FIN 503: Advanced derivatives **(Due: 10/11/20)**

Assignment #8

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PS 3

ex 1

$$\frac{dS_t}{S_t} = (R - q + \lambda\gamma)\,dt + \sigma\, dW_t^Q - \gamma\, dN_t$$

Derive the process for $\log(S_t)$

Ito lemma when S follows a jump-Diffusion process

$$dg(S) = \frac{dg}{dS}\,dS^c + \frac{1}{2}\frac{d^2 g}{dS^2}(dS^c)^2 + (g^+ - g^-)\,dN_t$$

$$dS^c = (R - q + \lambda\gamma)\,dt + \sigma\, dW_t^Q$$

$g^+ - g^-$ jump in g due to a jump in S

$$g(S) = \log(S) \qquad \frac{dg}{dS} = 1/S \qquad \frac{d^2 g}{dS^2} = -1/S^2$$

$$g^+ - g^- = \log[(1-\gamma)S] - \log(S)$$

$$= \log(1-\gamma) + \log(S) - \log(S) = \log(1-\gamma)$$

$$d(\log(S)) = (R - q + \lambda\gamma)\,dt + \frac{1}{2}\left(-\frac{1}{S^2}\cdot\sigma^2 S^2\right)dt + \sigma\, dW_t^Q + \log(1-\gamma)\,dN_t$$

$$= \left(R - q + \gamma\lambda - \frac{1}{2}\sigma^2\right)dt + \sigma\, dW_t^Q + \log(1-\gamma)\,dN_t$$

integrating over $[T, 0]$ and assuming $N_0 = \emptyset$

$$\log\left(\frac{S_T}{S_0}\right) = \left(R - q + \lambda\gamma - \frac{1}{2}\sigma^2\right)T + \sigma W_T + \log(1-\gamma) N_T$$

Compute

$$E^{q}\left[\log\left(\frac{S(T)}{S(0)}\right)\right]$$

$$=\sum_{J=0}^{\infty} P[N_T=J]\cdot E^{q}\left[\left(R-q+\lambda^{q}\gamma-\frac{1}{2}\sigma^2\right)T+\sigma W_T+\log(1-\gamma)J\right]$$

$$=\sum_{J=0}^{\infty} P[N_T=J]\cdot\left[\left(R-q+\lambda\gamma-\frac{1}{2}\sigma^2\right)T+E[\sigma W_T]+\log(1-\gamma)J\right]$$

$$E[\sigma W_T]=\emptyset$$

$$=\left(R-q+\lambda\gamma-\frac{1}{2}\sigma^2\right)T+\sum_{J=0}^{\infty} P[N_T=J]\cdot\log(1-\gamma)\cdot J$$

$$\sum_{J=0}^{\infty} P[N_T=J]\cdot J = E[X] \quad \text{where } X\sim \text{Poisson}(\lambda T)$$

expected value of a Poisson random variable $=\lambda T$

So

$$E^{q}\left[\log\left(\frac{S(T)}{S(0)}\right)\right]=\left(R-q+\lambda\gamma-\frac{1}{2}\sigma^2\right)T+\lambda T\cdot\log(1-\gamma)$$

Compute quadratic variation of $\log(S)$ over $[0,T]$:

$$\int_0^T d[\log(S)]_t$$

Since W_t is independent from N_t

$$=\int_0^T \sigma^2\, d[W]_t+\int_0^T(\log(1-\gamma))^2\, d[N]_t$$

Since $\int_0^T d[N]_t = N_T-N_0$ (from stochastic calculus course)

$$\int_0^T d[\log(S)]_d=\sigma^2 T+(\log(1-\gamma))^2 N_T$$

SHOW

$$2\left(RT - E^{Q}\left[\log\left(\frac{S(T)}{S(0)}\right)\right]\right)$$ perfectly replicates the QUADRATIC VARIATION

$$2\left(RT - E^{Q}\left[\log\left(\frac{S(T)}{S(0)}\right)\right]\right)$$

$$= 2\left(RT - \left(R - q + \lambda\gamma - \frac{1}{2}\sigma^2\right)T + \lambda T\log(1-\gamma)\right)$$

$$= 2\left(\left(q - \lambda\gamma + \frac{1}{2}\sigma^2\right)T + \lambda T\log(1-\gamma)\right) \quad *$$

