Exercise 1

a) Black Scholes Model

$(S_t)_{t\in[0,T]}$: price process $\quad \mathbb{P}$

$\mu, \hat{\sigma} \in \mathbb{R}$
W standard BM

$$(*) \quad dS_t = S_t(\mu dt + \sigma dW_t)$$

- Show that the process $\overset{(*)}{S_t} = S_0\, e^{(\mu - \frac{\sigma^2}{2})t + \sigma W_t}$ is a solution of $(*)$:

By definition, a stochastic process S_t is said to follow a Geometric Brownian motion if it satisfy the PDE: $dS_t = \mu S_t dt + \sigma dW_t$.
Given a certain initial value S_0, the solution of this PDE is given by the process $S_t = S_0 \exp((\mu - \frac{\sigma^2}{2})t + \sigma W_t)$ ← explicit formulation

Using Itô lemma we have that, if X_t is a Itô diffusion process (satisfies $dX_t = \mu_t dt + \sigma_t dW_t$) and $f(t,x)$ is a twice differentiable scalar function, then

$$df = \frac{\partial f}{\partial t}dt + \frac{\partial f}{\partial x}dx + \frac{1}{2}\frac{\partial^2 f}{\partial x^2}dx^2$$

In our case, X_t is S_t (and S_t is a Itô diffusion process) and $f(x) = \log x$. Then

$$df(S_t) = d(\log S_t) = \underset{\frac{\partial f}{\partial t}=0}{0} + \frac{1}{S_t}dS_t + \frac{1}{2}\left(-\frac{1}{S_t^2}\right)(dS_t)^2$$

$$(dS_t)^2 = [S_t(\mu dt + \sigma dW_t)]^2 \underset{\substack{dtdt=0\\ dW_tdW_t=dt}}{=} S_t^2\sigma^2 dt$$

$$\Rightarrow d\log(S_t) = \frac{dS_t}{S_t} - \frac{1}{2}\sigma^2 dt$$

Integrating

$$\log S_t - \log S_0 = \int_0^t d(\log S_t) = \int_0^t (\mu - \tfrac{1}{2}\sigma^2)dt + \int_0^t \sigma dW_t$$

$$\log\frac{S_t}{S_0} = (\mu - \tfrac{1}{2}\sigma^2)t + \sigma W_t$$

$\downarrow$ exponential

$$* \longrightarrow S_t/S_0 = e^{(\mu - \frac{1}{2}\sigma^2)t + \sigma W_t} \qquad \boxtimes$$

- Compute $\mathbb{E}[S_T]$:

$$\mathbb{E}[S_T] = \mathbb{E}[S_0\, e^{(\mu - \frac{\sigma^2}{2})T + \sigma W_T}] = S_0\, e^{(\mu - \frac{\sigma^2}{2})T}\,\mathbb{E}[e^{\sigma W_T}]$$

In order to find $\mathbb{E}[e^{\sigma W_T}]$, we need to think about W_t properties :

W_t is a BM $\Rightarrow W_t - W_s \sim N(0, t-s)$ $t>0$ consider $s=0$ and the fact that $W_0=0$

$\Rightarrow W_t \sim N(0,t)$

$\Rightarrow e^{W_t}$ has a lognormal distribution

$\Rightarrow e^{\sigma W_t}$ " " " " for which $\sigma W_t \sim N(0, \sigma^2 t)$

From theory we know that if $X \sim N(\mu, \sigma^2) \Rightarrow \mathbb{E}(e^X) = e^{\mu + \frac{\sigma^2}{2}}$

$\Rightarrow$ In our case: $\mathbb{E}(e^{\sigma W_T}) = e^{\frac{\sigma^2}{2}T}$

$\Rightarrow \mathbb{E}(S_T) = S_0 \, e^{(\mu - \frac{\sigma^2}{2})T} \, e^{\frac{\sigma^2}{2}T} = S_0 \, e^{\mu T}$

b) Law of one price: if 2 portfolios P_1 and P_2 have the same value $V_T(P_1) = V_T(P_2)$ $\begin{matrix} T > t \\ T > 0 \end{matrix}$

$\Rightarrow \forall t \le T \quad V_t(P_1) = V_t(P_2)$

(a) - Payoff European derivative: $D_T = |S_T - K|$

- Payoff Call option strike K, expiration T: $C_T = (S_T - K)^+$
- Payoff Put " " " " ": $P_T = (K - S_T)^+$

Why the value of the derivative D_t is equal to the value of the sum of a Put and a Call (strike K, expir date T)?

Mathematically: $|S_T - K| = \begin{cases} S_T - K & S_T \geq K \\ K - S_T & S_T < K \end{cases}$

$$(S_T - K)^+ = \begin{cases} S_T - K & S_T \geq K \\ 0 & S_T < K \end{cases}$$

$$(K - S_T)^+ = \begin{cases} K - S_T & S_T < K \\ 0 & S_T \geq K \end{cases}$$

Portofio that longs both the put and the call:

$$(S_T - K)^+ + (K - S_T)^+ = \begin{cases} S_T - K + 0 & S_T \geq K \\ 0 + K - S_T & S_T < K \end{cases} = \begin{cases} S_T - K & S_T \geq K \\ K - S_T & S_T < K \end{cases} = |S_T - K|$$

$$\Rightarrow (S_T - K)^+ + (K - S_T)^+ = |S_T - K|$$

This proves that, at the maturity date T, the equality is true.
Considering the definition of Law price, which consider P_1, P_2, we have that

