Problem set 3

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Exercise 1

By Ito forumla for jump diffusion, we have

$$dg(s) = \frac{\partial g}{\partial S} dS + 0.5 \frac{\partial^2}{\partial^2 S} d < S > + (g^+ - g^-) dN_t + \frac{\partial g}{\partial t} \tag{1}$$

let g(s) = log(s)

- $$\begin{aligned} dlog(S(t)) &= \frac{dS_t}{S_t} - 0.5 \frac{d < S >}{S^2} + (g^+ - g^-) dNt + 0 \\ &= (r - q + \lambda\gamma) dt + \sigma dW_t - 0.5\sigma^2 dt + (ln(S(1-\gamma)) - ln(S)) dN_t \\ &= (r - q + \lambda\gamma - 0.5\sigma^2) dt + \sigma dW_t + ln(1-\gamma) dN_t \end{aligned} \tag{2}$$

- $$\begin{aligned} E[log(\frac{S_t}{S_0})] &= E[(r - q + \lambda\gamma - 0.5\sigma^2) dt + \sigma dW + ln(1-\gamma) dN_t)] \\ &= (r - q + \lambda\gamma - 0.5\sigma^2) t + ln(1-\gamma)\lambda t \end{aligned} \tag{3}$$

- $$\begin{aligned} \int_0^T d < log(S) > &= \int_0^T (\sigma^2 d < W >_t + (log(1-\gamma))^2) d < N >_t) \\ &= \sigma^2 T + (log(1-\gamma))^2 N_T \end{aligned}$$

- The claim follows from (2) and (3). Note that if q=0 and there are no jumps, the replication would work.

Exercise 2

$$E[g(S_T)] = \int_0^{S\star} \frac{\partial^2 P}{\partial^2 K} g(K)dK + \int_{S\star}^{\infty} \frac{\partial^2 C}{\partial^2 K} g(K)dK \tag{4}$$

Using integral by parts :

$$\int u'v = uv - \int uv' \tag{5}$$

we can rewrite the first term as

$$\begin{aligned}
&= \frac{\partial P}{\partial K} g(K)|_0^{S\star} - \int_0^{S\star} \frac{\partial P}{\partial K} g'(K) \\
&= \frac{\partial P}{\partial K} g(K)|_0^{S\star} - [Pg'(K)|_0^{S\star} - \int_0^{S\star} Pg''(K)dK] \\
&= \frac{\partial P}{\partial S^\star} g(S\star) - Pg'(S\star) + \int_0^{S\star} Pg''(K)dK
\end{aligned} \tag{6}$$

similarly, we can rewrite the second term as

$$\begin{aligned}
&= \frac{\partial C}{\partial K} g(K)|_{S\star}^{\infty} - \int_{\star}^{\infty} \frac{\partial C}{\partial K} g'(K) \\
&= \frac{\partial C}{\partial C} g(K)|_{S\star}^{\infty} - [Cg'(K)|_{S\star}^{\infty} - \int_{S\star}^{\infty} Cg''(K)dK] \\
&= \frac{\partial C}{\partial S^\star} g(S\star) - Cg'(S\star) + \int_{S\star}^{\infty} Cg''(K)dK
\end{aligned} \tag{7}$$

Note that :
$\frac{\partial P}{\partial K} = 0$ evaluated at K=0
$\frac{\partial C}{\partial K} = 0$ evaluated at K=∞

Then, (4) is :

$$E[g(S_T)] = \frac{\partial P}{\partial S^\star} g(S\star) - Pg'(S\star) + \int_0^{S\star} Pg''(K)dK + \frac{\partial C}{\partial S^\star} g(S\star) - Cg'(S\star) + \int_{S\star}^{\infty} Cg''(K)dK$$

Using put call parity for undiscounted calls and puts :

$$\frac{\partial C}{\partial S} - \frac{\partial P}{\partial S} = 1 \tag{8}$$

$$C - P = S_t e^{r(T-t)} - K \tag{9}$$

we finally get

$$E[g(S_T)] = g(S\star) + g'(S\star)(S_t e^{r(T-t)} - S\star) + \int_0^{S\star} P(K)g''(K)dK + \int_{S\star}^{\infty} C(K)g''(K)dK \tag{10}$$

Exercise 3

For this exercice, I refer to the appendix A of "More than you ever wanted to know about volatility swaps" (Demeterfi et al. [1999]) that can be found on moodle.

The variance swap has the form

$$
\begin{aligned}
V - K &= \frac{2}{T}\left(\int_0^T \frac{dS_t}{S_t} - log(\frac{S_T}{S_0})\right) - K \\
&= \frac{2}{T}\left(\int_0^T \frac{dS_t}{S_t} - \frac{S_T - S_0}{S_0} + \int_0^{S_0} \frac{1}{K^2}(K - S_T)^+ dK + \int_{S_0}^{\infty} \frac{(S_T - K)^+}{K^2} dK\right) - K \quad (11)
\end{aligned}
$$

With only a set of options available (i.e $K_0, K_1, ..., K_n$), we need to discretize the integrals with respect to K. This gives the following payoff :

$$
\frac{2}{T}\left(\int_0^T \frac{dS_t}{S_t} - \frac{S_T - S_0}{S_0}\right) + \frac{2}{T}\left(\sum_{i=0}^{j} \frac{K_i - Ki - 1}{K_i^2}(K_i - S_T)^+ + \sum_{i=j+1}^{n} \frac{K_i - K_{i-1}}{K_i}(S_T - K_i)\right) - K \quad (12)
$$

Demeterfi et al. [1999] proposes to replicate the log contract with a set of out of the money puts and calls and a forward contract on the underlying.

FIGURE 12. Log-payoff and options portfolio at maturity.