Take Home Exam 5 Computational Finance

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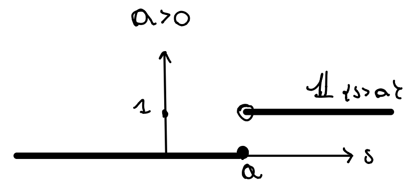
European digital option payoff : $\varphi: \mathbb{R}^+ \longrightarrow \mathbb{R}$

$s \quad \mathbb{1}_{\{s>a\}}$



log-price process $(X_t)_{0\leq t\leq T}$: $X_t = \gamma t + \sigma W_t + \sum_{j=1}^{N_t} J_k$ $\quad r=0$

- N_t: Poisson distributed RV with parameter λt $\quad \ln\left(\frac{S_t}{S_0}\right) = X_t$
- $J_k \overset{iid}{\sim} N(\alpha, \beta^2)$
- $\gamma = -\frac{1}{2}\sigma^2 - \lambda\left(e^{\alpha+\frac{1}{2}\beta^2} - 1\right)$ $\Rightarrow$ risk neutral model

a) From exercise 1c of homework 3:

$$\hat{P}_{X_t}(z) = \mathbb{E}\left[e^{iX_t z}\right] = \mathbb{E}\left[e^{iz\left(\gamma t + \sigma W_t + \sum_{k=1}^{N_t} J_k\right) z}\right]$$

$$= e^{iz\gamma t - \frac{1}{2}\sigma^2 z^2 t + \lambda t \int_{\mathbb{R}}\left(e^{ixz}-1\right)\frac{1}{\sqrt{2\pi}\beta} e^{-\frac{(x-\alpha)^2}{2\beta^2}} dx}$$

$$= \exp\left\{t\left(iz\gamma - \tfrac{1}{2}\sigma^2 z^2 + \lambda\left(\hat{P}_{J_k}(z) - 1\right)\right)\right\}$$

$$(*) \quad \hat{P}_{X_t}(z) = \exp\left\{t\left(iz\gamma - \tfrac{1}{2}\sigma^2 z^2 + \lambda\left(e^{iz\alpha - \frac{1}{2}\beta^2 z^2} - 1\right)\right)\right\}$$

characteristic function of Fourier transform of X : $\hat{P}_X(z) = \mathbb{E}\left[e^{i\langle z, X\rangle}\right]$

Stock price process $S_t = S_0 e^{X_t} \Rightarrow S_t = e^{X_t + x_0}$

In particular $\mathbb{E}\left[e^{-rt} S_t\right] = e^{-rt} S_0 \underbrace{\mathbb{E}\left(e^{X_t}\right)}_{=\hat{P}_{X_t}(-i)} = S_0$ (from Homework 3)

Price of an option at $t=0$: $\Pi_0(S_0) = \mathbb{E}\left[\phi(S_T)\right] = \mathbb{E}\left[\phi\left(S_0 e^{X_T}\right)\right]$

Payoff φ : $\mathbb{1}_{S_0 e^{X_t} > a}$ and it can be rewritten as $\mathbb{E}\left[g(X_T + x_0)\right]$

After this introduction, let us remind the 3 conditions of theorem 22:

$g_\eta(x) = e^{\eta x} g(x)$ $(*)$

Hp)

$P^{-\eta}_{X_T}(dx) = e^{-\eta x} P_{X_T}(dx)$ $(*)$

(A1) $g_\eta \in L^1(\mathbb{R})$

(A2) $\mathbb{E}\left[e^{-\eta X_T}\right] < \infty$

(A3) $u \longmapsto \hat{g}_\eta(u)\,\hat{P}^{-\eta}_{X_T}(-u) \in L^1(\mathbb{R})$

If they are respected, we have $\mathbb{E}\left[g(x^0 + X_T)\right] = \frac{e^{-\eta x^0}}{2\pi}\int_{\mathbb{R}} e^{iux^0}\,\hat{P}_{X_T}(u+i\eta)\,\hat{g}(-u-i\eta)\,du$

for a certain g to be found.

