FIN 503: Advanced derivatives **(Due: 03/10/20)**

Assignment #3

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Problem 1

Given Merton jump-diffusion model:

$$\frac{dS_t}{dt} = (r - q + \lambda^Q \gamma)dt + \sigma dW_t - \gamma dN_t$$

Price contract paying $(S_T)^n$ at maturity.

According to Slide 24 and 25 (lecture 2), we have the price of asset S at maturity as:

$$S_T = S_t e^{(r-q+\lambda^Q\gamma-\frac{\sigma^2}{2})(T-t)+\sigma(W_T^Q - W_t^Q)}(1-\gamma)^{N_T - N_t}$$

Now, the price of a call at $t = 0$ and with payoff S_T^n will be:

$$\begin{aligned}
C(S_0) &= E^Q[e^{-rT}S_T^n] \\
&= \sum_{j=0}^{\infty} P(N_T = j)E^Q[e^{-rT}S_T^n | N_T = j] \\
&= \sum_{j=0}^{\infty} P(N_T = j)E^Q[e^{-rT}S_0^n \exp(n(r - q + \lambda^Q\gamma - \frac{\sigma^2}{2})T + n\sigma W_T^Q)(1-\gamma)^{jn}] \\
&= \sum_{j=0}^{\infty} P(N_T = j)e^{-rT}S_0^n e^{(r-q+\lambda^Q\gamma-\frac{\sigma^2}{2})Tn}(1-\gamma)^{jn}E^Q[e^{n\sigma W_T^Q}] \\
&= \sum_{j=0}^{\infty} P(N_T = j)e^{-rT}S_0^n e^{(r-q+\lambda^Q\gamma-\frac{\sigma^2}{2})Tn}(1-\gamma)^{jn}e^{0.5n^2\sigma^2 T} \\
&= S_0^n e^{-rT+(r-q+\lambda^Q\gamma-\frac{\sigma^2}{2})Tn+0.5n^2\sigma^2 T} \sum_{j=0}^{\infty} P(N_T = j)(1-\gamma)^{nj} \\
&= S_0^n e^{-rT+(r-q+\lambda^Q\gamma-\frac{\sigma^2}{2})Tn+0.5n^2\sigma^2 T - \lambda^Q T} \sum_{j=0}^{\infty} \frac{(\lambda^Q T(1-\gamma)^n)^j}{j!} \\
&= S_0^n e^{-rT+(r-q+\lambda^Q\gamma-\frac{\sigma^2}{2})Tn+0.5n^2\sigma^2 T - \lambda^Q T + \lambda^Q T(1-\gamma)^n}
\end{aligned}$$

The fourth to last line used the moment generating function of the normal distribution since $W_T^Q \sim N(0,T)$, i.e $E^Q[e^{n\sigma W_T^Q}] = e^{\frac{1}{2}n^2\sigma^2 T}$. The second last line used the fact that $P(N_T = j) = \exp(-\lambda T)\frac{(\lambda^Q T)^j}{j!}$ (slide 23 lecture 2) and the last line used the identity $\sum_j^\infty \frac{x^j}{j!} = e^x$.

Problem 2

EX2

$$\frac{dS_t}{S_t} = (R - q + \lambda^a \gamma)\,dt + \sigma\, dW_t - \gamma\, dN_t$$

JUMP INTENSITY 0,2 FOR 1Y $\lambda = 0{,}2$

- PROBABILITY THAT THE TIME TO THE NEXT JUMP WILL BE • LONGER • THAN TWO YEARS

  THIS IS EQUAL TO THE PROBABILITY OF NO JUMPS FOR TWO YEARS

  THE SURVIVAL PROBABILITY IS GIVEN BY $p^s(t) = e^{-\lambda t}$

  IN THIS CAS $p^s(2) = e^{-\lambda \cdot 2} = 0{,}67032$

- SHORTER THAN 3 YEARS

  IS EQUAL TO THE COMPLEMENTARY PROBABILITY OF THE EVENT THAT WE DO NOT HAVE JUMPS FOR 3Y

  SO $= 1 - p^s(3) = 1 - e^{-0{,}2 \cdot 3} = 0{,}45$

- BETWEEN 2 AND 3 YEARS

  $=$ TO THE INTERSECTION OF NOW JUMPS FOR 2 YEAR AND THE PROBABILITY OF A JUMP BEFORE THE END OF THE YEAR AFTER!

  $$P[N_2 - N_0 = \emptyset,\ N_3 - N_2 = x] \qquad x \geq 1$$

  $$= p^s(2) * (1 - p^s(1)) = e^{-\lambda \cdot 2}(1 - e^{-\lambda \cdot 1}) = 0{,}12$$

Problem 3

To find the price of the undiscounted call, we use the fact that the Poisson distribution with parameter λ is this:

$$P_\lambda(n) = \frac{\lambda^n}{n!}e^{-\lambda}. \tag{0.1}$$

From slide 22 of Lecture2, if N_t is the process that counts the number of events occurred up to t, we have that:

$$P(N_t = n) = e^{-\lambda t}\frac{(t\lambda)^n}{n!}. \tag{0.2}$$

See the code for more details.We obtain the plot showed in Figure 1.

