ADVANCED DERIVATIVES - Assignment 1

Outperformance option price: $O(t, S_1(t), S_2(t)) = ?$

$$\text{Payoff}: X(T, S_1(T), S_2(T)) = \max\left\{0, \frac{S_1(T)}{S_1(0)} - \frac{S_2(T)}{S_2(0)}\right\}$$

$S_1(0), S_2(0)$ constant

where
$$dS_1 = \mu_1 S_1 dt + \sigma_1 S_1 dW_1$$
$$dS_2 = \mu_2 S_2 dt + \sigma_2 S_2 dW_2$$
$$E(dW_1 dW_2) = \rho dt$$

Solution:

risk-free asset

Replicating portfolio: $P = \Delta_1 S_1 + \Delta_2 S_2 + B$

We set $P = X$ $\quad dP = dX$. Using the Itô lemma:

option X

see last page

$$dX_t \overset{*}{=} \frac{\partial X}{\partial t} dt + \frac{\partial X}{\partial S_1} dS_1 + \frac{\partial X}{\partial S_2} dS_2 + \left(\frac{1}{2}\frac{\partial^2 X}{\partial S_1^2}\sigma_1^2 S_1^2 + \frac{1}{2}\frac{\partial^2 X}{\partial S_2^2}\sigma_2^2 S_2^2 + \frac{\partial^2 X}{\partial S_1 \partial S_2}\rho\sigma_1\sigma_2 S_1 S_2\right) dt$$

$$dP = \Delta_1 dS_1 + \Delta_2 dS_2$$

In order to preserve $dX = dP$, it must be that
$$\begin{cases} \Delta_1 = \dfrac{\partial X}{\partial S_1} \\ \Delta_2 = \dfrac{\partial X}{\partial S_2} \end{cases} \quad *$$

*

[To obtain this result, we have applied the Euler theorem:

Euler theorem: Suppose $f: \mathbb{R}^n \setminus \{0\} \to \mathbb{R}$ is continuously differentiable. Then,
f is positive homogeneous of degree k $\iff$ $x \cdot \nabla f(x) = k f(x)$

Therefore, considering that X is a homogeneous function of degree 1, we must have

$$\frac{\partial X}{\partial S_1} S_1 + \frac{\partial X}{\partial S_2} S_2 = 1 \cdot X$$

$$\Rightarrow dX = \frac{\partial X}{\partial S_1} dS_1 + \frac{\partial X}{\partial S_2} dS_2$$
$$\parallel$$
$$dP = \Delta_1 dS_1 + \Delta_2 dS_2 \qquad \square \ ]$$

This also proves that $B = 0$.

In order to cancel out the drift part (dt), we also impose:

$$\frac{\partial X}{\partial t} + \frac{1}{2}\frac{\partial^2 X}{\partial S_1^2}\sigma_1^2 S_1^2 + \frac{1}{2}\frac{\partial^2 X}{\partial S_2^2}\sigma_2^2 S_2^2 + \frac{\partial^2 X}{\partial S_1 \partial S_2}\rho\sigma_1\sigma_2 S_1 S_2 = 0 \quad \leftarrow X \text{ which solves the PDE}$$

Define $y = \dfrac{S_1}{S_2}$ and f such that $X(S_1, S_2, t) = S_2 f(y, t)$ (see slide 27)

$$\Rightarrow f(y, t) = X\left(\frac{S_1}{S_2}, 1, t\right)$$

Deriving these formulas: $\frac{\partial X}{\partial t} = S_2 \frac{\partial f}{\partial t}$ $\quad \frac{\partial X}{\partial S_1} = \frac{\partial f}{\partial y}$ (because $\frac{\partial y}{\partial S_1} = \frac{1}{S_2}$)

$= S_2 \frac{\partial f}{\partial S_1}$ but $\frac{\partial y}{\partial S_1} = \frac{1}{S_2}$

$\Rightarrow \frac{\partial f}{\partial S_1} = \frac{\partial f}{\partial S_1} \frac{\partial y}{\partial y} = \frac{\partial f}{\partial y} \frac{1}{S_2} \Rightarrow S_2 \frac{\partial f}{\partial S_1} = \frac{\partial f}{\partial y}$

$\frac{\partial X}{\partial S_2} = f - y \frac{\partial f}{\partial y}$ $\quad \frac{\partial^2 X}{\partial S_1^2} = \frac{1}{S_2} \frac{\partial^2 f}{\partial y^2}$ $\quad \frac{\partial X^2}{\partial S_1 \partial S_2} = -\frac{S_1}{S_2^2} \frac{\partial^2 f}{\partial y^2}$ $\quad \frac{dX^2}{dS_2^2} = \frac{S_1^2}{S_2^3} \frac{\partial^2 f}{\partial y^2}$