We want to derive the Fourier pricing formula. Therefore, according to the previous formula, g is the log payoff :

$$g(x) = \mathbb{1}_{\{x > \log a\}}$$

(A1) Considering $*$, $g_\eta \in \mathcal{L}^1(\mathbb{R}) \Leftrightarrow g \in \mathcal{L}_\eta^1(\mathbb{R})$

$$g(x) = \mathbb{1}_{\{x > \log(a)\}} \Rightarrow g_\eta(x) = e^{\eta x}\mathbb{1}_{\{x > \log(a)\}}$$

To prove that $g_\eta \in \mathcal{L}^1(\mathbb{R})$, we need to show that the following integral is bounded:

$$I_g = \int_{\mathbb{R}} \underbrace{g_\eta(x)}_{\text{always} \geq 0}\, dx = \int_{\mathbb{R}} e^{\eta x}\mathbb{1}_{x > \log(a)}\, dx = \int_{\log a}^{+\infty} e^{\eta x} = \frac{1}{\eta} e^{\eta x}\Big|_{\log a}^{+\infty}$$

- if $\eta = 0 \Rightarrow I_g \to \infty$
- if $\eta > 0 \Rightarrow I_g \to \infty$
- if $\eta < 0 \Rightarrow \eta = -|\eta| \longrightarrow I = \frac{1}{-|\eta|} e^{-|\eta| x}\Big|_{\log a}^{+\infty} = \frac{1}{-|\eta|}\left[0 - e^{-|\eta|\log a}\right] < \infty$ ok!

Therefore $I_g < +\infty \Leftrightarrow \eta < 0$ (not infinity)

(A2) $\mathbb{E}[e^{-\eta X_T}] < \infty$ if η does not tend to infinity, the assumption is respected

(A3) $u \longmapsto \hat{g}_\eta(u)\hat{P}_{X_T}^{-\eta}(-u) \in \mathcal{L}^1(\mathbb{R})$

Let us find $\hat{g}(z) = \int_{-\infty}^{+\infty} g(x) e^{ixz}\, dx = \int_{\log a}^{+\infty} e^{ixz}\, dx =$

$$= \int_{\log a}^{+\infty} \cos(zx)\, dx + i\int_{\log a}^{\infty} \sin(xz)\, dx =$$

$$= \frac{1}{z}\left[\sin(zx) - i\cos(xz)\right]_{\log a}^{+\infty} = \frac{1}{z}\left[0 - \frac{e^{i\log(a) z}}{i}\right] =$$

$$= -\frac{1}{zi} e^{i\log(a) z} = \frac{i}{z} e^{i\log(a)\cdot z} \quad *$$

$$-\frac{1}{i} = \frac{-i}{i^2} = +i$$

In $*$ we also have the definition of $\hat{P}_{X_T}(u)$

$$\Rightarrow \hat{P}_{X_T}^{-\eta}(-u) = e^{-\eta u}\hat{P}_{X_T}(u)$$

$$= e^{\eta u} e^{T(-iu\gamma - \frac{1}{2}\sigma^2 u^2 + \lambda(e^{-iu\alpha - \frac{1}{2}\beta^2 u^2} - 1))}$$

$$\Rightarrow \hat{g}_\eta(u)\hat{P}_{X_T}^{-\eta}(-u) = e^{\eta u}\frac{i}{u} e^{i\log(a) u} e^{\eta u} e^{T(-iu\gamma - \frac{1}{2}\sigma^2 u^2 + \lambda(e^{-iu\alpha - \frac{1}{2}\beta^2 u^2} - 1))}$$

This is in $\mathcal{L}^1$ if the integral is finite. This is true if $\eta < \infty$

To make things easier, we can use the Remark 24 (slide 47 lecture 3) to substitude (A3) with (A3'): Assume that $\hat{g}_\eta \in \mathcal{L}^1(\mathbb{R})$. This is true if $\eta < 0$

$$\hat{g}_\eta(u) = \frac{i}{u} e^{\eta u + i\log(a) u}$$

$\Rightarrow$ we have proved (A1), (A2) and (A3)

$\Rightarrow$ we find the Fourier pricing formula using the theorem:

$S_0 = e^{x_0} \Rightarrow \log S_0 = x_0$

$$\Pi = \frac{1}{\pi} \cdot e^{-\eta \log S_0} \int_{\mathbb{R}} \operatorname{int}(u)\, du$$

we will use these results in part d)

where $\operatorname{int}(u) = e^{iux^0} \hat{P}_{X_T}(u+i\eta)\, \hat{g}(-u-i\eta)\, du$

b) Explain how you can compute the cumulative distribution function $F_{X_T}(x)$ of X_T using a)

CDF

Sol

From point a) we have the Fourier pricing formula Π and the characteristic function of the Fourier trasform $\hat{P}_{X_T}(z)$.