P_1 is the Eur. derivative payoff s.t. $V_T(P_1) = |S_T - K|$

P_2 is the sum of the call and the put s.t. $V_T(P_2) = (S_T - K)^+ + (K - S_T)^+$

and we have proved that $V_T(P_1) = V_T(P_2) \xRightarrow{\text{no arbitrage + law}} \forall t < T \quad V_t(P_1) = V_t(P_2)$

$$|S_t - K| = (S_t - K)^+ + (K - S_t)^+$$

This ends our proof $\square$

(b) P_1: discount certificate on the underlying S with cap K

$\Rightarrow$ Payoff $V_T(P_1) = \min\{S_T, K\}$

P_2: Portfolio whose price is given by

S_0 minus the price of a call option with strike K and expiration date T.

S_0 is constant $\Rightarrow$ value of S_t $\forall t$

$\Rightarrow$ Payoff $V_T(P_2) = S_T - \text{payoff call} = S_T - (S_T - K)^+$

the price S_0 when you buy the portfolio becomes S_T at the maturity date

Mathematically: $V_T(P_1) = \begin{cases} S_T & S_T \leq K \\ K & K < S_T \end{cases}$

$$V_T(P_2) = \begin{cases} S_T - (S_T - K) & S_T > K \\ S_T - 0 & S_T \leq K \end{cases}$$

Therefore, $V_T(P_2) = \begin{cases} K & S_T > K \\ S_T & S_T \leq K \end{cases} = V_T(P_1)$

As in the previous point, given that $V_T(P_1) = V_T(P_2)$, we can conclude that the price of the two portfolios is the same $V_t(P_1) = V_t(P_2)$ $t<T$

If $t=0$ $\quad V_0(P_1) = S_0 - \underbrace{\text{Call}(S_0, 0)}_{\text{Call price at } t=0}$ $\quad\square$

Exercise 2 OU process

An OU process is defined as the solution of the SDE: $dX_t \overset{*}{=} \kappa(\theta - X_t)dt + \lambda dW_t$ $\quad t \in [0,T]$

$\kappa > 0 \quad \lambda > 0$ const

$\theta \in \mathbb{R}$

x_0 : starting value

a) Solves $*$:

This SDE is a linear SDE, whose general form is $\begin{cases} dX_t = (AX_t + b)dt + \sigma dW_t \\ X_0 = x_0 \end{cases}$

$\Rightarrow$ The unique solution of the linear SDE is given by:

$$X_t = e^{At}x_0 + \int_0^t e^{A(t-s)} b\, ds + \int_0^t e^{A(t-s)} \sigma dW_s$$

In our case, $A = -\kappa \quad b = \kappa\theta \quad \sigma = \lambda$

$\Rightarrow$ The solution is:

$$X_t = e^{-\kappa t} x_0 + \int_0^t e^{-\kappa(t-s)} \kappa\theta\, ds + \int_0^t e^{-\kappa(t-s)} \lambda dW_s$$

$$= e^{-\kappa t} x_0 + \theta e^{-\kappa t} \int_0^t e^{\kappa s} \kappa\, ds + \lambda \int_0^t e^{-\kappa(t-s)} dW_s$$

$$= e^{-\kappa t} x_0 + \theta e^{-\kappa t} [e^{\kappa t} - 1] + \lambda \int_0^t e^{-\kappa(t-s)} dW_s$$

$$\Rightarrow X_t = e^{-\kappa t} x_0 + \theta(1 - e^{-\kappa t}) + \lambda \int_0^t e^{-\kappa(t-s)} dW_s$$

b) Using a), show $\lim_{t \to \infty} \mathbb{E}[X_t] = \theta$:

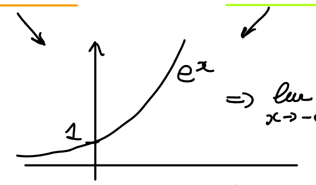
From a): $\mathbb{E}[X_t] = \mathbb{E}\left[e^{-\kappa t} x_0 + \theta(1 - e^{-\kappa t}) + \lambda \int_0^t e^{-\kappa(t-s)} dW_s \right]$

$$= e^{-\kappa t} x_0 + \theta(1 - e^{-\kappa t}) + \lambda \underbrace{\mathbb{E}\left[\int_0^t e^{-\kappa(t-s)} dW_s \right]}_{\text{the expected value of a stochastic integral is zero}}$$

$$= e^{-\kappa t} x_0 + \theta(1 - e^{-\kappa t})$$

$$\lim_{t\to\infty} \mathbb{E}[X_t] = \lim_{t\to\infty} e^{-\kappa t} x_0 + \theta(1 - e^{-\kappa t}) \underset{-t = u}{=} \lim_{u \to -\infty} e^{u\kappa} x_0 + \theta(1 - e^{\kappa u})$$

$$= 0 + \theta(1 - 0) = \theta$$



$\Rightarrow \lim_{x \to -\infty} e^x = 0$

considering non infinite κ and x_0:

$\lim_{u \to -\infty} x_0 e^{u\kappa} = 0$

and

$\lim_{u \to -\infty} e^{\kappa u} = 0$

c) Compute $\mathbb{E}[X_T^2]$ in 2 ways $\qquad dX_t = \kappa(\theta - X_t)dt + \lambda dW_t$, $X_0 = x_0$