WHICH DYNAMICS ARE

$$2\left[\left(q - \lambda\gamma + \frac{1}{2}\sigma^2 + \lambda\log(1-\gamma)\right)dt\right]$$

$$\int_0^T (d\log(S(t))) = \sigma^2 T + (\log(1-\gamma))^2 N_T$$

DYNAMICS!

$$\sigma^2 dt + (\log(1-\gamma))^2 dN_T \quad **$$

SINCE * AND ** HAVE two Different DYNAMICS we CAN Affirm that!

$$2\left(RT - E^{Q}\left[\log\left(\frac{S(T)}{S(0)}\right)\right]\right)$$ Does not replicate the QUADRATIC VARIATION of $\log(S)$ over $[0, T]$

Problem 2

Breeden-Litzenberger formula for the pdf of the stock price S_T:

$$p(S_T, T; S_t, t) = \left.\frac{\partial^2 \tilde{C}(S_t, K, \tau)}{\partial K^2}\right|_{K=S_T} = \left.\frac{\partial^2 \tilde{P}(S_t, K, \tau)}{\partial K^2}\right|_{K=S_T}$$

And we can write:

$$E[g(S_T)|S_t] = \int_0^{S^*} \frac{\partial^2 \tilde{P}(S_t, K, \tau)}{\partial K^2} g(K)dK + \int_{S^*}^{\infty} \frac{\partial^2 \tilde{C}(S_t, K, \tau)}{\partial K^2} g(K)dK$$

In order to prove that the expression above is equivalent to the formula obtained in class, we will work out each of the integrals that we will call (1) and (2) respectively, using integration by part.

(1)

$$\int_0^{S^*} \frac{\partial^2 \tilde{P}(S_t, K, \tau)}{\partial K^2} g(K)dK \tag{0.1}$$

$$= g(K)\tilde{P}'(K)\Big|_0^{S^*} - \int_0^{S^*} g'(K)\tilde{P}'(K)dK \tag{0.2}$$

$$= g(K)\tilde{P}'(K)\Big|_0^{S^*} - \left(g'(K)\tilde{P}(K)\Big|_0^{S^*} - \int_0^{S^*} g''(K)\tilde{P}(K)dK \right) \tag{0.3}$$

$$\tag{0.4}$$

(2)

$$\int_{S^*}^{\infty} \frac{\partial^2 \tilde{C}(S_t, K, \tau)}{\partial K^2} g(K)dK \tag{0.5}$$

$$= g(K)\tilde{C}'(K)\Big|_{S^*}^{\infty} - \left(g'(K)\tilde{C}(K)\Big|_{S^*}^{\infty} - \int_{S^*}^{\infty} g''(K)\tilde{C}(K)dK \right) \tag{0.6}$$

Now, we can make use of the following put-call parity relation:

$$C(S_t, K, \tau) - P(S_t, K, \tau) = S_T - K \tag{0.7}$$

$$\frac{\partial C}{\partial K} = \frac{\partial P}{\partial K} - 1 \tag{0.8}$$

$$\frac{\partial^2 C}{\partial K^2} = \frac{\partial^2 P}{\partial K^2} \tag{0.9}$$

Combining (1), (2) and the relations above, we have:

$$
\begin{aligned}
E[g(S_T)|S_t] &= g(S^*)\left[\tilde{P}'(K) - \tilde{C}'(K)\right]\Big|_{K=S^*} - g'(S^*)\left[\tilde{P}(K) - \tilde{C}(K)\right]\Big|_{K=S^*} + \\
&\int_0^{S^*} g''(K)\tilde{P}(K)dK + \int_{S^*}^{\infty} g''(K)\tilde{C}(K)dK \qquad (0.10)\\
&= g(S^*)\times 1 - g'(S^*)(S^* - S_T) + \int_0^{S^*} g''(K)\tilde{P}(K)dK + \int_{S^*}^{\infty} g''(K)\tilde{C}(K)dK \qquad (0.11)\\
&= g(S^*) - g'(S^*)S^* + g'(S^*)S_T + \int_0^{S^*} g''(K)\tilde{P}(K)dK + \int_{S^*}^{\infty} g''(K)\tilde{C}(K)dK \qquad (0.12)
\end{aligned}
$$

which is the identity needed proving.

PROBLEM 3

Stock price: $\frac{dS_t}{S_t} = (r-q)dt + \sigma dW_t$ ← pure diffusion process

- We want to replicate the variance swap only when a discrete set of options with strikes $\{K_0, K_1, \dots K_n\}$ are available.
How would you build a portfolio which replicates the payoff of the variance swap using these options?