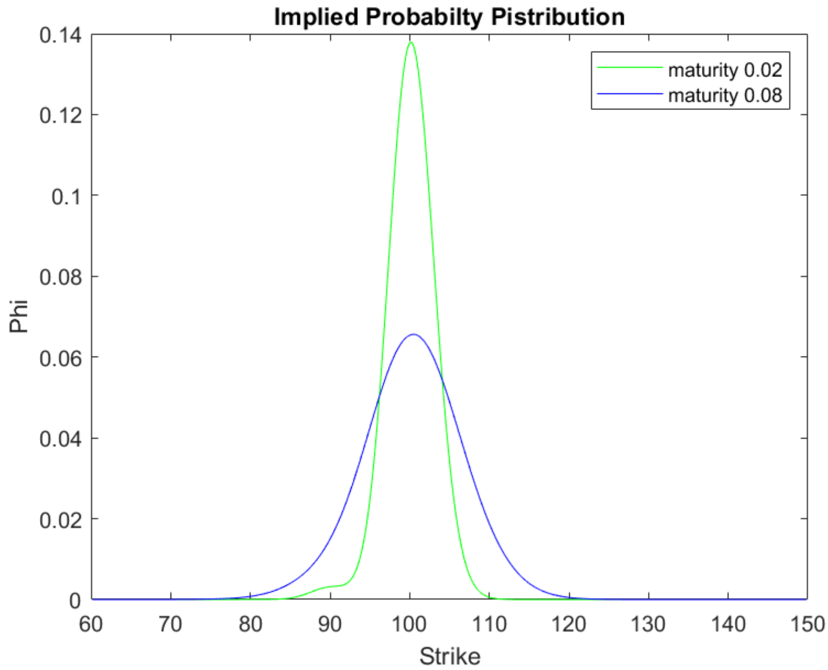

Figure 1: Problem 3 plot.

Bonus

Show: The prob. distribution matches the one you obtain using the solution S_T/S_0 of the SDE.

From slide 25 of the second lecture, we know that the solution of an SDE with jumps is given by

$$\frac{S_T}{S_0} = e^{(r-q+\lambda^{Q}\gamma-\frac{1}{2}\sigma^2)T+\sigma W_T}(1-\gamma)^{N_T} \qquad (*)$$

From probability theory: $P(A) = \sum_{B_i} P(B_i)\cdot P(A|B_i)$

Therefore, in our case, $P(K \le S_T \le K+dS) = \sum_{J=1}^{\infty} P(\underbrace{\#\text{jumps}=J}_{\text{number of jumps}})P(K \le S_T \le K+dS / \#\text{jumps}=J)$

What we would like to do is to express the increments in terms of S as increments in terms of K. Therefore, K will become something else.

From $K \le S_T \le K+dS$ to $\alpha \le W_T \le \alpha + dW$

Using formula $(*)$ in this expression:

$$(r-q+\lambda^{Q}\gamma-\tfrac{1}{2}\sigma^2)T+\sigma W_T = \log\left(\frac{S_T}{S_0(1-\gamma)^{N_T}}\right) \quad N_T=J$$

$$\Rightarrow \alpha(J) = \frac{\log\left(\frac{S_T}{S_0(1-\gamma)^{J}}\right)-(r-\frac{\sigma^2}{2})T}{\sqrt{T}\sigma}$$

$P(K \le S_T \le K+dt / \#\text{jumps}=J) = P(\alpha(J) \le W_T \le \alpha(J)+dW_T)$

Therefore: $\underbrace{P(K \le S_T \le K+dS)}_{=\phi(K)dS} = \sum_{J=1}^{\infty} P(\#\text{jumps}=J)\,P(\alpha(J) \le W_T \le \alpha(J)+dW_T)$

ϕ: density function of S

+ We want to find ϕ:

$$\phi(K) = \sum_{J=1}^{\infty}\frac{P(\#\text{jumps}=J)\,P(\alpha(J) \le W_T \le \alpha(J)+dW_T)}{dS}$$

$$P(\alpha(J) \le W_T \le \alpha(J)+dW_T) = \phi^{norm}(\alpha(J))\,dW_T$$

$$\Rightarrow \phi(K) = \sum_{J=1}^{\infty}\frac{P(\#\text{jumps}=J)\,\phi^{norm}(\alpha(J))}{\left.\frac{\partial S}{\partial W}\right|_{S=K}} \qquad \frac{\partial S}{\partial W} = S\sigma\sqrt{T}$$

$$= \sum_{J=1}^{\infty}\frac{P(\#\text{jumps}=J)\,\phi^{norm}(\alpha(J))}{K\sigma\sqrt{T}}$$

$\square$

```
clear all
clc

% set parameters
q = 0;
lambda = 1; %one jump per year
gamma = 0.1; % jump size
sigma = 0.2;
r = 0.04;
S0 = 100;

% expiration dates
T1 = 0.02;
T2 = 0.08;
T = [T1; T2];
t = 0;

K = linspace(60,150, 500)'; %strikes

C = zeros(length(K), 2); % initial price call

% discounted C with a jump process
for i=1:length(K)        % for each strike
    for j=1:length(T)    % for each maturity date
        temp = 0;
        for k=1:100
            S0_bs = S0*(1 - gamma)^(k-1);
            q_bs = q-lambda*gamma;
            payoff_bs = black_scholes(S0_bs , t, sigma, K(i), T(j), r,↙
q_bs);

            % probability k-1 jumps up to the maturity:
            jumps = exp(-lambda*T(j))*((lambda*T(j))^(k-1)/(factorial↙
(k-1)));

            C(i,j) = C(i,j) + jumps * payoff_bs;
        end
    end
end

% undiscounted C
% Computing C tild the undiscouted price
C_und = C.*exp(r*(T'-t));


% We compute phi by differenetation of C
der_C = C_und(1:end-2,:) - 2 * C_und(2:end-1,:) + C_und(3:end,:);
phi = der_C/(K(2)-K(1))^2;
```

```matlab
% We plot the result
figure
plot(K(2:end-1,1), phi(:,1), 'g')
hold on
plot(K(2:end-1,1), phi(:,2), 'b')
hold off
xlabel('Strike')
ylabel('Phi')
title('Implied Probabilty Pistribution')
legend('maturity 0.02','maturity 0.08')
```