① Using a): $X_t = e^{-\kappa t} x_0 + \theta(1 - e^{-\kappa t}) + \lambda \int_0^t e^{-\kappa(t-s)} dW_s$

$$\Rightarrow X_T = e^{-\kappa T} x_0 + \theta(1 - e^{-\kappa T}) + \lambda \int_0^T e^{-\kappa(T-s)} dW_s$$

If f is square integrable on $[0,t] \Rightarrow \int_0^t f(s)\,dW_s \overset{*}{\sim} N\left(0, \int_0^t f^2(s)\,ds\right)$

From $*$, substituting t with T and considering $f(s) = e^{-K(T-s)}$ $\leftarrow$ square integrable

$$\int_0^T e^{-K(T-s)}\,dW_s \sim N\left(0, \int_0^T e^{-2K(T-s)}\,ds\right)$$

Multiplying by the constant λ:

$$\lambda \int_0^T e^{-K(T-s)}\,dW_s \sim N\left(0, \lambda^2 \int_0^T e^{-2K(T-s)}\,ds\right)$$

Adding $e^{-KT}x_0 + \theta(1-e^{-KT})$ in order to obtain X_T:

$$\underbrace{e^{-KT}x_0 + \theta(1-e^{-KT}) + \lambda \int_0^T e^{-K(T-s)}\,dW_s}_{} \sim N\left(e^{-KT}x_0 + \theta(1-e^{-KT}), \lambda^2 \int_0^T e^{-2K(T-s)}\,ds\right)$$

$$\Rightarrow X_T \sim N\left(e^{-KT}x_0 + \theta(1-e^{-KT}), \lambda^2 \int_0^T e^{-2K(T-s)}\,ds\right)$$

$$Var(X_T) = \mathbb{E}(X_T^2) - (\mathbb{E}(X_T))^2$$

$$\Rightarrow \mathbb{E}(X_T^2) = Var(X_T) + (\mathbb{E}(X_T))^2 = \lambda^2 \underbrace{\int_0^T e^{-2K(T-s)}\,ds} + \left[e^{-KT}x_0 + \theta(1-e^{-KT})\right]^2$$

$$\int_0^T e^{-2K(T-s)}\,ds = e^{-2KT}\int_0^T e^{2Ks}\,ds = e^{-2KT}\left.\frac{e^{2Ks}}{2K}\right|_0^T$$

$$= \frac{e^{-2KT}}{2K}\left(e^{2KT}-1\right) = \frac{1}{2K}\left(1-e^{-2KT}\right)$$

$$\mathbb{E}[X_T^2] = \frac{\lambda^2}{2K}\left(1-e^{-2KT}\right) + \left[e^{-KT}x_0 + \theta(1-e^{-KT})\right]^2$$

$$= \frac{\lambda^2}{2K}\left(1-e^{-2KT}\right) + \left[e^{-2KT}x_0^2 + \theta^2\underbrace{(1-e^{-KT})^2}_{1+e^{-2KT}-2e^{-KT}} + 2\theta x_0 \underbrace{e^{-KT}(1-e^{-KT})}_{e^{-KT}-e^{-2KT}}\right] =$$

$$\mathbb{E}[X_T]^2 = \frac{\lambda^2}{2K}\left(1-e^{-2KT}\right) + \left[x_0^2 e^{-2KT} + \theta^2 + \theta^2 e^{-2KT} - 2\theta^2 e^{-KT} + 2\theta x_0\left(e^{-KT}-e^{-2KT}\right)\right]$$

(2) Define $v(t,x) = \mathbb{E}\left[(X_T^{t,x})^2\right]$

where $X_s^{t,x}$ is solution of $*\begin{cases} dX_s = K(\theta - X_s)ds + \lambda\,dW_s \\ X_t = x \end{cases}$ $\quad t \le s \le T$

- Derive the following PDE satisfied by v: $\quad *\begin{cases} v_t + \mathcal{G}v_t = 0 \\ v(T,x) = x^2 \end{cases}$ $\quad \mathcal{G}v \overset{*}{=} K(\theta - x)v_x + \frac{\lambda^2}{2}v_{xx}$

From theory, see slides 20/21 of lecture 2b, we know that:

denoting $X^{t,x} = (X_s^{t,x})_{s\in[t,T]}$ a solution of $dX_s = b(s,X_s)ds + \sigma(s,X_s)dW_s$, we define

$$\mathcal{G}f(t,x) = \sum_{i=1}^{m} b_i(t,x)\frac{\partial f}{\partial x^i} + \frac{1}{2}\sum_{i,j=1}^{m}(\sigma\sigma^T)_{i,j}(t,x)\frac{\partial^2 f}{\partial x_i \partial x_j}$$

m: dimension of X

The Feynman-Kac theorem state that, if h and c are sufficiently integrable functions and if u is a sufficiently smooth solution solution to the PDE

$$\begin{cases} \frac{\partial u}{\partial t} + \mathcal{G}u + cu = 0 \\ u(T-x) = h(x) \end{cases}$$

Then u has the stochastic representation $u(t,x) = \mathbb{E}\left[h(X_T^{t,x}) \exp\left(\int_t^T c(s, X_s^{t,x})ds\right)\right]$

The theory can be applied to our exercise. In fact, $X_s^{t,x}$ is solution of *

$\Rightarrow b(s,X_s) = k(\theta - X_s)$ and $\sigma(s,X_s) = \lambda$

We can also apply the theorem substituting u with v and considering:

$h(x) = x^2$ $\quad c = 0$ $\qquad$ (satisfy the Hp)

$\Rightarrow$ The theorem states that v, whose stochastic representation is $v(t,x) = \mathbb{E}[(X_T^{t,x})^2]$, satisfies the PDE

$$\begin{cases} \frac{\partial v}{\partial t} + \mathcal{G}v = 0 \\ v(T,x) = x^2 \end{cases} \quad \rightsquigarrow \text{equal to } *$$

Where $\mathcal{G}v = k(\theta - x)\frac{\partial v}{\partial x} + \frac{\lambda^2}{2}\frac{\partial^2 v}{\partial x^2}$ $\quad \rightsquigarrow$ equal to *

$\Rightarrow$ we have derived the PDE * $\quad \square$

- Solve the PDE and derive $\mathbb{E}[X_T^2]$

In order to solve the PDE, we use the Ansatz $\quad V(t,x) \overset{*}{=} a(T-t) + b(T-t)x + c(T-t)x^2$, (derive the equations for a, b and c and solve them).

$$\frac{\partial v}{\partial t} = -a'(T-t) - b'(T-t)x - c'(T-t)x^2$$

$$\frac{\partial v}{\partial x} = b(T-t) + 2c(T-t)x \qquad \frac{\partial v^2}{\partial x^2} = 2c(T-t)$$

Substitute in $\quad \frac{\partial v}{\partial t} + k(\theta - x)\frac{\partial v}{\partial x} + \frac{1}{2}\lambda^2\frac{\partial^2 v}{\partial x^2} = 0$:

$$-a'(T-t) - b'(T-t)x - c'(T-t)x^2 + k(\theta - x)\left[b(T-t) + 2c(T-t)x\right] + \frac{1}{\cancel{2}}\lambda^2 \cancel{2}c(T-t) = 0$$

$a = a(T-t) \qquad b = b(T-t) \qquad c = c(T-t)$

$$-a' - b'x - c'x^2 + k(\theta - x)(b + 2cx) + \lambda^2 c = 0$$
$$-a' - b'x - c'x^2 + k(\theta b - xb + 2\theta c x - 2cx^2) + \lambda^2 c = 0$$
$$-a' - b'x - c'x^2 + k\theta b - kxb + 2k\theta c x - 2kcx^2 + \lambda^2 c = 0$$

All the components (without x, with x, with x^2) must be zero:

$$\begin{cases} -a' + k\theta b + \lambda^2 c = 0 \\ -b' - kb + 2k\theta c = 0 \\ +c' + 2kc = 0 \end{cases} \qquad \begin{cases} 1)\ a' = k\theta b + \lambda^2 c \\ 2)\ b' + kb = 2k\theta c \\ 3)\ c' = -2kc \end{cases}$$

3) $c' = -2kc \Rightarrow$ exponential solution $c(T-t) = \alpha e^{-2k(T-t)}$ $\alpha \in \mathbb{R}$

we need to find $\alpha \Rightarrow$ initial conditions: $t = T \Rightarrow T - t = 0$

$$v(T, x) = x^2$$

$$a(0) + b(0)x + c(0)x^2 = x^2$$

$$\Rightarrow a(0) = 0 \qquad \underline{b(0) = 0} \qquad c(0) = 1$$

Substituting: $c(0) = \alpha = 1 \Rightarrow c(T-t) \overset{*}{=} e^{-2k(T-t)}$

2) $b' + kb = \theta(\underbrace{2kc}_{-c' \text{ (from 3)}}) \Rightarrow b' + kb + \theta c' = 0$

$$b = \frac{-\theta c' - b'}{k}$$

Also in this case, we use initial conditions $(t = T \Rightarrow T - t = 0)$

$$\underset{\overset{\parallel}{0}}{b(0)} = \frac{1}{k}(-\theta c'(0) - b'(0))$$

$$\Rightarrow -\theta c'(0) - b'(0) = 0$$

we already know that $c'(0) = -2k\,c(0) = -2k$

$$\Big\}\ b'(0) = -\theta c'(0) = +\theta 2k$$
$$\underline{b'(0) = +2\theta k}$$

1) $a' = k\theta b + \lambda^2 c \Rightarrow a'(0) = k\theta \underset{\overset{\parallel}{0}}{b(0)} + \lambda^2 \underset{\overset{\parallel}{1}}{c(0)} = \lambda^2$

$$\Rightarrow \underline{a'(0) = \lambda^2}$$

Now let's see the explicit formula of $a(T-t)$, $b(T-t)$, $c(T-t)$:

- $c(T-t) = e^{-2k(T-t)}$
- $b'(T-t)$ $\qquad b' + kb = 2k\theta c$

$$b'(T-t) = 2k\theta e^{-2k(T-t)} - kb$$

Divide the problem in subproblems:
$$\begin{cases} \bar{b}'(T-t) = -k\bar{b} \Rightarrow \bar{b}(T-t) = \bar{\alpha} e^{-k(T-t)} & \bar{\alpha} \in \mathbb{R} \\ \bar{\bar{b}}'(T-t) = 2k\theta e^{-2k(T-t)} \Rightarrow \bar{\bar{b}}(T-t) = -\bar{\bar{\alpha}}\theta e^{-2k(T-t)} & \bar{\bar{\alpha}} \in \mathbb{R} \end{cases}$$

$$b(T-t) \overset{**}{=} \bar{\alpha} e^{-k(T-t)} - \bar{\bar{\alpha}}\theta e^{-2k(T-t)}$$