Substituting: $\frac{\partial X}{\partial t} + \frac{1}{2} \frac{\partial^2 X}{\partial S_1^2} \sigma_1^2 S_1^2 + \frac{1}{2} \frac{\partial^2 X}{\partial S_2^2} \sigma_2^2 S_2^2 + \frac{\partial^2 X}{\partial S_1 \partial S_2} \rho \sigma_1 \sigma_2 S_1 S_2 = 0$ and setting $\frac{\partial f}{\partial y} = f'$:

$$S_2 \frac{\partial f}{\partial t} + \frac{1}{2} \frac{1}{S_2} f'' \sigma_1^2 S_1^2 + \frac{1}{2} \frac{S_1^2}{S_2^{\cancel{3}}} f'' \sigma_2^2 \cancel{S_2^2} - \frac{S_1}{S_2^{\cancel{2}}} f'' \rho \sigma_1 \sigma_2 S_1 \cancel{S_2} = 0$$

$$S_2 \frac{\partial f}{\partial t} + \frac{1}{2} f'' \sigma_1^2 y S_1 + \frac{1}{2} y S_1 f'' \sigma_2^2 - y f'' \rho \sigma_1 \sigma_2 S_1 = 0$$

divide by S_2

$$\frac{\partial f}{\partial t} + \frac{1}{2} f'' \sigma_1^2 y^2 + \frac{1}{2} y^2 f'' \sigma_2^2 - y^2 f'' \rho \sigma_1 \sigma_2 = 0$$

$$\frac{\partial f}{\partial t} = \left[\frac{1}{2} (\sigma_1^2 + \sigma_2^2) - \rho \sigma_1 \sigma_2 \right] y^2 f''$$

with $\sigma^2 = \sigma_1^2 + \sigma_2^2 - 2 \sigma_1 \sigma_2$ we obtain

$$\frac{\partial f}{\partial t} = \frac{1}{2} \sigma^2 y^2 f''$$

We remember that $X(T, S_1(T), S_2(T)) = \left(\frac{S_1(T)}{S_1(0)} - \frac{S_2(T)}{S_2(0)} \right)^+$ and that $f(y,t) = X(t, \frac{S_1}{S_2}, 1)$

$\Rightarrow$ for f: $\quad f(Y,T) = \left(\frac{1}{S_1(0)} \frac{S_1}{S_2} - \frac{1}{S_2(0)} \right)^+ = \left(\frac{Y}{S_1(0)} - \frac{1}{S_2(0)} \right)^+$

$Y = y(T)$

Hence we have that $f(y,t) = \frac{y}{S_1(0)} N(d_1) - \frac{1}{S_2(0)} N(d_2)$

and $X(t, S_1, S_2) = \frac{S_1}{S_1(0)} N(d_1) - \frac{S_2}{S_2(0)} N(d_2)$

with

$$d_1 = \left(\log \left(\frac{\frac{S_1}{S_1(0)}}{\frac{S_2}{S_2(0)}} \right) + \frac{1}{2} \sigma^2 (T-t) \right) \Big/ \left(\sigma \sqrt{T-t} \right) \qquad d_2 = d_1 - \sigma \sqrt{T-t}$$

$$\sigma^2 = \sigma_1^2 + \sigma_2^2 - 2 \rho \sigma_1 \sigma_2$$

(*) This formula can be found on slide 26 (in this case the BM is W).
However it can be easily derived:

$$dX_t = \frac{\partial X_t}{\partial t} dt + \frac{\partial X_t}{\partial S_1} dS_1 + \frac{\partial X_t}{\partial S_2} dS_2 + \cancel{\frac{\partial^2 X_t}{\partial t^2} \langle dt \rangle} + \frac{\partial^2 X_t}{\partial S_1^2} \langle S_1 \rangle_t +$$

$$+ \frac{\partial^2 X_t}{\partial S_2^2} \langle S_2 \rangle_t + \frac{1}{2} \frac{\partial^2 X_t}{\partial S_1 \partial S_2} \langle S_1, S_2 \rangle_t$$

$$\langle S_1 \rangle_t = dS_1 dS_1 = (\mu_1 S_1 dt + \sigma_1 S_1 dW_1)(\mu_1 S_1 dt + \sigma_1 S_1 dW_1) \underset{\langle dW_1 \rangle_t = dt}{=} \sigma_1^2 S_1^2 dt$$

$$\langle S_2 \rangle_t = \sigma_2^2 S_2^2 dt$$

$$\langle S_1, S_2 \rangle_t = (\mu_1 S_1 dt + \sigma_1 S_1 dW_1)(\mu_2 S_2 dt + \sigma_2 S_2 dW_2) \overset{dW_1 dW_2 = dt}{=} \sigma_1 \sigma_2 S_1 S_2 dt$$

$$\Rightarrow dX_t = \frac{\partial X_t}{\partial t} dt + \frac{\partial X_t}{\partial S_1} dS_1 + \frac{\partial X_t}{\partial S_2} dS_2 + \frac{\partial^2 X_t}{\partial S_1^2} \sigma_1^2 S_1^2 dt + \frac{\partial^2 X_t}{\partial S_2^2} \sigma_2^2 S_2^2 dt +$$

$$+ \frac{1}{2} \frac{\partial^2 X_t}{\partial S_1 \partial S_2} \sigma_1 \sigma_2 S_1 S_2 dt$$

which is equal to what we find in the slides $\square$