

# Computational Finance

## Take Home Exam 2

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### Exercise 2 - Ornstein Uhlenbeck process

OU process is the solution of the SDE:  $dX_t = K(\theta - X_t) dt + \lambda dW_t$

If we let  $v(t, x) := \mathbb{E} [e^{ivX_t} | X_t = x]$ , then  $v$  satisfies

$$\begin{cases} v_t + \mathcal{L}v = 0 \\ v(T) = e^{ivx} \end{cases} \quad \text{with } \mathcal{L}v = K(\theta - x)v_x + \frac{\lambda^2}{2} v_{xx}$$

Suppose that  $v(t, x) = e^{\varphi(t-t, v) + \psi(t-t, v)x}$  (\*)

a) Deduce the system of ODEs for the functions  $\varphi$  and  $\psi$ :

First, we can observe that  $X$  is an affine process because its conditional characteristic function has this \* form.

Thanks to the fact we are working with an affine process, the solution is a relatively simple ODE.

We can recall the slide 24 of lecture 4: the functions  $\varphi$  and  $\psi$  solve the system of Riccati equations

Apply the Ito formula to  $Y_t = v(t, X_t)$ :

$$Y_t = Y_0 + \int_0^t \underbrace{\partial_s v(s, X_s)}_{\partial_t v(t, x)} ds + \int_0^t v_x(s, X_s) dX_s$$

$$\partial_t v(t, x) = v_t(t, x) + v_x(t, x) \mathcal{L}x + \frac{1}{2} v_{xx}(t, x) \lambda^2$$

$$\begin{aligned} dX_t &= K(\theta - X_t) dt + \lambda dW_t \\ dX_t &= (\alpha + bX_t) dt + (\sqrt{\lambda} X_t) dB_t \\ \alpha &= K\theta \\ b &= -\lambda \end{aligned}$$

Using the law of iterated expectations:  $Y_t = \mathbb{E}_s [e^{ivX_t}] = \mathbb{E}_s [\mathbb{E}_t [e^{ivX_t}]] = \mathbb{E}_s [Y_t]$   
 $\Rightarrow Y$  is a martingale  $\Rightarrow Y$  has zero drift a.e.

Hence, we obtain this PDE:  $v_t(t, x) + v_x(t, x) \mathcal{L}x + \frac{1}{2} v_{xx}(t, x) \lambda^2 = 0$

Boundary condition:  $v(x, T) = e^{ivx}$

We can now substitute using \*  $(v(t, x) = e^{\varphi(t-t, v) + \psi(t-t, v)x})$ :

$$\begin{aligned} v_t(t, x) &= (-\varphi' - \psi' x) v(t, x) \\ v_x(t, x) &= \psi v(t, x) \end{aligned}$$

$$v_{xx}(t, x) = \psi^2 v(t, x)$$

To simplify the notation:  
 $\varphi(t-t, v) = \varphi$   
 $\psi(t-t, v) = \psi$

Plugging into the PDE:

$$(-\varphi' - \psi' x) v(t, x) + \psi v(t, x) \mathcal{L}x + \frac{1}{2} \lambda^2 \psi^2 v(t, x) = 0$$

$$v(t, x) \left( -\varphi' - \psi' x + \psi K(\theta - x) + \frac{1}{2} \lambda^2 \psi^2 \right) = 0 \quad \text{Divide by } v(t, x)$$

$$-\varphi' - \psi' x + \psi K\theta - \psi K x + \frac{1}{2} \lambda^2 \psi^2 = 0 \quad (*)$$

We collect terms in  $\kappa$ : 
$$\begin{cases} \psi^1 + \psi_K = 0 \\ \psi^1 - \psi_K \kappa - \frac{1}{2} \lambda^2 \psi^2 = 0 \end{cases} \quad \begin{array}{l} \leftarrow \text{with } \kappa \\ \leftarrow \text{without } \kappa \end{array}$$

$$\begin{cases} \psi^1(T-t, v) = -\psi(T-t, v)\kappa \\ \psi^1(T-t, v) = \psi(T-t, v)\kappa - \frac{\lambda^2}{2} \psi^2(T-t, v) \end{cases}$$

Boundary conditions: from  $\psi(x, T) = e^{ivx}$  we get:  $e^{\psi(T-t, v) + \psi(T-t, v)\kappa} = e^{ivx}$   
 $\Rightarrow \psi(0, v) = iv$   
 $\psi(0, v) = 0$

b) Solve the system and write explicitly the form of the characteristic function of  $x_0$  given  $x_0 = se$ :