To find $\bar{\alpha}$ and $\bar{\bar{\alpha}}$ use $\underline{b(0) = 0}$, $\underline{b'(0) = +2\theta k}$:
$$\begin{cases} b(0) = \bar{\alpha} - \theta\bar{\bar{\alpha}} = 0 \\ b'(0) = -k\bar{\alpha} + 2k\bar{\bar{\alpha}}\theta = 2\theta k \end{cases}$$

$$\begin{cases} \bar{\alpha} = \theta\bar{\bar{\alpha}} \\ -\bar{\alpha} + 2\theta\bar{\bar{\alpha}} = 2\theta \end{cases} \qquad \begin{cases} \text{———} \\ -\theta\bar{\bar{\alpha}} + 2\theta\bar{\bar{\alpha}} = 2\theta \end{cases} \rightarrow \begin{array}{l} \bar{\alpha} = 2\theta \\ \bar{\bar{\alpha}} = 2 \end{array}$$

$$b(T-t) = 2\theta e^{-k(T-t)} - 2\theta e^{-2k(T-t)}$$

$$b(T-t) = 2\theta\left[e^{-k(T-t)} - e^{-2k(T-t)}\right]$$

- $a' = k\theta b + \lambda^2 c$

Substituting b and c:

$$a' = 2k\theta^2\left[e^{-k(T-t)} - e^{-2k(T-t)}\right] + \lambda^2 e^{-2k(T-t)}$$

$$a' = 2k\theta^2 e^{-k(T-t)} + (\lambda^2 - 2k\theta^2)\, e^{-2k(T-t)}$$

Divide part with or without λ^2 to simplify things later:

$$a'(T-t) = 2k\theta^2 e^{-k(T-t)}\left(1 - e^{-k(T-t)}\right) + \lambda^2 e^{-2k(T-t)}$$

$$\begin{aligned}\bar{a}'(T-t) &= 2k\theta^2 e^{-k(T-t)}\left(1 - e^{-k(T-t)}\right)\\ &= -2\theta^2\left(-k e^{-k(T-t)}\right) + \theta^2\left(-2k\, e^{-2k(T-t)}\right)\end{aligned}$$

$$\begin{aligned}\bar{a}(T-t) &= -2\theta^2 e^{-k(T-t)} + \theta^2 e^{-2k(T-t)}\\ &= \theta^2\left(-2\, e^{-k(T-t)} + e^{-2k(T-t)}\right)\\ &= -\theta^2 + \theta^2\left(1 - e^{-k(T-t)}\right)^2\end{aligned}$$

$$\bar{\bar{a}}(T-t) = \lambda^2 e^{-2k(T-t)}$$

$$a(T-t) = \bar{\beta}\left(\lambda^2 e^{-2k(T-t)}\right) + \bar{\bar{\beta}}\left(-\theta^2\right) + \bar{\bar{\bar{\beta}}}\theta^2\left(1 - e^{-k(T-t)}\right)^2$$

$$\begin{cases} a(0) = \bar{\beta}\lambda^2 - \bar{\bar{\beta}}\theta^2 = 0 \\ a'(0) = -\bar{\beta}\lambda^2 2k = \lambda^2 \end{cases} \longrightarrow \bar{\beta} = -\frac{1}{2k}$$

$$\bar{\bar{\beta}} = \frac{\bar{\beta}\lambda^2}{\theta^2} = \frac{\frac{\lambda^2}{-2k}}{\theta^2} \quad (\ast)$$

$$-\bar{\bar{\beta}}\theta^2 \overset{\ast}{=} \frac{\lambda^2}{2k}$$

$\bar{\bar{\bar{\beta}}} = 1 \leftarrow$ both in $a(0)$ and $a'(0)$ the part goes away

$\Rightarrow$ we try with 1 and we check if everything works correctly

$$a(T-t)^{\ast\ast\ast} = \frac{\lambda^2}{2k} - \frac{1}{2k}\lambda^2 e^{-2k(T-t)} + \theta^2\left(1 - e^{-k(T-t)}\right)^2$$

Therefore, the functions are:

$$\ast\ast\ast \quad a(T-t) = \frac{\lambda^2}{2k} - \frac{\lambda^2}{2k} e^{-2k(T-t)} + \theta^2 + \theta^2 e^{-2k(T-t)} - 2\theta^2 e^{-k(T-t)}$$

$$\ast\ast \quad b(T\ t) = 2\theta\left[e^{-k(T-t)} - e^{-2k(T-t)}\right]$$

$$\ast \quad c(T-t) = e^{-2k(T-t)}$$

Now we want to check 2 things: that this equations actually solve the system and that the expectation $\mathbb{E}(X_T^2)$ maches with what we have found in the first part.

- The system is given by 1), 2) and 3)

3) $c' = -2Kc$ → we can easily see that this eq is respected

2) $b' = -Kb - \theta c'$: substituting b, b' and c.

$$\hookrightarrow b' = 2\theta\left[K e^{-K(T-t)} - 2K e^{-K(T-t)}\right]$$

$$c' = 2K e^{-2K(T-t)}$$

$$\cancel{2\theta}\left[\cancel{K} e^{-K(T-t)} - 2\cancel{K} e^{-K(T-t)}\right] = -\cancel{K}\cancel{2\theta}\left[e^{-K(T-t)} - e^{-2K(T-t)}\right] - \cancel{\theta}\cancel{2K} e^{-2K(T-t)}$$

$$e^{-2K(T-t)} - 2e^{-K(T-t)} = -e^{-K(T-t)} + e^{-2K(T-t)} - e^{-2K(T-t)}$$ yes!

