Problem set 3

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Exercise 1

$$\frac{dS_t}{S_t} = (r - q + \lambda^Q \gamma)dt + \sigma dW_t^Q - \gamma dN_t \tag{1}$$

$$S_T = S_t e^{(r-q+\lambda^Q\gamma-\frac{\sigma^2}{2})T+\sigma(W_T^Q - W_t^Q)} * (1-\gamma)^{(N_T - N_t)} \tag{2}$$

The price of a security paying S_T^n at T is

$$\begin{aligned} P_0 &= exp(-rT)E^Q[S_T^n] \\ &= \sum_{j=0}^{\infty} P[N_T = j] * E^Q[e^{-rT}[S_0 e^{(r-q+\lambda^Q\gamma-\frac{\sigma^2}{2})T+\sigma W_T^Q} * (1-\gamma)^j]^n \\ &= \sum_{j=0}^{\infty} P[N_T = j] * e^{-rT} S_0^n e^{(r-q+\lambda^Q\gamma-\frac{\sigma^2}{2})Tn)} * (1-\gamma)^{jn} * E^Q[e^{n\sigma W_T^Q}] \end{aligned} \tag{3}$$

Using the moment generating function for a normal random variable $x \sim N(\mu, \sigma^2)$:

$$E[e^{tx}] = e^{\mu t + \frac{1}{2}\sigma^2 t^2} \tag{4}$$

We have that

$$E^Q[e^{n\sigma W_T^Q}] = e^{\frac{1}{2}n^2\sigma^2 T} \tag{5}$$

Hence,

$$
\begin{aligned}
P_0 &= \sum_{j=0}^{\infty} P[N_T = j] * e^{-rT} S_0^n e^{(r-q+\lambda^Q\gamma-\frac{\sigma^2}{2})Tn)} * (1-\gamma)^{jn} * e^{\frac{1}{2}n^2\sigma^2 T} \\
&= S_0^n e^{-rT+(r-q+\lambda^Q\gamma-\frac{\sigma^2}{2})Tn-\lambda^Q T+\frac{n^2\sigma^2 T}{2}} * \sum_{j=0}^{\infty} \frac{(\lambda^Q T * (1-\gamma)^n)^j}{j!} \\
&= S_0^n e^{-rT+(r-q+\lambda^Q\gamma-\frac{\sigma^2}{2})Tn-\lambda^Q T+\frac{n^2\sigma^2 T}{2}+\lambda^Q T*(1-\gamma)^n}
\end{aligned} \tag{6}
$$

Recall that by definition of the exponential,

$$
\sum_{j=0}^{\infty} \frac{x^j}{j!} = e^x \tag{7}
$$

Exercise 2

- The probability that the time to next jump will be longer than 2 years is the probability that no jumps occur until two years. That is the 2 years survival probability. Denote by $P^S(t)$ the survival probability from now up to time t. We have:

$$
P[N_2 - N_0 = 0] = P^S(2) = exp(-\lambda * 2) = 0.67 \tag{8}
$$

- The probability that the time to next jump is shorter than 3 years is the probability that one (or more) jumps happen up to time 3.

$$
\begin{aligned}
P[N_3 = 1] + P[N_3 = 2] + P[N_3 = 3] + ... &= 1 - P[N_3 = 0] \\
&= 1 - P^S(3) = 1 - exp(-0.2 * 3) = 0.45
\end{aligned} \tag{9}
$$

- The probability that the time to next jump is bigger than 2 but smaller than 3 years is the intersection between the probability of survival until 2 years from now and the probability of having at least one jump before the end of the following year. Because of the Independence property, we have:

$$
\begin{aligned}
&P[N_2 - N_0 = 0, N_3 - N_2 = 1] + P[N_2 - N_0 = 0, N_3 - N_2 = 2] + ... \\
&= P[N_2 - N_0 = 0] * (P[N_3 - N_2 = 1] + P[N_3 - N_2 = 2] + ...) \\
&= P^S(2) * (1 - P^S(1)) = 0.12
\end{aligned} \tag{10}
$$