① Solve the system:  $\begin{cases} \psi^1(s, v) = -\psi(s, v)\kappa \\ \psi(0, v) = iv \end{cases} \quad \begin{array}{l} \leftarrow \frac{\partial \psi}{\partial s} \\ T-t=s \end{array} \quad \begin{array}{l} \psi(s) = C e^{-ks} \\ \psi(0) = C \cdot 1 = iv \end{array} \quad \begin{array}{l} C \in \mathbb{R} \\ \Rightarrow C = iv \end{array}$

$$\psi^1(s, v) = \psi(s, v)\kappa - \frac{\lambda^2}{2} \psi^2(s, v) = \kappa v - iv e^{-ks} - \frac{\lambda^2}{2} \frac{-v^2}{(iv)^2} e^{-2ks}$$

$$\begin{cases} \psi^1(s, v) = iv \cancel{\kappa e^{-ks}} + \frac{1}{2} \lambda^2 v^2 e^{-2ks} \\ \psi(0, v) = 0 \end{cases}$$

considering the boundary conditions require that  $\psi(0, v) = 0$ , to obtain a derivative equal to  $\kappa e^{-ks}$ , we will not  $\cancel{ve^{-ks}}$ , but  $\cancel{1 - e^{-ks}}$ .  
 Same reasoning for  $e^{-2ks}$ .

$$\psi(s, v) = d_1 iv \cancel{(1 - e^{-ks})} + d_2 \frac{1}{2} \lambda^2 v^2 \frac{1 - e^{-2ks}}{2k} \quad d_1, d_2 \in \mathbb{R}$$

$$\psi(0, v) = d_1 iv \cancel{(1 - 1)} + d_2 \lambda^2 v^2 \frac{1 - 1}{4k} = 0 \quad \text{satisfied if } d_1, d_2 \in \mathbb{R}$$

Therefore, we need other conditions to find  $d_1$  and  $d_2$ . We decide to get back to ② and plot our results for  $\psi(s, v)$  and  $\psi(s, v)$ :

$$-\psi^1 - \psi^1 \kappa + \psi_K \kappa - \psi_K \kappa + \frac{1}{2} \lambda^2 \psi^2 = 0 \quad \psi(s) = iv e^{-ks}$$

$$-\psi^1 + iv \kappa e^{-ks} + iv e^{-ks} \kappa - iv e^{-ks} \kappa - \frac{1}{2} v^2 \lambda^2 e^{-2ks} = 0$$

$$d_1 iv \kappa e^{-ks} + d_2 \lambda^2 v^2 \frac{2k e^{-2ks}}{2k} =$$

$$\cancel{iv \kappa e^{-ks}} + \cancel{iv e^{-ks} \kappa} - \cancel{iv e^{-ks} \kappa} - \frac{1}{2} v^2 \lambda^2 e^{-2ks}$$

$$\underline{d_1 iv \kappa e^{-ks}} + \underline{d_2 \frac{1}{2} v^2 e^{-2ks}} = \underline{iv e^{-ks} \kappa} - \underline{\frac{1}{2} v^2 \lambda^2 e^{-2ks}}$$

$$\Rightarrow d_1 = 1 \text{ and } d_2 = -1$$

$$\Rightarrow \begin{cases} \psi(s, v) = iv e^{-ks} \\ \psi(s, v) = iv v (1 - e^{-ks}) - \lambda^2 v^2 \frac{1 - e^{-2ks}}{4k} \end{cases}$$

$$\text{Therefore, } v(t, x) = \exp \left[ i\theta v (1 - e^{-kt}) - \lambda^2 v^2 \frac{1 - e^{-2kt}}{4K} + x i v e^{-kt} \right]$$

② we want to find an explicit formula of  $E[e^{ivX_t} | X_0 = x]$

$$v(t, x) := E[e^{ivX_t} | X_t = x]$$

$$\Rightarrow v(0, x) = E[e^{ivX_t} | X_0 = x]$$

$$\Rightarrow E[e^{ivX_t} | X_0 = x] = \exp [i\theta v (1 - e^{-kt}) - \lambda^2 v^2 \frac{1 - e^{-2kt}}{4K} + x i v e^{-kt}]$$

c) Deduce that  $X_t$  is normally distributed. Write explicitly the mean and variance of  $X_t$ .

First, we recall the characteristic function of the normal distribution  $N(\mu, \sigma^2)$ :

$$\phi_{\text{normal}}(t) = e^{i\mu t - \frac{1}{2} \sigma^2 t^2}$$

In general, by definition, the characteristic function is given by  $\ell_X(t) = E[e^{itX}]$ .  
For this exercise,  $v(t, x) = E[e^{ivX_t} | X_0 = x] \Rightarrow$  the characteristic function of  $X_t$   
is a function of  $v$

$$iv(e^{-kt} x + \theta(1 - e^{-kt})) - \frac{v^2 \lambda^2}{4K} (1 - e^{-2kt})$$

Then, if  $X_t$  is the previous OUP process:  $\phi_{X_t}(v) = e^{iv(e^{-kt} x + \theta(1 - e^{-kt})) - \frac{v^2 \lambda^2}{4K} (1 - e^{-2kt})}$

$$\text{For } X_t: \phi_{X_t}(v) = \exp \{ iv(e^{-kt} x + \theta(1 - e^{-kt})) - \frac{v^2 \lambda^2}{4K} (1 - e^{-2kt}) \}$$