1) $a' = K\theta b + \lambda^2 c$

$$\Rightarrow \lambda^2 e^{-2K(T-t)} - 2K\theta^2 e^{-2K(T-t)} + 2K\theta^2 e^{-K(T-t)} = 2K\theta^2 e^{-K(T-t)} - 2K\theta^2 e^{-2K(T-t)} + \lambda^2 e^{-2K(T-t)}$$ yes!

- $\mathbb{E}[(X_T^{t,x})^2] = v(t,x) = a(T-t) + b(T-t)x + c(T-t)x^2 =$

$$= \frac{\lambda^2}{2K} - \frac{\lambda^2}{2K} e^{-2K(T-t)} + \theta^2 + \theta^2 e^{-2K(T-t)} - 2\theta^2 e^{-K(T-t)} + 2\theta x\left[e^{-K(T-t)} - e^{-2K(T-t)}\right] + x^2 e^{-2K(T-t)}$$

From part 1 of this exercise we have obtained

$$\mathbb{E}[X_T]^2 = \frac{\lambda^2}{2K}(1 - e^{-2KT}) + \left[x_0^2 e^{-2KT} + \theta^2 + \theta^2 e^{-2KT} - 2\theta^2 e^{-KT} + 2\theta x_0\left(e^{-KT} - e^{-2KT}\right)\right]$$

The solutions from part 1 and 2 match!

Exercise 3 - Heston Model

Price process $(S_t)_{0\le t\le T}$ $\qquad dS_t = r\,S_t\,dt + S_t\,\rho\sqrt{V_t}\,dW_t^{(1)} + S_t\sqrt{1-\rho^2}\sqrt{V_t}\,dW_t^{(2)}$

spot variance V_t modelled by $\quad dV_t = k(\theta - V_t)dt + \sigma\sqrt{V_t}\,dW_t^{(1)}$ $\qquad dW_t^{(1)}dW_t^{(2)} = \rho dt$

$U(t,x,y)$: price at t of a European option payoff Ψ, when $S_t = x$ and $V_t = y$

$$U(t,x,y) \overset{*}{=} e^{-r(T-t)}\,\mathbb{E}^{\mathbb{Q}}[\Psi(S_T^{t,x,y})]$$

Derive the PDE satisfied by U:

To solve this exercise we can use 2 approaches: ① ②

① Martingala has zero drift

From derivatives' theory, we know that the discounted price of a derivative is a martingala under the risk-neutral measure.
In this exercise we know that $\mathbb{Q}$ is a risk-neutral measure (equivalent martingala measure) $\rightarrow$ i.e. a probability measure which brings to a fair game $\mathbb{E}^{\mathbb{Q}}[U_t|\mathcal{F}_s] = U_s \quad s<t$

$\Rightarrow$ The discounted value of the derivative U under $\mathbb{Q}$ is a martingala
$\Rightarrow$ To find the PDE satisfied by U we must set to zero the drift part.

First of all we use Ito's theorem to find the discounted value of U:

$$d(e^{-r(T-t)}\underbrace{U(t,\underset{x}{S_t},\underset{y}{V_t})}_{U:=U(t,x,y)}) = -re^{-r(T-t)}dt\,U + e^{-r(T-t)}dU$$

$$dU(t,x,y) = \frac{dU}{dt} + \frac{dU}{dx}dS_t + \frac{dU}{dy}dV_t + \frac{1}{2}\frac{d^2U}{dx^2}d\langle S\rangle_t + \frac{1}{2}\frac{d^2U}{dy^2}d\langle V\rangle_t + \frac{d^2U}{dxdy}d\langle S,V\rangle_t$$

Now we find $d\langle S\rangle_t$, $d\langle V\rangle_t$ and $d\langle S,V\rangle_t$

$$(*)\begin{cases} d\langle S\rangle_t = \left(r\,S_t\,dt + S_t\,\rho\sqrt{V_t}\,dW_t^{(1)} + S_t\sqrt{1-\rho^2}\sqrt{V_t}\,dW_t^{(2)}\right)^2 = \\ \quad = \rho^2 S_t^2 V_t\,dt + S_t^2(1-\rho^2)V_t\,dt = S_t^2 V_t\,dt \\ d\langle V\rangle_t = \left(k(\theta - V_t)dt + \sigma\sqrt{V_t}\,dW_t^{(1)}\right)^2 = \sigma^2 V_t\,dt \\ d\langle S,V\rangle_t = \left(r\,S_t\,dt + S_t\,\rho\sqrt{V_t}\,dW_t^{(1)} + S_t\sqrt{1-\rho^2}\sqrt{V_t}\,dW_t^{(2)}\right)\left(k(\theta - V_t)dt + \sigma\sqrt{V_t}\,dW_t^{(1)}\right) = \\ \quad = S_t\,\rho\sigma V_t\,dt \end{cases}$$