One thing to keep in mind is that there is bijection between CDF and CF (CF: characteristic function): two distinct probability distributions never share the same CF.

To understand the link between CF and CDF or PDF, let us remind that

$$\underbrace{\phi_X(u)}_{CF} = \mathbb{E}[e^{iux}] = \int_{-\infty}^{+\infty} e^{iux} \underbrace{f_X(x)}_{PDF} dx = \int_{\Omega} e^{iux}\, d\underbrace{F_X(x)}_{CDF}$$

$\Rightarrow$ Given the CF, it is possible to reconstruct the CDF:

$$F_X(y) - F_X(x) = \lim_{\tau\to\infty} \frac{1}{2\pi} \int_{-\tau}^{+\tau} \frac{e^{-iux} - e^{iuy}}{iu} \phi_X(u)\, dx$$

Otherwise

$$F_X(x) = \frac{1}{2} + \frac{1}{2\pi} \int_0^{\infty} \frac{1}{iu} \left(e^{iux}\phi_X(-u) - e^{-iux}\phi_X(u)\right) du$$

In other words, by definition, the CF of a real random variable X is the Fourier trasform of its distribution. Therefore, to find the distribution (or cdf) we need to invert the CF.

The key message is that the Fourier transfor characterizes the probability measures

Getting back to our exercise specifically,

$$\Pi_0(S_0) = \mathbb{E}[\mathbb{1}_{e^{X_T+x^0} > a}] = \mathbb{P}(X_T + x^0 > \log(a)) = \mathbb{P}(X_T > \log(a) - x^0)$$
$$= 1 - F_{X_T}(\log(a) - x^0)$$

$$F_{X_T}(\log(a) - x^0) = 1 - \Pi_0(S_0)$$

c) $T > 0$, $S_0, \lambda, \sigma, \alpha, B$, $\eta < 0$ - Consider the function $\overset{o}{\hat{P}}: [a_{min}, a_{max}] \to \mathbb{R}$

$P(a) := \mathbb{E}[\mathbb{1}_{\{S_T > a\}}]$

Fourier pricing formula of a)

$I_n(P(\cdot), a)$: Chebychev interpolation of order n of $P(a)$.

Derive:
- appropriate error bound
- convergence rate $n \to \infty$

Someone may ask why the exercise recalls the Fourier pricing formula or, more in general, why we make use of Fourier transforms. There are many answers:

- for many option prices, the Fourier representation is explicit
- Though Fourier representation, we can study the analytic properties of option prices (payoff structure, distribution of the underlying ...)

In other words, Fourier can help to detect hidden properties.

First, note that our payoff for this exercise Y is not continuous, then it is not smooth and not analytic.

We define the Bernstein ellips for our interpolation:

$$B([a_{min}, a_{max}], \rho) = \tau_{[a_{min}, a_{max}]} \circ B([-1,1], \rho)$$
$$\tau_{[a_{min}, a_{max}]}(x) = a_{max} + \frac{a_{max} - a_{min}}{2}(1-x)$$

According to corollary 8 at slide 33 of lecture 10, if the function $p \mapsto Price^p$ is a real valued function that has an analytic extension to Bernstein ellips B_ρ for a certain vector $\rho \in (1,\infty)^P$ and $\sup_{p \in B[P,\rho]} |Price^p| < V$

$$\Rightarrow \text{the error bound of the Cheb. interpolation is } \|P - I_n(P)\|_\infty < C\rho^{-n}$$

In our case we have
- $a \mapsto P(a))$
- $B([a_{min}, a_{max}], \rho)$
- $\sup_{a \in B([a_{min}, a_{max}], \rho)} |P(a)| < V$

It signifies exponential decay.
We need to prove that the price function has an analytic expansion in the Bernstein ellipse.

We have the price function $a \mapsto P(a) = \frac{e^{-\eta x^0}}{2\pi} \int_{\mathbb{R}} e^{-i\xi x^0} \frac{a e^{\eta + i\xi}}{\eta + i e} \hat{P}_{X_T}(\xi + i\eta) d\xi$

Note that $\eta < 0$

satisfy analyticity in x_0

To prove that the integrant is analytic, we need to show that the integrant over ξ is also analytic. We can use the example before theorem 10 of lecture 10 to understand how to proceed. In particular, we use the Theorem of Morera: we need to check:

- Is $a \mapsto P(a)$ continuous?
- $\int_\gamma P(x) dx = 0$ for each closed triangle γ in the inner of the domain $U \in \mathbb{C}$

