\frac{P(t,T+\tau)}{K}\right) + \int_t^T r + \sigma_p^T \sigma_p^u - \frac{(\sigma_p^{T+\tau})^2}{2} ds}{\left(\int_t^T |\sigma_p^{T+\tau}|^2 ds\right)^{1/2}} \qquad u = T \text{ or } T+\tau$$

where $\sigma_p^u = -\sigma B(t,u)$ but also $\sigma_p^u = -\sigma \frac{1}{K}(1-e^{-K(u-t)})$

$$\Rightarrow ZBC(t,T,T+\tau;K) = P(t,T+\tau)\,\phi(d_{T+\tau}) - K\,P(t,T)\,\phi(d_T)$$

PROBLEM 3

Call option in the BS model. Stock price under $\mathbb{Q}$: $\frac{dS}{S} = r dt + \sigma dW^{\mathbb{Q}}$ $\quad r \in \mathbb{R}$

- Show that the value of the option can be written as

$$C_t = S_t \, \mathbb{E}^{\mathbb{R}}[I_{S_T > K} \mid \mathcal{F}_t] - e^{-r(T-t)} K \, \mathbb{E}^{\mathbb{Q}}[I_{S_T > K} \mid \mathcal{F}_t]$$

and specify the RN derivative associated with the new measure $\mathbb{R}$:

$$C_t = e^{-r(T-t)} \, \mathbb{E}^{\mathbb{Q}}[(S_T - K)^+ \mid \mathcal{F}_t] =$$
$$= e^{-r(T-t)} \, \mathbb{E}^{\mathbb{Q}}[(S_T - K) I_{S_T > K} \mid \mathcal{F}_t]$$
$$= \underbrace{e^{-r(T-t)} \, \mathbb{E}^{\mathbb{Q}}[S_T I_{S_T > K} \mid \mathcal{F}_t]} - e^{-r(T-t)} \cdot K \cdot \mathbb{E}^{\mathbb{Q}}[I_{S_T > K} \mid \mathcal{F}_t]$$

$$S_t \, \mathbb{E}^{\mathbb{Q}}\left[\frac{B_t}{S_t} \frac{S_T}{B_T} I_{S_T > K} \mid \mathcal{F}_t\right] \underset{Z_s = S_s / B_s}{=} S_t \, \mathbb{E}^{\mathbb{R}}\left[\frac{Z_T}{Z_t} I_{S_T > K} \mid \mathcal{F}_t\right] = S_t \, \mathbb{E}^{\mathbb{R}}[I_{S_T > K} \mid \mathcal{F}_t]$$

$$= S_t \, \mathbb{E}^{\mathbb{R}}[I_{S_T > K} \mid \mathcal{F}_t] - e^{-r(T-t)} K \, \mathbb{E}^{\mathbb{Q}}[I_{S_T > K} \mid \mathcal{F}_t]$$

where $Z_T = \frac{d\mathbb{R}}{d\mathbb{Q}}$ is the RN derivative.

- What is the relationship between $dW^{\mathbb{Q}}$ and $dW^{\mathbb{R}}$?

Remind that, for the Girsanov theorem, $Z_t$ is a strictly positive martingale.

$$dW^{\mathbb{R}} = dW^{\mathbb{Q}} - \sigma dt$$

- What is the process followed by $S_t$ under $\mathbb{R}$?

$$\frac{dS}{S} = r dt + \sigma dW^{\mathbb{Q}} = r dt + \sigma (dW^{\mathbb{R}} + \sigma dt) = (r + \sigma^2) dt + \sigma dW^{\mathbb{R}}$$

Show that $\mathbb{E}^{\mathbb{R}}[I_{S_T > K} \mid \mathcal{F}_t]$ is the famous $N(d_1)$:

Solution to $dS$: $\quad S_t = S_0 \, e^{(r + \frac{\sigma^2}{2}) t + \sigma \sqrt{t} \, z} \qquad z \sim N(0,1)$

$$\mathbb{E}^{\mathbb{R}}[I_{S_T > K} \mid \mathcal{F}_t] = \mathbb{R}[S_T > K \mid \mathcal{F}_t] = \mathbb{R}[S_t \, e^{(r + \frac{\sigma^2}{2})(T-t) + \sigma \sqrt{T-t} \, z} > K \mid \mathcal{F}_t] =$$
$$= \mathbb{R}\left[(r + \sigma^2/2)(T-t) + \sigma \sqrt{T-t} \, z > \log\left(\frac{K}{S_t}\right) \mid \mathcal{F}_t\right] =$$
$$= \mathbb{R}\left[z > \frac{\log(K/S_t) - (r + \frac{\sigma^2}{2})(T-t)}{\sigma \sqrt{T-t}} \mid \mathcal{F}_t\right]$$
$$= \mathbb{R}\left[z > -\left[\underbrace{\frac{\log(S_t/K) + (r + \frac{\sigma^2}{2})(T-t)}{\sigma \sqrt{T-t}}}_{d_1}\right] \mid \mathcal{F}_t\right]$$
$$= 1 - \Phi(-d_1) = \Phi(d_1)$$

$W_t - W_s \sim N(0, t-s)$
$t > s$

where $\Phi$ is the cdf of $N(0,1)$

$$d_1 = \frac{\log(S_t/K) + (r + \frac{\sigma^2}{2})(T-t)}{\sigma \sqrt{T-t}}$$