Exercise 3

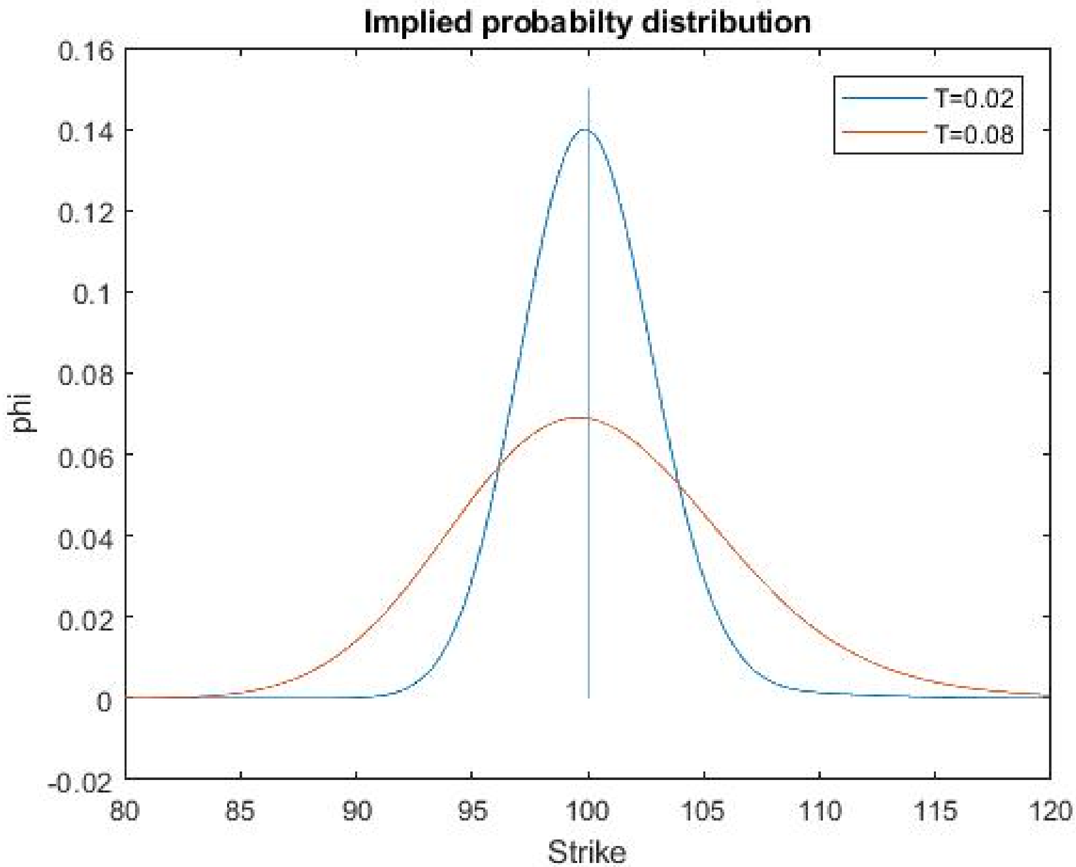

Figure 1: Implied probability distribution

Bonus part

We can do the bonus part either by simulation or analytically. Since it is possible to have a closed form solution, it is better to do it analytically. You were awarded some bonus points if you did this part by simulation, and full points if you did it analytically.

We can express the probability that S_T is an interval dS around K as

$$P(S_T \in [K, K+dS]) = \phi(K)dS = \Sigma_{j=1}^{\infty} P(n_{jumps} = j) P(S_T \in [K, K+dS] | n_{jumps} = j) \quad (11)$$

where $\phi(x)$ is the probability density function for S.
From the formula

$$\frac{S_T}{S_0} = e^{(r-q+\lambda^Q \gamma - \frac{1}{2}\sigma^2)T + \sigma W_T} (1-\gamma)^{N_T} \quad (12)$$

we see that

$$P(S_T \in [K, K+dS] | n_{jumps} = j) = P(W_T \in [d_2^j, d_2^j + dW]) \quad (13)$$

where

$$d_2^j = \frac{log\left(\frac{S_T}{S_0(1-\gamma)^j}\right) - (r - \sigma^2/2)T}{\sigma\sqrt{T}} \tag{14}$$

and dW should be the increment in the Wiener process that results in an increment dS (around dK) in the stock price. So

$$\phi(K) = \Sigma_{j=1}^{\infty} \frac{P(n_{jumps} = j)\phi^{norm}(d_2^j)}{\left.\frac{\partial S}{\partial W}\right|_{S=K}} \tag{15}$$

where $\phi^{norm}(x)$ is the standard normal probability density. Since $\frac{\partial S}{\partial W} = S\sigma\sqrt{T}$ we conclude that

$$\phi(K) = \frac{\Sigma_{j=1}^{\infty} e^{-\lambda T}\frac{(\lambda T)^j}{j!}\phi^{norm}(d_2^j)}{K\sigma\sqrt{T}} \tag{16}$$