Observe that
$$\int_\gamma P(x) dx = \frac{e^{-\eta x_0}}{2\pi} \int_\gamma \int_{\mathbb{R}} -e^{-i\xi x^0} \frac{x e^{\eta + i\xi}}{\eta + i\xi} \hat{P}_{X_T}(\xi + i\eta) d\xi dx$$

The above has integrals that can be interchanged if there exists an integral bound. In order to do that, note that $e^{ix\xi}$ over a closed form is zero for any $\xi \in \mathbb{R}$ (Homework 10).
Once these integrals are interchanged, we can apply the Fubini theorem and get $\int_\gamma P(x) dx = 0$.

d) In the function MertonDigitalEurOptPrincing we derive the price of the European digital option in Merton's model.
Steps

- Find γ as gamma $= -\frac{1}{2}\sigma^2 - \lambda\left(e^{\alpha+\frac{1}{2}\beta^2} - 1\right)$

- Calculate the characteristic function of the Fourier trasform of X FP following $*$.

- Create the function $\hat{g}$:

$$\hat{g} := Fg(z) = i\,\frac{e^{i\log(a)z}}{z} \leftarrow \text{derived in } (*)$$

- Find the integrand function: (see equation slide 50 lecture 3)

$$\text{int}(u) = \text{Re}\left(e^{i\log(S_0)u}\right) \underbrace{\hat{P}_{X_t}(u+i\eta)}_{FP}\,\underbrace{\hat{g}(-u-i\eta)}_{Fg}$$

- Finally, get the price thanks to: price $= \frac{1}{\pi} \cdot e^{-\eta\log S_0} \int_0^L \text{int}(u)\,du$

e) The function ChebInterpol(f, x, n, a, b) computes the Cheb. interpolation of order n of f, defined in $[a,b]$, in x.
$f: [a,b] \to \mathbb{R}$ $x \in [a,b]$

From c, $P: [a_{min}, a_{max}] \to \mathbb{R}$
$P(a) = \mathbb{E}[\mathbb{1}_{\{S_T > a\}}]$

We need to find the options prices using Chebychev and Fourier. We follow these steps

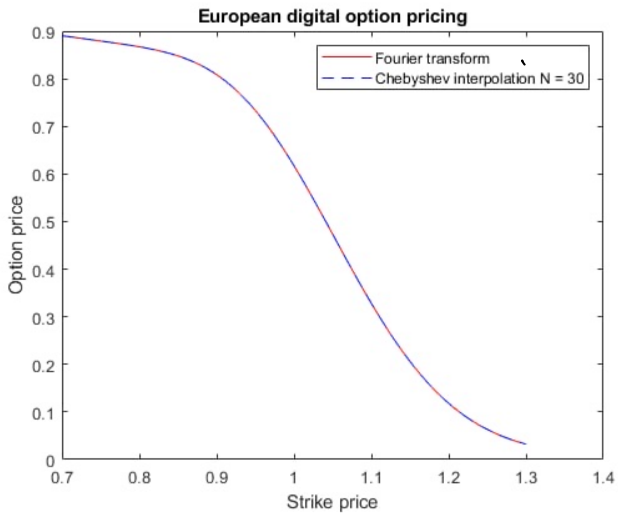
- Set the parameters
- create the vector a of strike prices

