Problem set 9

Due on November 17, 2020, at 11:15am

Exercise 1

Consider a 3-month put option on a 5-year zero-coupon bond, with strike $K = 0.805$. The short-term rate follows a Vasicek process

$$dr = k(\theta - r(t))dt + \sigma_r dW_t \tag{1}$$

Use $k = 0.15$, $\sigma_r = 0.01$, $\theta = 0.05$ and $r_0 = 0.042$

Price the option in 3 different ways

- Using the analytical formula

$$ZBP(0, T_1, T_2, K) =)KP(0, T_1; r_0)N(-h + \tilde{\sigma}) - P(0, T_2; r_0)N(-h), \tag{2}$$

  where T_1 is the option expiration (3-months in our case), T_2 is the bond expiration (5 years and 3 months), $P(0, T; r_0)$ is the bond price at time 0 with expiration T, given the initial short rate value r_0 (see formula below, eq(8)-(10)), $N(x)$ is the standard normal cdf, and finally

$$\tilde{\sigma} = \sigma_r \sqrt{\frac{1 - e^{-2kT_1}}{2k}} B(T_1, T_2) \tag{3}$$

$$h = \frac{1}{\tilde{\sigma}} ln \frac{P(0, T_2; r_0)}{P(0, T_1; r_0)K} + \frac{\tilde{\sigma}}{2} \tag{4}$$

$$B(T_1, T_2) = \frac{1}{k}(1 - e^{-k(T_2 - T_1)}) \tag{5}$$

- Solving the option PDE

$$\frac{\partial V(t, r)}{\partial t} + \frac{\partial V(t, r)}{\partial r} k(\theta - r(t)) + \frac{1}{2} \frac{\partial^2 V(t, r)}{\partial r^2} \sigma_r^2 = r(t)V(t, r) \tag{6}$$

  using an explicit method

- Solving the option PDE (7) using a Crank-Nicholoson method.

Remember that the boundary condition $V(T_1, r(T_1))$ is

$$V(T_1, r(T_1)) = max(0, P(T_1, T_2; r(T_1)) - K) \tag{7}$$

where the bond price $P(T_1, T_2; r(T))$ as a function of the short-term interest rate at expiration $r(T_1)$ is equal to

$$A(T_1, T_2) exp(-B(T_1, T_2) r(T_1)) \tag{8}$$

with

$$
\begin{aligned}
A(T_1, T_2) &= exp\left(\left(\theta - \frac{\sigma^2}{2k^2}\right)(B(T_1, T_2) - (T_2 - T_1)) - \frac{\sigma^2}{4k} B(T_1, T_2)^2\right) && (9) \\
B(T_1, T_2) &= \frac{1}{k}(1 - e^{-k(T_2 - T_1)}) && (10)
\end{aligned}
$$

Although in theory r could become smaller than 0 in the Vasicek model, set $r_{min} = 0$ on the grid, and pick an appropriately large r_{max}. What are appropriate boundary conditions for $V(t, r_{min})$ and $V(t, r_{max})$?

Experiment with different time and "space" intervals δ_t and δ_r. What intervals do you need in the explicit and implicit method for the final price to be within 1% of the analytical price?