Computational Finance
Take Home Exam 2

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Exercise 2 - Ornstein Uhlenbeck process

OU process is the solution of the SDE: $dX_t \overset{*}{=} k(\theta - X_t)\,dt + \lambda\, dW_t$

If we let $v(t,x) := \mathbb{E}\left[e^{i\nu X_T} \mid X_t = x\right]$, then v satisfies
$= \mathbb{E}_t\left[e^{i\nu X_T}\right]$

$$\begin{cases} v_t + \mathcal{G}v = 0 \\ v(T) = e^{i\nu x} \end{cases} \qquad \text{with} \quad \mathcal{G}v = k(\theta - x)v_x + \frac{\lambda^2}{2} v_{xx}$$

Suppose that $v(t,x) = e^{\varphi(T-t,\nu) + \psi(T-t,\nu)x}$ (*)

a) Deduce the system of ODEs for the functions φ and ψ:

First, we can observe that X is an affine process because its conditional characteristic function has this * form.
Thanks to the fact we are working with an affine process, the solution is a relatively simple ODE.
We can recall the slide 24 of Lecture 4: the functions φ and ψ solve the system of RICCATI equations

$$dX_t = k(\theta - X_t)\,dt + \lambda\, dW_t$$
$$dX_t = (a + bX_t)\,dt + c\sqrt{X_t}\,dB_t$$
$$a = k\theta$$
$$b = -k$$

Apply the Ito formula to $Y_t = v(t, X_t)$:

$$Y_t = Y_0 + \int_0^t \underbrace{\gamma(s, X_s)}\,ds + \int_0^t v_x(t, X_s)$$

$$\gamma(t,x) = v_t(t,x) + v_x(t,x)\,k(\theta - x) + \frac{1}{2} v_{xx} \lambda^2$$

Using the law of iterated expectations: $Y_s = \mathbb{E}_s\left[e^{i\nu X_T}\right] = \mathbb{E}_s\left[\mathbb{E}_t\left[e^{i\nu X_T}\right]\right] = \mathbb{E}_s\left[Y_t\right]$
$\Rightarrow Y$ is a martingale $\Rightarrow Y$ has zero drift a.e.

Hence, we obtain this PDE: $v_t(t,x) + v_x(t,x)\,k(\theta - x) + \frac{1}{2} v_{xx} \lambda^2 = 0$

Boundary condition: $v(x,T) = e^{i\nu x}$

We can now substitute using * $\left(v(t,x) = e^{\varphi(T-t,\nu) + \psi(T-t,\nu)x}\right)$:

To simplify the notation:
$\varphi(T-t,\nu) = \varphi$
$\psi(T-t,\nu) = \psi$

$$v_t(t,x) = \left(-\varphi' - \psi' x\right) v(t,x) \qquad v_{xx}(t,x) = \psi^2 v(t,x)$$
$$v_x(t,x) = \psi\, v(t,x)$$

Plugging into the PDE:

$$(-\varphi' - \psi' x)\,v(t,x) + \psi\, v(t,x)\,k(\theta - x) + \frac{1}{2}\lambda^2 \psi^2 v(t,x) = 0$$

$$v(t,x)\left(-\varphi' - \psi' x + \psi k(\theta - x) + \frac{1}{2}\lambda^2\psi^2\right) = 0 \qquad \text{Divide by } v(t,x)$$

$$-\varphi' \underline{- \psi' x} + \psi k \theta \underline{- \psi k x} + \frac{1}{2}\lambda^2 \psi^2 = 0 \quad (*)$$

We collect terms in x: $\begin{cases} \psi' + \psi K = 0 & \leftarrow \text{with } x \\ \varphi' - \psi K\theta - \frac{1}{2}\lambda^2\psi^2 = 0 & \leftarrow \text{without } x \end{cases}$

$$\begin{cases} \psi'(T-t,\nu) = -\psi(T-t,\nu)K \\ \varphi'(T-t,\nu) = \psi(T-t,\nu)K\theta - \frac{\lambda^2}{2}\psi^2(T-t,\nu) \end{cases}$$

Boundary conditions: from $v(x,T) = e^{i\nu x}$ we get: $e^{\varphi(T-t,\nu)+\psi(T-t,\nu)x} = e^{i\nu x}$

$$\Rightarrow \quad \psi(0,\nu) = i\nu$$
$$\varphi(0,\nu) = 0$$

b) Solve the system (1) and write (2) explicitly the form of the characteristic function of X_t given $X_0 = x$:

(1) Solve the system: $T-t=s$

$$\begin{cases} \psi'(s,\nu) = -\psi(s,\nu)K \quad \left(\psi' = \frac{\partial\psi}{\partial s}\right) \\ \psi(0,\nu) = i\nu \end{cases} \longrightarrow \psi(s) = c\,e^{-Ks} \quad c \in \mathbb{R}$$

$$\psi(0) = c\,1 = i\nu \Rightarrow c = i\nu$$

$$\Rightarrow \psi(s) \overset{*}{=} i\nu e^{-Ks}$$

$$\varphi'(s,\nu) = \psi(s,\nu)K\theta - \frac{\lambda^2}{2}\psi^2(s,\nu) = K\theta i\nu e^{-Ks} - \frac{\lambda^2}{2}\overbrace{(i\nu)^2}^{-\nu^2} e^{-2Ks}$$

$$\begin{cases} \varphi'(s,\nu) = i\theta\nu(Ke^{-Ks}) + \frac{1}{2}\lambda^2\nu^2 e^{-2Ks} \\ \varphi(0,\nu) = 0 \end{cases}$$

considering the boundary conditions require that $\varphi(0,\nu)=0$, to obtain a derivative equal to Ke^{-Ks}, we will not use $-e^{-Ks}$, but $1-e^{-Ks}$.
Same reasoning for e^{-2Ks}.

$$\varphi(s,\nu) = d_1 i\theta\nu(1-e^{-Ks}) + d_2\frac{1}{2}\lambda^2\nu^2\frac{1-e^{-2Ks}}{2K} \qquad d_1, d_2 \in \mathbb{R}$$

$$\varphi(0,\nu) = d_1 i\theta\nu(1-1) + d_2\lambda^2\nu^2\frac{1-1}{4K} = 0 \qquad \text{satisfied } \forall\, d_1, d_2 \in \mathbb{R}$$

Therefore, we need other conditions to find d_1 and d_2. We decide to get back to (*) and plot our results for $\psi(s,\nu)$ and $\varphi(s,\nu)$:

$$-\varphi' - \psi' x + \psi K\theta - \psi K x + \frac{1}{2}\lambda^2\psi^2 = 0 \qquad \psi(s) \overset{*}{=} i\nu e^{-Ks}$$

$$-\varphi' + i\nu K e^{-Ks}x + i\nu e^{-Ks}K\theta - i\nu e^{-Ks}x - \frac{1}{2}\nu^2\lambda^2 e^{-2Ks} = 0$$

$$d_1 i\theta\nu K e^{-Ks} + d_2\lambda^2\nu^2\frac{\cancel{2K}e^{-2Ks}}{2\cancel{4K}} =$$

$$\cancel{i\nu K e^{-Ks}x} + i\nu e^{-Ks}K\theta - \cancel{i\nu e^{-Ks}x} - \frac{1}{2}\nu^2\lambda^2 e^{-2Ks}$$

$$d_1 i\theta\nu K e^{-Ks} + d_2\frac{\lambda^2}{2}\nu^2 e^{-2Ks} = i\nu e^{-Ks}K\theta - \frac{\lambda^2}{2}\nu^2 e^{-2Ks}$$

$$\Rightarrow d_1 = 1 \quad \text{and} \quad d_2 = -1$$

$$\Rightarrow \begin{cases} \psi(s,\nu) = i\nu e^{-Ks} \\ \varphi(s,\nu) = i\theta\nu(1-e^{-Ks}) - \lambda^2\nu^2\frac{1-e^{-2Ks}}{4K} \end{cases}$$

Therefore, $v(t,x) = \exp\left[i\theta\nu(1-e^{-\kappa(T-t)}) - \lambda^2\nu^2\frac{1-e^{-2\kappa(T-t)}}{4\kappa} + x i\nu e^{-\kappa(T-t)}\right]$

(2) we want to find an explicit formula of $\mathbb{E}[e^{i\nu X_T} | X_0 = x]$

$$v(t,x) := \mathbb{E}\left[e^{i\nu X_T} | X_t = x\right]$$

$$\Rightarrow v(0,x) = \mathbb{E}\left[e^{i\nu X_T} | X_0 = x\right]$$

$$\Rightarrow \mathbb{E}[e^{i\nu X_T} | X_0 = x] = \exp\left[i\theta\nu(1-e^{-\kappa T}) - \lambda^2\nu^2\frac{1-e^{-2\kappa T}}{4\kappa} + x i\nu e^{-\kappa T}\right]$$

c) Deduce that X_t is normally distributed. Write expicitly the mean and variance of X_t.