Remember that $W^{(1)}$ and $W^{(2)}$ are indipendent

Therefore,

$$dU(t,x,y) = -re^{-r(T-t)}U\,dt + e^{-r(T-t)}\left[\frac{dU}{dt}dt + \frac{dU}{dx}dS_t + \frac{dU}{dy}dV_t + \frac{1}{2}\frac{d^2U}{dx^2}(x^2y\,dt) + \frac{1}{2}\frac{d^2U}{dy^2}\sigma^2 y\,dt + \right.$$
$$\left. + \frac{d^2U}{dxdy}\rho\sigma xy\,dt\right]$$

$$dS_t = r\,S_t\,dt + S_t\,\rho\sqrt{V_t}\,dW_t^{(1)} + S_t\sqrt{1-\rho^2}\sqrt{V_t}\,dW_t^{(2)}$$
$$dV_t = k(\theta - V_t)dt + \sigma\sqrt{V_t}\,dW_t^{(1)}$$

Now, we impose drift $=0$ and we take away $e^{-r(T-t)}$ because it is not relevant:

$$-rU+\frac{dU}{dt}+\frac{dU}{dx}rx+\frac{dU}{dy}\kappa(\theta-y)+\frac{1}{2}\frac{d^2U}{dx^2}x^2y+\frac{1}{2}\frac{d^2U}{dy^2}\sigma^2y+\frac{d^2U}{dxdy}\rho\sigma xy=0$$

$$\left[rU=\frac{dU}{dt}+\frac{dU}{dx}rx+\frac{dU}{dy}\kappa(\theta-y)+\frac{1}{2}\frac{d^2U}{dx^2}x^2y+\frac{1}{2}\frac{d^2U}{dy^2}\sigma^2y+\frac{d^2U}{dxdy}\rho\sigma xy\right]$$

Boundary conditions: $U(T,x,y)=\varphi(x)$ $\leftarrow$ From $U(t,x,y)=\underbrace{e^{-r(T-t)}}_{=1 \text{ if } T=t}\underbrace{\mathbb{E}^{\mathbb{Q}}}_{\text{goes away}}[\varphi(S_T^{t,x,y})]$

② Feynman-Kac Theorem

At first glance, equation $*$ recalls the solution of a PDE given by the Feynman-Kac theorem.
However, in this case, we would like to work conversely: from the solution we need to find the PDE.
Therefore, we can apply the converse version of the FK theorem:
(from the Stochastic calculus course)

Let $r(x,t)$ be a continuous function satisfying

$$\mathbb{E}_t^{\mathbb{Q}}[e^{-\int_t^T r(s,X_s)ds}|\phi(X_T)|]<\infty \quad \text{for all } t\le T \text{ and } x$$

Assume $f(t,x)=\mathbb{E}_t^{\mathbb{Q}}[e^{-\int_t^T r(s,X_s)ds}\phi(X_T)]$ is $C^{1,2}$.
Then $f(t,X_t)$ is a martingale and f solves the PDE

$$\frac{\partial f(t,x)}{\partial t}+\mathcal{G}f(t,x)-r(t,x)f(t,x)=0$$
$$f(T,x)=\phi(x)$$

See the definition of operator $\mathcal{G}$ at slide 20 of lecture 2b:

$$\mathcal{G}f(t,x)=\sum_{i=1}^{m}b_i(t,x)\frac{\partial f}{\partial x^i}+\frac{1}{2}\sum_{i,j=1}^{m}(\sigma\sigma^T)_{ij}(t,x)\frac{\partial^2 f}{\partial x^i\partial x^j}$$

In our case $m=2$ and

$$dS_t=\underbrace{rS_t}_{b_1}dt+\underbrace{S_t\rho\sqrt{V_t}}_{\sigma_{11}}dW_t^{(1)}+\underbrace{S_t\sqrt{1-\rho^2}\sqrt{V_t}}_{\sigma_{12}}dW_t^{(2)}$$
$$dV_t=\underbrace{\kappa(\theta-V_t)}_{b_2}dt+\underbrace{\sigma\sqrt{V_t}}_{\sigma_{21}}dW_t^{(1)}$$

Therefore, let us apply this contrary version of the F-K theorem: $r(s,X_s)=r$ $\leftarrow$ constant, $\phi=\varphi$, $f=U$

$\Rightarrow$ r is a continuous function because it is a constant
and $\mathbb{E}[e^{-r(T-t)}|\varphi(S_T^{t,x,y})|]<\infty \quad t\le T$

$\Rightarrow U(t,x,y)$ solves the PDE given by

$$\frac{\partial U}{\partial t}+\mathcal{G}U+(-r)U=0$$
$$U(T,x,y)=\varphi(x)$$

$$\mathcal{G}U(t,x,y)=\frac{dU}{\partial x}rx+\frac{dU}{\partial y}\kappa(\theta-y)+\cancel{\frac{1}{2}\overbrace{x^2\rho y}^{\sigma_{11}\sigma_{11}}\frac{\partial^2U}{\partial x^2}}+\frac{1}{2}\overbrace{x^2(1-\rho^2)y}^{\sigma_{12}\sigma_{12}}\frac{d^2U}{\partial x^2}+\frac{1}{2}\overbrace{\sigma^2y}^{\sigma_{21}\sigma_{21}}\frac{d^2U}{dy^2}+\overbrace{x\sigma\rho y}^{\sigma_{11}\sigma_{21}}\frac{\partial^2U}{\partial x\partial y}$$