First, let us remind what are variance swaps:
<u>Variance swaps</u> are future contracts on future realised annualized variance. They are, as volatility swaps, an easy way for investors to gain exposure to the future level of volatility. They provide pure exposure to volatility alone, not contaminated by the stock price path. Therefore, they are usefull in many situations.

It is important that a variance swap can be theoretically replicated by a hedged portfolio of standard options with suitable strikes, given that the underlying has no jumps.
The cost of this replicating portfolio is the <u>fair value</u> of the variance swap
The fair value of variance is the delivery price that makes the swap have zero value.
The payoff at expiration is equal to $(\sigma^2 - K)\cdot N$
↳ notional amount of the swap in $ per annualised volatility point squared
In other words, the investor at maturity will receive N for each point by which the stock's realized variance σ^2 has exceed strike K.

In next steps, consider $v = \sigma^2 \tau$ (time to maturity) as the total variance of the stock to expiration and V the exposure of an option to the stock's variance.

To replicate a variance swap, we need a portfolio whose variance is independent of moves in the stock price. A single option is not suitable, we need to combine many options.
We can prove that a portfolio with weights inversely proportional to K^2 produces a V virtually indipendent from S (if the strikes are distributed closely and constantly). See more details below. The idea is that an option with higer strike will produce a V contribution that increases with S and we need to offset this.

⇒ To obtain an exposure to variance indipendent from the stock price, you can own a portfolio Π of option of all strikes, weighted in inverse proportion to K^2.

Let us start with a portfolio $\Pi(S) = \int_0^\infty \rho(K)\, O(S,K,v)\, dK$

where $O(S,K,V)$ is a normal standard BS option
we need to prove that $\rho(K)$ is inversely proportional to K^2.

This portfolio must be such that V does not depend on S ⇒ $\frac{\partial V}{\partial S} = 0$

Therefore we need to find V:
we start from the V of a single option V_0: $V_0 = \tau \frac{\partial O}{\partial v} = \tau S f\left(\frac{K}{S}, v\right)$

From Black Scholes $f(S,K,v) = \frac{1}{2\sqrt{v}} \frac{e^{-d_1^2/2}}{\sqrt{2\pi}}$ $\qquad d_1 = \frac{\ln(S/K)+v/2}{\sqrt{v}}$

$$\Rightarrow V(S) = \int_0^\infty \tau \rho(K) S f\left(\frac{K}{S}, v\right) dK$$

Consequently, $\frac{\partial V}{\partial S} = \tau \int_0^\infty S\left[2\rho(xS) + xS\rho'(xS)\right] f(x,v)\, dx$ with $x = K/S$

$$\frac{dV}{dS} = 0 \iff 2\rho + K\frac{\partial \rho}{\partial K} = 0 \Rightarrow \rho \propto \frac{1}{K^2}$$

This proves what we have stated above

Now, we focus on the log payoff replication with a discrete set of options.
We use the given SDE: $\frac{dS_t}{S_t} = (r-q)dt + \sigma dW_t$

$$d\log(S) \overset{\text{Itô formula}}{=} \frac{1}{S_t} dS_t - \frac{1}{2S_t^2} \underbrace{d\langle S_t, S_t\rangle}_{S_t^2\sigma^2 dt} = (r-q)dt + \sigma dW_t - \sigma^2 dt$$
$$= \left(r-q-\frac{\sigma^2}{2}\right)dt + \sigma dW_t$$

$$\Rightarrow \frac{dS}{S} - d\log S = -\frac{1}{2}\sigma_t^2 dt$$

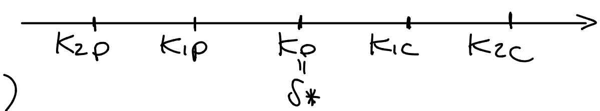
This subtraction permits to delete the diffusion part regarding dW_t and simplify the process. Then, consider
$$f(S_T) = \frac{2}{T}\left[\frac{S_T - S_*}{S_*} - \log\frac{S_T}{S_*}\right]$$

where S_* is the reference price.
Remind that we have only a discrete set of option available to replicate $f(S_T)$.
We need to determine, for each strike, how many options we need.