each different option corresponds to a different option price

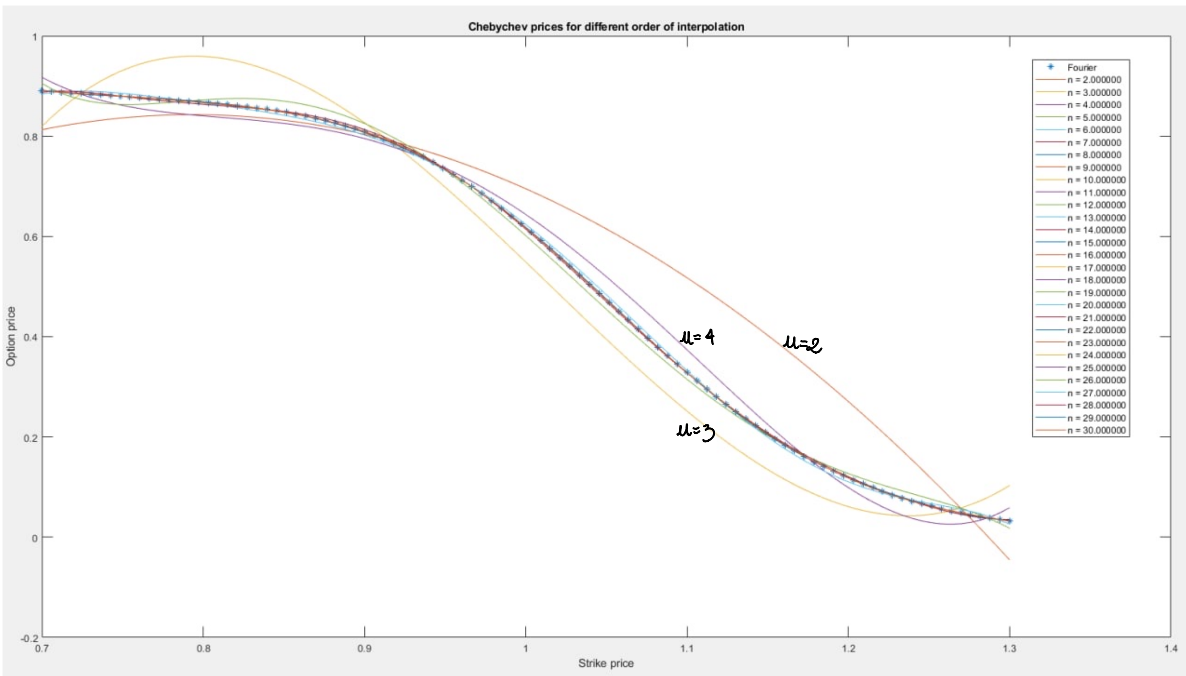
- 1) Fourier method: we call the function MertonDigitalEurOptPrincing, one time for each strike, in order to find the vector of prices using Fourier

- 2) Chebychev interpolation: also in this case we need the function MertonDigitalEurOptPrincing. However, in this case, this is only the function f we need to pass to ChebInterpol to compute the interpolation algorithm.
In this case we do not only have different strikes, but also many interpolation orders, saved in the vector n. We create a loop in which, for each n, we find a vector of prices (which lenght corresponds to length(a)).

We plot the two vector prices obtained with the Fourier method and with the Chebichev method (considering the maximum order of interpolation).
The resulting figure shows that the two prices are almost the same, proving that the two methods are almost identical in terms of result.

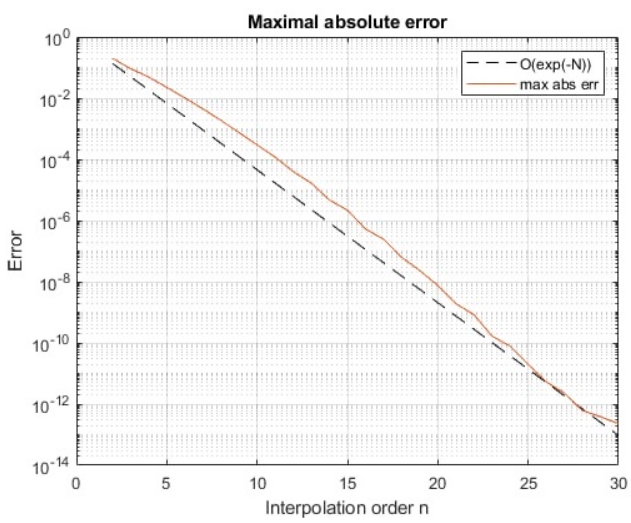


We may ask if this is the case for each interpolation order. Therefore, we plot the prices obtained using different orders:



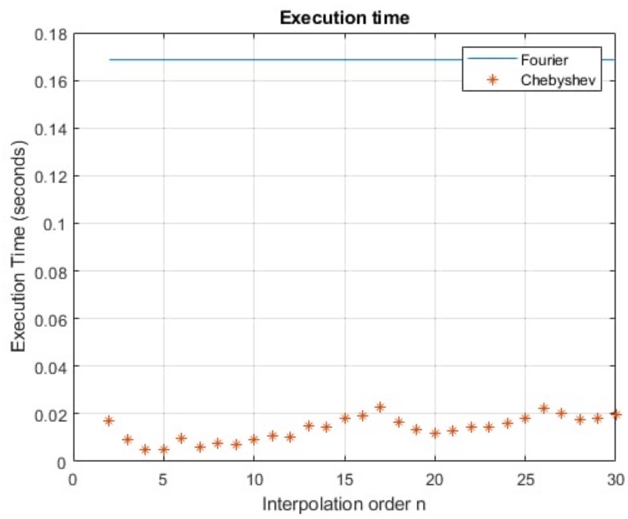
As expected, increasing the order, the interpolation gets closer to Fourier prices

This is showed also by the maximal absolute error, that decreases when the order grows:



=> the error decreases increasing the order

Regarding the execution time, we observe that the Fourier method is slower. This is true for any given order. Focusing on Chebychev method, we can see that the execution time changes with n. However, it does not have a stable trend: it goes up and down. In all cases it is under 0,03, while the Fourier execution time is around 0,17.
We can conclude that the Chebychev method, as anticipated during the class, is remarkably faster.