First, we recall the characteristic function of the normal distribution $N(\mu,\sigma^2)$:

$$\phi_{normal}(t) = e^{+i\mu t - \frac{1}{2}\sigma^2 t^2}$$

In general, by definition, the characteristic function is given by $\varphi_X(t) = \mathbb{E}[e^{itX}]$.
For this exercise, $v(t,x) = \mathbb{E}[e^{i\nu X_T} | X_t = x]$ $\Rightarrow$ the characteristic function of OU is a function of ν

Then, if X_t is the previous OU process: $\phi_{X_t}(\nu) = e^{i\nu(e^{-\kappa t}x + \theta(1-e^{-\kappa t})) - \frac{\nu^2\lambda^2}{4\kappa}(1-e^{-2\kappa t})}$

we need to prove that for certains μ and σ, X_t is normally distributed
To simplify the notation, we work with the exponentials:

normal: $i\,t\,(\mu) - \frac{1}{2}\,\sigma^2\,t^2$

OU: $i\,\nu\,(e^{-\kappa t}x + \theta(1-e^{-\kappa t}))) - \frac{1}{2}\,\nu^2\,\frac{\lambda^2}{2\kappa}(1-e^{-2\kappa t}))$

$$\Rightarrow * \begin{cases} \mu = e^{-\kappa t}x + \theta(1-e^{-\kappa t}) \\ \sigma^2 = \frac{\lambda^2}{2\kappa}(1-e^{-2\kappa t}) \end{cases}$$

To conclude that this correspondence is sufficient to show that X_t is normally distributed, we need to remember that the charateristic function uniquely determines the distribution function. Therefore, recognising the characteristic function of a random variable identifies its distribution function. For the OU process, we recognised the characteristic function of a normal distribution if * is respected, then the OU process is normally distributed, such that

$$X_t | X_0 = x \sim N\left(e^{-\kappa t}x + \theta(1-e^{-\kappa t}), \frac{\lambda^2}{2\kappa}(1-e^{-2\kappa t})\right)$$

$$dX_t \overset{*}{=} \kappa(\theta - X_t)\,dt + \lambda\, dW_t$$

d) Using Itô formula solve * expicitly and explain why this is consistent with the results in the previous part.

To solve * consider $f(t, X_t) = X_t e^{\kappa t}$ where X_t is the solution.
Applying the Ito formula, we obtain

$$
\begin{aligned}
df(t, X_t) &= \kappa X_t e^{\kappa t} dt + e^{\kappa t} dX_t \\
&= \kappa X_t e^{\kappa t} dt + e^{\kappa t}\left(\kappa(\theta - X_t)\,dt + \lambda\, dW_t\right) \\
&= \kappa e^{\kappa t}\left[\cancel{X_t} + \theta - \cancel{X_t}\right] dt + e^{\kappa t}\lambda\, dW_t
\end{aligned}
$$

$$d(X_t e^{\kappa t}) = \kappa\theta e^{\kappa t} dt + e^{\kappa t}\lambda\, dW_t$$

$$\int_0^t d(X_t e^{\kappa t}) = \theta \int_0^t \kappa e^{\kappa s} ds + \lambda \int_0^t e^{\kappa s} dW_s$$

$$X_t e^{\kappa t} = X_0 + \theta(e^{\kappa t} - 1) + \lambda \int_0^t e^{\kappa s} ds$$

divided by $e^{-\kappa t}$

$$X_t = X_0 e^{-\kappa t} + \theta(1 - e^{-\kappa t}) + \lambda \int_0^t e^{\kappa(s-t)} ds$$

Note that the solution is a sum of deterministic terms and an integral of a deterministic function with respect to a Wiener process with normally distributed increments (*). Therefore, the process is Gaussian.

(*) NB: a Wiener process has Gaussian increments by definition: $W_t - W_s \sim N(0, t-s)$ $t \geq s$
Moreover,
if f is square integrable on $[0,t] \Rightarrow \int_0^t f(s)\,dW_s \sim N\left(0, \int_0^t f^2(s)\,ds\right)$

In our case: $f(s) = \lambda e^{(s-t)}$ and it is square integrable

$$\Rightarrow \int_0^t \lambda e^{\kappa(s-t)} dW_s \sim N\left(0, \int_0^t \lambda^2 e^{2\kappa(s-t)} ds\right)$$