Assume $S_* = K_0$ and that you can trade

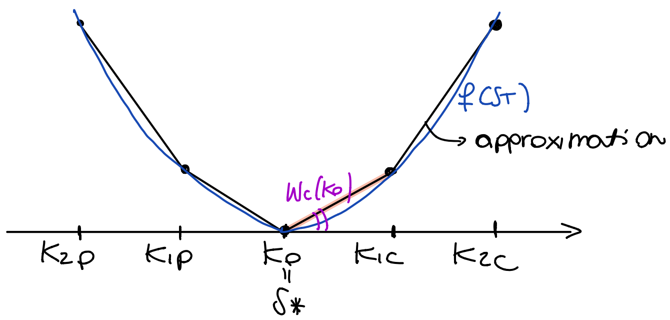
- call options with strikes $K_0 < K_{1c} < K_{2c} \dots$
- put options " " $K_0 > K_{1p} > K_{2p} \dots$

$\Rightarrow \dots K_{2p} < K_{1p} < K_0 < K_{1c} < K_{2c} < \dots$



Using this as x axis we can approximate $f(S_T)$ with a piecewise linear function:

The first segment to the right of S^* is equivalent to the payoff of a call price with K_0 as strike.
The number of options determines the slopes of the segments.



For this first segment

$$W_c(K_0) = \frac{f(K_{1c}) - f(K_0)}{K_{1c} - K_0}$$

We can proceed in the same way to the second segment, which is a combination of call prices with payoff K_0 and K_1.

$$W_c(K_1) = \frac{f(K_{2c}) - f(K_{1c})}{K_{2c} - K_{1c}} - W_c(K_0)$$

Going on, for a general $K_{n,c}$ (call option with strike K_n) we have

$$W_c(K_{n,c}) = \frac{f(K_{n+1,c}) - f(K_{n,c})}{K_{n+1,c} - K_{n,c}} - \sum_{i=0}^{n-1} W_c(K_{i,c})$$

The same applies for put options.

We build the entire curve one step at a time - How many segments we have depends on how many options we have.

Imagine that you are short the variance swap and long this portfolio. For which values of the final stock price is the payoff of the variance swap higher than the payoff of this portfolio, and for which values of ST is it lower?

Note that in practice, we are always in the case of a finite number of options available $\forall T \Rightarrow$ a finite number of strikes fails to create an uniform V_c because the stock price goes outside the range of available strikes.

When the stock price moves outside the strike range, the reduced vega of the imperfectly replicated log contract will make it less reactive than the variance swap.

To answer to the question, we must remember that a variance swap has a payoff proportional to realized variance. Also continuous hedging of a log contract produces a payoff whose value is proportional to future realised variance.

We are asked for which values of the final stock price S_T the payoff of the variance swap is higher than the payoff of the portfolio, ie

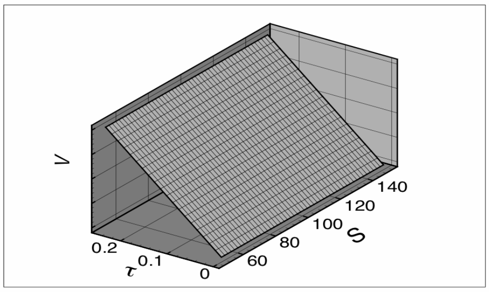
Find S_T s.t $\Psi_{\text{var. swap}} > \Psi_{\text{portfolio}}$

We suppose this happens when S_T moves outside the range of possible strikes.

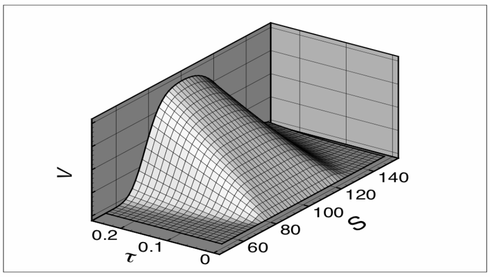
On the other hand, it never happens that $\Psi_{\text{var swap}} < \Psi_{\text{portfolio}}$ - At maximum we can have an equality.

If we look at Figure 3 of page 14 of the "More Than you ever wanted to know about volatility swaps" paper, we have a graphical interpretation of our idea :

(a)



(b)



a) Ideal case with infinite strikes.
The vega does not depend on S, but it decreases linearly when we approach to the maturity:

$\tau \to 0 \quad \Rightarrow \quad V \downarrow$

b) On the other hand, if $K \in [75, 125]$, V depends on S. In particular, V decreases when S_T goes outside the possible strikes interval.
As in the previous case, V linearly decreases when the time to maturity goes to zero