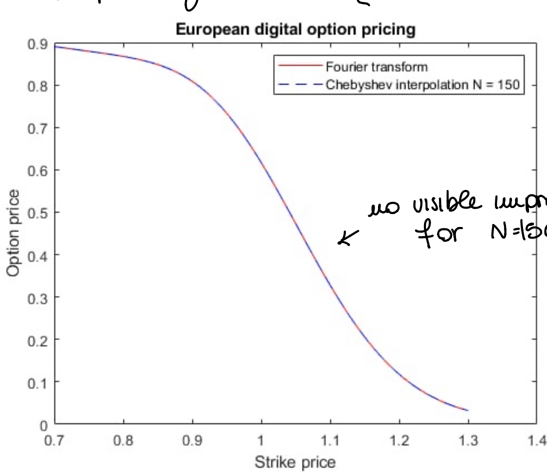


In conclusion, we can state that, if the chebychev order of interpolation is sufficiently high, we obtain a good approximation. Indeed, we can chose the order according to the error we are willing to accept.
Therefore, the Chebychev interpolation method is a excellent solution, which allows to save a lot of time.

Note:
In this case, the order arrives to 30 only. If we proceed to higher orders, the execution time rapidly raises. For this exercise, this has no sense, because the error is quite low also with n=30. In fact, raising the order does not bring additional value in terms of error.

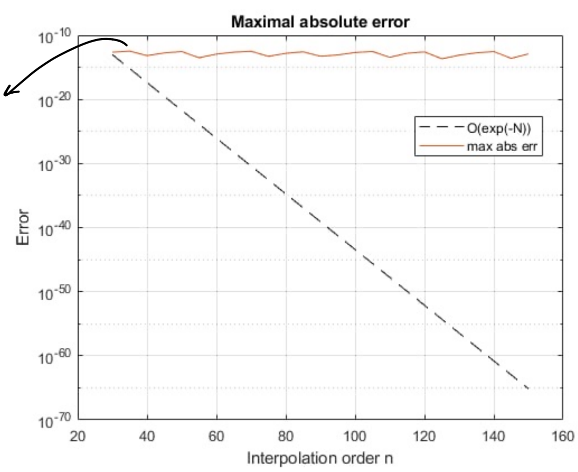
We only want to point at that, for high orders of interpolation, it may happen that the Chebychev method is not so convenient in terms of time. However, most of the times we can be satisfied for relatively low orders.

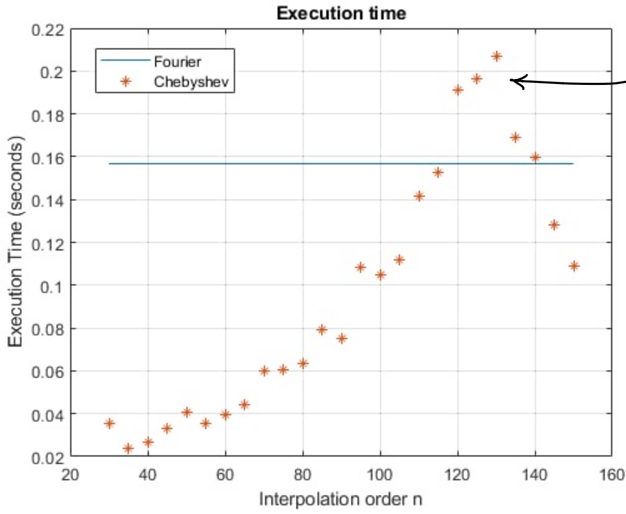
Example for n = [30 : 5 : 150]



no visible improvements for N=150

this is reflected by the error, which does not decrease when we increase the order





However, we have significant worsening for the execution time

⇒ We need to be caution when we decide the order of interpolation. A too higer order will throw away the time advantages of the Chebychev model.