Computational Finance
FIN-472
Homework 4 and Take Home Exam 2

October 09, 2020

- **The exercises of Take Home Exam 2 are marked by (*). They are Exercises 2 and 4.**
- **Please upload your solutions on moodle by Friday 23.10.2020 at 8h15.**
- **You should upload one PDF file containing the solutions to the theoretical exercises as well as one PDF file with the Matlab code and results you obtained for Exercises 4.**
- **In addition, for Exercise 4, do not forget to upload in moodle the Matlab codes. You should submit only one Matlab file for each part.**
- **For the theoretical exercises you can either scan your handwritten solutions or write them using a math typesetting software such as LaTeX.**

Exercise 1: For the

- call option (payoff function defined as $g(x) = (S_0 e^x - K)^+$ for a certain initial price $S_0 = e^{x_0}$ and strike K), and the
- put option (payoff function defined as $g(x) = (K - S_0 e^x)^+$),

find the real values for the dampening parameter η such that $g_\eta := e^{\eta x} g(x) \in L^1 \cap L^2(\mathbb{R})$ and derive the corresponding Fourier transform $\hat{g}_\eta(z)$.

Exercise 2: (*) part of Take Home Exam 2 (*)
The *Ornstein Uhlenbeck process* is defined as a solution of the Stochastic Differential Equation (SDE)

$$dX_t = \kappa(\theta - X_t)dt + \lambda dW_t. \tag{1}$$

If we let

$$v(t,x) := \mathbb{E}[\exp(i\nu X_T)|X_t = x]$$

then v satisfies the PDE

$$v_t + \mathcal{G}v = 0$$

with terminal condition $v(T,x) = \mathrm{e}^{i\nu x}$, where

$$\mathcal{G}v = \kappa(\theta - x)v_x + \frac{\lambda^2}{2}v_{xx}.$$

Suppose that

$$v(t,x) = \exp(\varphi(T-t,\nu) + \psi(T-t,\nu)x).$$

a) Deduce a system of Ordinary Differential Equations (ODEs) for the functions φ and ψ.

b) Solve this system and write explicitly the form of the characteristic function of X_t given $X_0 = x$.

c) Deduce that X_t is normally distributed. Write explicitly the mean and variance of X_t.

d) Using Itô's formula solve (1) explicitly and explain why this is consistent with the results in the previous part.

Exercise 3: Using the Variance Gamma model and the following parameters

$$S_0 = 100, \qquad \nu = 0.2, \qquad \theta = -0.14, \qquad r = 0.1, \qquad \sigma = 0.12$$

a) Compute the price of the European put options with maturities

$$T \in \{1/12, 1/6, 1/4, 1/2, 1, 2, 3, 5\}$$

and strikes

$$K \in \{50, 80, 90, 95, 100, 105, 110, 120, 150\},$$

using the FFT approach proposed by Carr and Madan (1999) with $e^{\alpha k}$ as damping factor. Here e^k denotes the strike of the option.

b) Graph the implied volatility surface obtained in the previous part. This is the graph of the implied volatilities as a function of K and T. Recall that the implied volatility is the value $\sigma(K, T)$ such that

$$P(K, T) = P^{BS}(K, T; \sigma(K, T))$$

where $P(K, T)$ is the price of the put (in this case obtained by the FFT method) and $P^{BS}(K, T; \sigma^{BS})$ is the price of a put in the Black-Scholes model with parameter σ^{BS}. If you want, you can use the function `blsimpv` already implemented in Matlab.

c) Redo a) and b), but this time use the Simpson rule instead of the trapezoidal rule. Compare the results.

Exercise 4: (*) part of Take Home Exam 2 (*)
Calibrate the Heston model using the prices of the European call options on the S&P 500 observed on January 3, 2005. The prices are available in the file `call_20050103.mat`, where the call prices, strikes, time-to-maturities (in days) and implied volatilities are in the first, second, third and fourth column, respectively. To calibrate the model, find the Heston parameters that minimize the root-mean-squared error of the differences between the Heston prices and the observed prices. For the interest rate and the price of the S&P 500, take $r = 0.015$ and $S = 1202.10$. If needed, you can use the Matlab functions `Call_Heston.m` and `fminsearchcon.m` for the pricing and optimization routine, respectively. Finally, depending on the optimization technique chosen, you can take the following initial values for the Heston parameters

$$\theta = 0.04, \quad \kappa = 1.5, \quad \sigma = 0.3, \quad \rho = -0.6, \quad V_0 = 0.0441.$$