• Mean: $\mathbb{E}[X_t] = \mathbb{E}\Big[e^{-\kappa t} X_0 + \theta(1 - e^{-\kappa t}) + \underbrace{\int_0^t \lambda e^{\kappa(s-t)} dW_s}_{\text{zero mean}}\Big] = e^{-\kappa t} X_0 + \theta(1 - e^{-\kappa t})$

• Variance: $V(X_t) = V\Big(\overbrace{e^{-\kappa t} X_0 + \theta(1 - e^{-\kappa t})}^{\text{deterministic}} + \int_0^t \lambda e^{\kappa(s-t)} dW_s\Big) =$

$$
\begin{aligned}
&= V\left(\int_0^t \lambda e^{\kappa(s-t)} dW_s\right) = \int_0^t \lambda^2 e^{2\kappa(s-t)} ds = \lambda^2 e^{-2\kappa t} \int_0^t e^{2\kappa s} ds = \\
&= \lambda^2 e^{-2\kappa t} \left.\frac{e^{2\kappa s}}{2\kappa}\right|_0^t = \lambda^2 e^{-2\kappa t} \frac{1}{2\kappa}\left(e^{2\kappa t} - 1\right) = \\
&= \frac{\lambda^2}{2\kappa}\left(1 - e^{-2\kappa t}\right)
\end{aligned}
$$

In our case we need the conditional mean and variance:

$$\mathbb{E}[X_t \mid X_0 = x_0] = e^{-kt} x_0 + \theta(1-e^{-kt})$$
$$V(X_t \mid X_0 = x_0) = \frac{\lambda^2}{2k}(1-e^{-2kt})$$

$$X_t \mid X_0 = x_0 \sim N\left(e^{-kt} x_0 + \theta(1-e^{-kt}), \frac{\lambda^2}{2k}(1-e^{-2kt})\right)$$

This is exactly the same result we have found using the characteristic function.

Exercise 4 - Calibration Heston Model

Heston Model: from slide 26 of Lecture 1

$$dS_t = r S_t + \sqrt{V_t}\, S_t W_t^{(1)}$$
$$dV_t = k(\theta - V_t)dt + \sigma\sqrt{V_t}\, dW_t^{(2)}$$

S_t: stock price
V_t: spot variance
$W_t^{(1)}, W_t^{(2)}$: BM with correlation ρ

Parameters of the Heston model: θ, k, ρ

We need to calibrate the model using the prices of the European call options

$\Rightarrow$ find the parameters that minimize the root-mean-squared error of the difference: Heston prices $-$ *observed prices*

In other words, using slides of lecture 4 regarding the calibration problem

- empirical market prices: $\hat{C}(T_j, k_j)$ $\quad j = 1:N$
- option price of the stoch parametric model S^q: $C^q(T, k)$

we need to find the parameter q

$$\Rightarrow q^* = \underset{q}{\operatorname{argmin}}\ \operatorname{dis}\left((\hat{C}(T_j,k_j))_{j=1:N}, (C(T_j,k_j))_{j=1:N}\right)$$

We start our problem loading the data and splitting the columns.
Then, we set r and S_0
Then, in order to use the function "fminsearchcon.m" to implement the optimization routine, we need to set the initial parameters of the model, the lower and the upper bound. In our case:

INITIAL VALUES	LOWER BOUND	UPPER BOUND
$\theta = 0{,}04$	$-\infty$	$+\infty$
$k = 1{,}5$	0	$+\infty$
$\sigma = 0{,}3$	0	$+\infty$
$\rho = -0{,}6$	-1	1
$V_0 = 0{,}0441$	0	$+\infty$

even if there is empirical evidence that $\rho < 0$ (slide 26, first lecture), we cannot conclude that a priori
$\Rightarrow \rho \in [-1, 1]$
However, we will probably get a negative value

We also need to pass as argument the function to minimize, the $\underbrace{\text{distance function}}_{\text{distance\_prices}}$.

distance_prices : function that calculate the $\underbrace{\text{square error}}$

minimization of the root-mean-squared error equivalent to minimization of the squared error

↳ This function has some restrictions regarding the form it can have: In order to be passed as argument to fminsearchcon, it has to accept a vector x as input and return a scalar function value evaluated at x.

⇒ function [error] = distance_prices (x)

Results:

Optimal parameters:
Nu = 0.013210
Kappa = 4.644733
Sigma = 0.245656
Rho = -0.672776
V = 0.011532

Optimization routine iterations: 509.000000

Message:
Optimization terminated:
the current x satisfies the termination criteria using OPTIONS.TolX of 1.000000e-04
and F(X) satisfies the convergence criteria using OPTIONS.TolFun of 1.000000e-04

See code in the PDF