Problem set 1

Marc-Aurele Divernois

FIN-503 Advanced Derivatives
EPFL

September 27, 2018

Exercise 1

The goal is to replicate the payoff of the option by a dynamic self-financing trading strategy involving the underlying stock (Δ) and the risk-free asset (B), and dynamically trading this portfolio until T. At T, the payoff of the replicating portfolio is equal to the payoff of the option.

- Option payoff :

$$max\{0, \frac{S_1(T)}{S_1(0)} - \frac{S_2(T)}{S_2(0)}\} \tag{1}$$

- Stocks follow a GBM (for simplicity, assume non-dividend stocks):

$$dS_{it} = \mu S_{it} dt + S_{it} \sigma dZ_{it} \tag{2}$$

  for i = 1,2

- Replicating portfolio's worth :

$$V_t = \Delta_1 S_{1t} + B + \Delta_2 S_{2t} \tag{3}$$

- Hence,

$$dV_t = \Delta_1 dS_{1t} + \Delta_2 dS_{2t} + Brdt \tag{4}$$

Option C depends on underlying S_1, S_2 and t : $C(S_1, S_2, t)$. Hence, by Ito :

$$dC(S_{1t}, S_{2t}, t) = \frac{\partial C}{\partial t} dt + \frac{\partial C}{\partial S_1} dS_1 + \frac{\partial C}{\partial S_2} dS_2 + \frac{1}{2} \frac{\partial^2 C}{\partial^2 S_1} d < S_1 >_t + \frac{1}{2} \frac{\partial^2 C}{\partial^2 S_2} d < S_2 >_t$$
$$+ \frac{\partial^2 C}{\partial S_1 \partial S_2} d < S_1, S_2 >_t$$

Using the fact that

$$d < S_1 >_t = S_{1t}^2 \sigma_1^2 dt \tag{5}$$

$$d < S_2 >_t = S_{2t}^2 \sigma_1^2 dt \tag{6}$$

$$d < S_1, S_2 >_t = S_{2t} S_{1t} \sigma_1 \sigma_2 \rho dt \tag{7}$$

We have that :

$$\begin{aligned} dC(S_{1t}, S_{2t}, t) = &\frac{\partial C}{\partial t} dt + \frac{\partial C}{\partial S_1} dS_1 + \frac{\partial C}{\partial S_2} dS_2 + \frac{1}{2} \frac{\partial^2 C}{\partial^2 S_1} S_{1t}^2 \sigma_1^2 dt + \frac{1}{2} \frac{\partial^2 C}{\partial^2 S_2} S_{2t}^2 \sigma_2^2 dt \\ &+ \frac{\partial^2 C}{\partial S_1 \partial S_2} S_{2t} S_{1t} \sigma_1 \sigma_2 \rho dt \end{aligned}$$

Now, we need to find Δ and B such that the value of the self-financing portfolio is equal to the option $\forall$ t.

$$V_t = \Delta_1 S_1 + B + \Delta_2 S_2 = C_t \tag{8}$$

$$\Rightarrow B = C_t - \Delta_1 S_1 - \Delta_2 S_2 \tag{9}$$

$$dV_t = dC_t \tag{10}$$

$$\begin{aligned} \Rightarrow \Delta_1 dS_{1t} + \Delta_2 dS_{2t} + Brdt = &\frac{\partial C}{\partial t} dt + \frac{\partial C}{\partial S_1} dS_1 + \frac{\partial C}{\partial S_2} dS_2 + \frac{1}{2} \frac{\partial^2 C}{\partial^2 S_1} S_{1t}^2 \sigma_1^2 dt \\ &+ \frac{1}{2} \frac{\partial^2 C}{\partial^2 S_2} S_{2t}^2 \sigma_2^2 dt + \frac{\partial^2 C}{\partial S_1 \partial S_2} S_{2t} S_{1t} \sigma_1 \sigma_2 \rho dt \end{aligned}$$

Stochastic components have to be equal on both sides of the equation :

$$\Delta_1 = \frac{\partial C}{\partial S_1} \tag{11}$$

$$\Delta_2 = \frac{\partial C}{\partial S_2} \tag{12}$$

Note that the payoff of the option is a homogeneous function of degree 1, because

$$C(t, \lambda S_1, \lambda S_2) = max\{0, \frac{\lambda S_1(T)}{S_1(0)} - \frac{\lambda S_2(T)}{S_2(0)}\} = \lambda C(t, S_1, S_2) \tag{13}$$