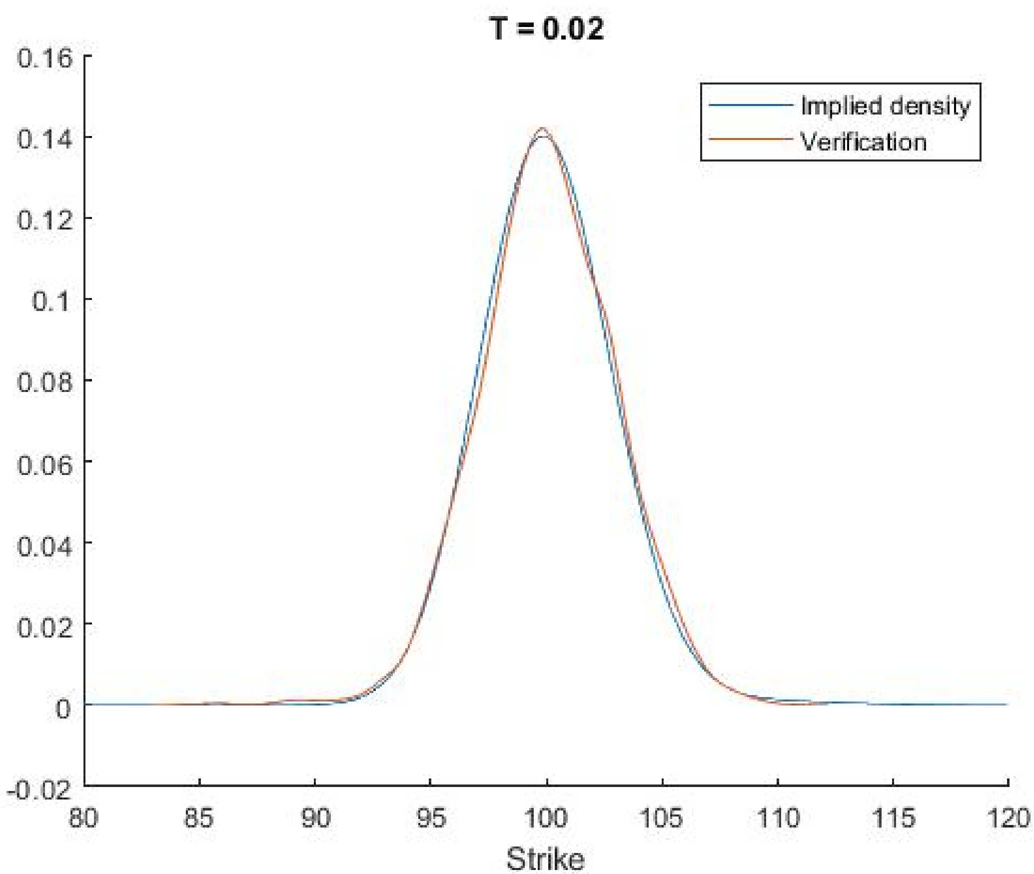

Figure 2: Verification

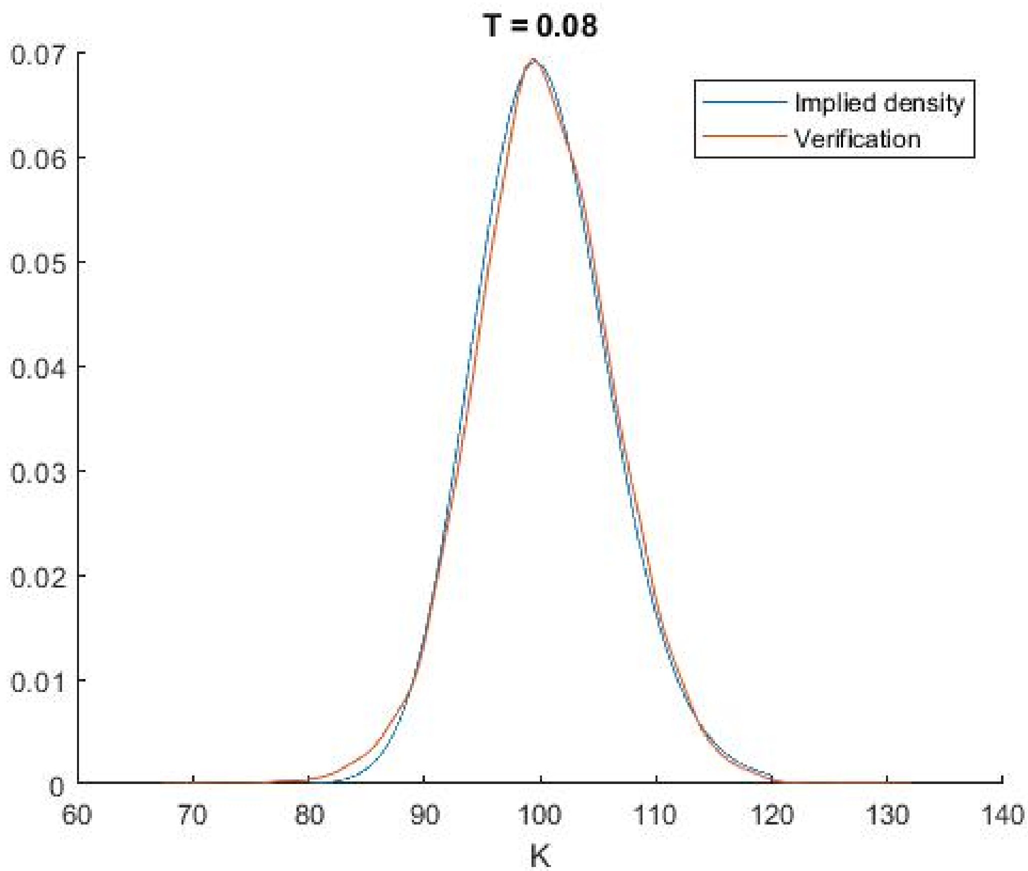

Figure 3: Verification