Which brings to:

$$\frac{\partial u}{\partial t} + \frac{\partial u}{\partial x} r x + \frac{\partial u}{\partial y} k(\theta - y) + \frac{1}{2} x^2 y \frac{\partial^2 u}{\partial x^2} + \frac{1}{2} \sigma^2 y \frac{\partial^2 u}{\partial y^2} + x \sigma \rho y \frac{\partial^2 u}{\partial x \partial y} = 0$$

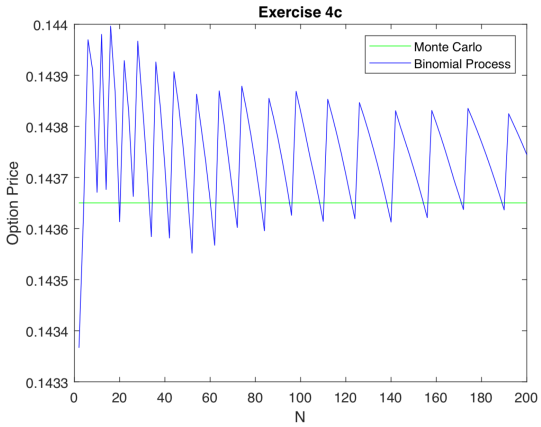
$$u(T, x, y) = \Psi(x)$$

$\Rightarrow$ PDEs of approaches 1 and 2 concide.

Exercise 4 - Up and Out option

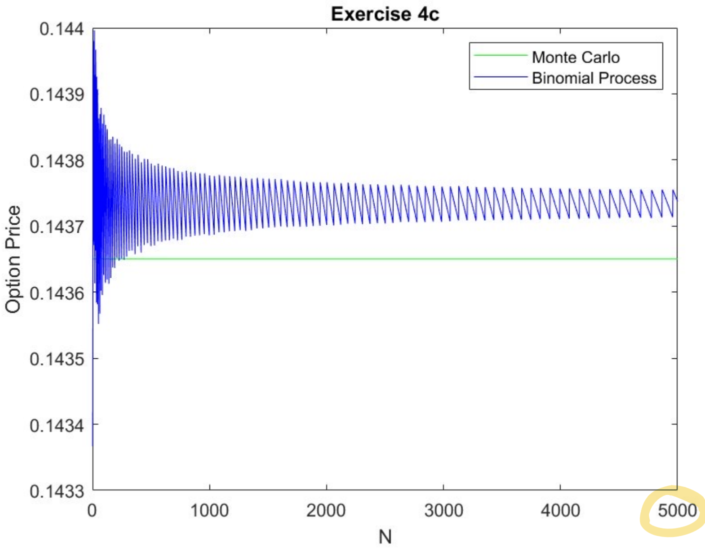
a) b) See code (both in matlab and pdf formats)

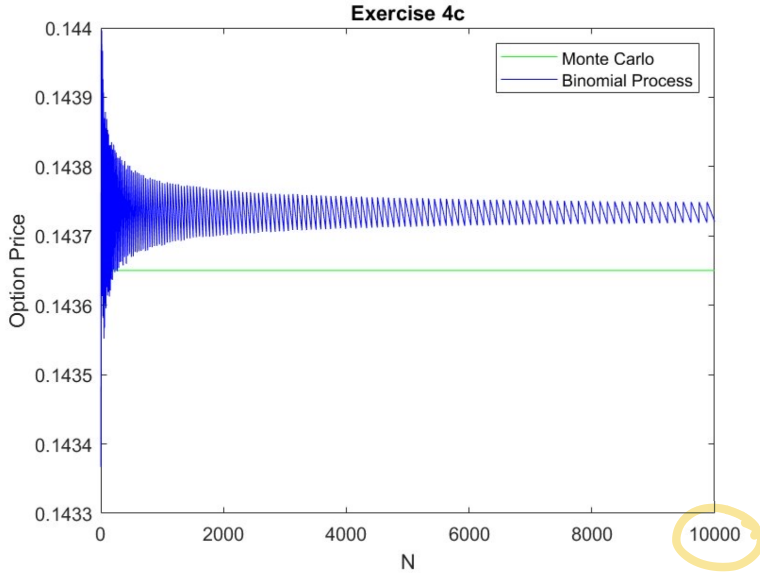
c) The plot is:



With the MC simulation we split the time interval $[0, T]$ in N_{time} Δt and we simulate N_{sim} paths. The payoff of all of these simulations contributes to find a singular value P_{MC}, which is constant. Therefore, the green constant line, that is shown above, is already the results of a simulation of N_{sim} different paths, with $N_{sim} = 10^6$.

On the other hand, the binomial price is the result of an iterative process in which step is a step in time (discrete model). Remembering the binomial model, at each step there is a certain probability of going down and another one of going up. To obtain a multiperiod binomial model we have to collect a certain number (200 in this case) of one period binomial models and we have to repeat the same for each of them. The blue line above is the result of this kind of model: at each step the option price depends on the previous step plus an up or down contribution.
With only 200 steps we cannot conclude that this model converges. However, increasing the number of steps, the noise should decay. In order to prove that, we run the same model substituting 200 with $N = 5000$ and $N = 10000$.
Even if the option price tends to stabilize between a certain interval, we do not obtain a numerical price in our plots as using a Monte Carlo simulation.
Increasing N even more it might converge


Exercise 4c
Monte Carlo
Binomial Process
Option Price
0.144
0.1439
0.1438
0.1437
0.1436
0.1435
0.1434
0.1433
0
1000
2000
3000
4000
5000
N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10000
N