The Euler theorem states that a function f is homogeneous of degree k if and only if:

$$\sum_{i=1}^{n} x_i f_i'(x_1, x_2, ..., x_n) = kf(x_1, ..., x_n) \tag{14}$$

Hence for $C(t, S_1, S_2)$ being a homogeneous function of degree 1 we have,

$$\frac{\partial C}{\partial S_1}S_1 + \frac{\partial C}{\partial S_2}S_2 = C(t, S_1, S_2) \tag{15}$$

$$\Delta_1 S_1 + \Delta_2 S_2 = C(t, S_1, S_2) \tag{16}$$

The replicating portfolio involves 0 cash because

$$B = C_t - \Delta_1 S_1 - \Delta_2 S_2 = 0 \tag{17}$$

Equalizing the drift parts of equation 10 gives the following PDE to solve :

$$0 = \frac{\partial C}{\partial t} + \frac{1}{2}\frac{\partial^2 C}{\partial^2 S_1}S_{1t}^2\sigma_1^2 + \frac{1}{2}\frac{\partial^2 C}{\partial^2 S_2}S_{2t}^2\sigma_2^2 + \frac{\partial^2 C}{\partial S_1 \partial S_2}S_{2t}S_{1t}\sigma_1\sigma_2\rho \tag{18}$$

Define y and f such that :

$$y = \frac{S_1}{S_2} \tag{19}$$

$$C(t, S_1, S_2) = C(t, \frac{S_1}{S_2}, \frac{S_2}{S_2})S_2 = S_2 f(t, y) \tag{20}$$

Hence,

$$\frac{\partial C}{\partial S_1} = \frac{\partial S_2 f(t,y)}{\partial S_1} = S_2\frac{\partial f(t, \frac{S_1}{S_2})}{\partial S_1} = \frac{S_2}{S_2}f' = f' \tag{21}$$

$$\frac{\partial C}{\partial S_2} = \frac{\partial S_2 f(t,y)}{\partial S_2} = f + S_2\frac{\partial f(t,y)}{\partial S_2} = f + S_2 * (-\frac{S_1}{S_2^2})f' = f - yf' \tag{22}$$

$$\frac{\partial^2 C}{\partial^2 S_2} = \frac{\partial (f - yf')}{\partial S_2} = -\frac{S_1}{S_2}f' + \frac{S_1}{S_2}f' - yf''(-\frac{S_1}{S_2^2}) = \frac{y^2}{S_2}f'' \tag{23}$$

$$\frac{\partial^2 C}{\partial^2 S_1} = \frac{1}{S_2}f'' \tag{24}$$

$$\frac{\partial^2 C}{\partial S_1 \partial S_2} = f'\frac{1}{S_2} - f'\frac{1}{S_2} - yf''\frac{1}{S_2} = -\frac{y}{S_2}f'' \tag{25}$$

$$\frac{\partial C}{\partial t} = S_2 f \tag{26}$$

The PDE becomes

$$0 = \frac{\partial f}{\partial t} + \frac{1}{2}f''y^2\sigma_1^2 + \frac{1}{2}f''y^2\sigma_2^2 - f''y^2\sigma_1\sigma_2\rho \tag{27}$$

It is the traditional Black Scholes PDE, except that there is no drift part (i.e r=0).

The boundary condition is

$$f(T, y) = max(0, \frac{y(T)}{S_1(0)} - \frac{1}{S_2(0)}) \tag{28}$$

The pricing equation for the option is thus

$$C = \frac{S_1(T)}{S_1(0)} N(d_1) - \frac{S_2(T)}{S_2(0)} N(d_2) \tag{29}$$

with

$$d_1 = \frac{log(\frac{S_2(0) * S_1}{S_1(0) * S_2}) + \frac{1}{2}\sigma^2(T-t))}{\sigma\sqrt{T-t}} \tag{30}$$

$$d_2 = d_1 - \sigma\sqrt{T-t} \tag{31}$$

$$\sigma = \sqrt{\sigma_1^2 + \sigma_2^2 - 2\rho\sigma_1\sigma_2} \tag{32}$$