We need to prove that for certain  $\mu$  and  $\sigma$ ,  $X_t$  is normally distributed.  
To simplify the notation, we work with the exponentials:

$$\text{normal: } it(\mu) - \frac{1}{2} \sigma^2 t^2$$

$$\sigma \cdot iv(e^{-kt} x + \theta(1 - e^{-kt})) - \frac{1}{2} v^2 \frac{\lambda^2}{2K} (1 - e^{-2kt})$$

$$\Rightarrow \begin{cases} \mu = e^{-kt} x + \theta(1 - e^{-kt}) \\ \sigma^2 = \frac{\lambda^2}{2K} (1 - e^{-2kt}) \end{cases}$$

To conclude that this correspondence is sufficient to show that  $X_t$  is normally distributed, we need to remember that the characteristic function uniquely determines the distribution function. Therefore, recognising the characteristic function of a random variable identifies its distribution function. For the OUP process, we recognised the characteristic function of a normal distribution if \* is respected, then the OUP process is normally distributed, such that

$$X_t | X_0 = x \sim N(e^{-kt} x + \theta(1 - e^{-kt}), \frac{\lambda^2}{2K} (1 - e^{-2kt}))$$

$$dX_t = K(\theta - X_t) dt + \lambda dW_t$$

- d) Using Itô formula solve \* explicitly and explain why this is consistent with the result in the previous part -

To solve \* consider  $f(t, X_t) = X_t e^{kt}$  where  $X_t$  is the solution.

Applying the Itô formula, we obtain

$$\begin{aligned} df(t, X_t) &= K X_t e^{kt} dt + e^{kt} dX_t \\ &= K X_t e^{kt} dt + e^{kt} (K(\theta - X_t) dt + \lambda dW_t) \\ &= K e^{kt} [X_t + \theta - X_t] dt + e^{kt} \lambda dW_t \end{aligned}$$

$$d(X_t e^{kt}) = K\theta e^{kt} dt + e^{kt} \lambda dW_t$$

$$\int_0^t d(X_s e^{ks}) = \theta \int_0^t K e^{ks} ds + \lambda \int_0^t e^{ks} dW_s$$

$$\left. \begin{array}{l} X_t e^{kt} = X_0 + \theta(e^{kt} - 1) + \lambda \int_0^t e^{ks} ds \\ X_t = \underline{X_0 e^{-kt}} + \underline{\theta(1 - e^{-kt})} + \underline{\lambda \int_0^t e^{k(s-t)} ds} \end{array} \right\} \text{divided by } e^{-kt}$$

Note that the solution is a sum of deterministic terms and an integral of a deterministic function with respect to a Wiener process with normally distributed increments. Therefore, the process is Gaussian.

(\*) NB: A Wiener process has Gaussian increments by definition:  $W_t - W_s \sim N(0, t-s)$   $\forall s \leq t$ . Moreover,

If  $f$  is square integrable on  $[0, t]$   $\Rightarrow \int_0^t f(s) dW_s \sim N(0, \int_0^t f^2(s) ds)$

In our case:  $f(s) = \lambda e^{ks}$  and it is square integrable

$$\Rightarrow \int_0^t \lambda e^{ks} dW_s \sim N(0, \int_0^t \lambda^2 e^{2ks} ds)$$

$$\bullet \text{Mean: } E[X_t] = E[e^{-kt} X_0 + \theta(1 - e^{-kt}) + \underbrace{\int_0^t \lambda e^{ks} dW_s}_{\text{zero mean}}] = e^{-kt} X_0 + \theta(1 - e^{-kt})$$

$$\begin{aligned} \bullet \text{Variance: } V(X_t) &= V(e^{-kt} X_0 + \theta(1 - e^{-kt}) + \int_0^t \lambda e^{ks} dW_s) = \\ &= V(\int_0^t \lambda e^{ks} dW_s) = \int_0^t \lambda^2 e^{2ks} ds - \lambda^2 e^{-2kt} \int_0^t e^{2ks} ds = \\ &= \lambda^2 e^{-2kt} \frac{e^{2kt}}{2k} \Big|_0^t = \lambda^2 e^{-2kt} \frac{1}{2k} (e^{2kt} - 1) = \\ &= \frac{\lambda^2}{2k} (1 - e^{-2kt}) \end{aligned}$$

You are we need the conditional mean and covariance:

$$E[X_t | X_0 = x_0] = e^{-kt} x_0 + \theta (1 - e^{-kt})$$

$$V(X_t | X_0 = x_0) = \frac{\lambda^2}{2k} (1 - e^{-2kt})$$

$$X_t | X_0 = x_0 \sim N \left( e^{-kt} x_0 + \theta (1 - e^{-kt}), \frac{\lambda^2}{2k} (1 - e^{-2kt}) \right)$$

This is exactly the same result we have found using the characteristic function.

## Exercise 4 - Calibration Heston Model

$S_t$ : stock price

$V_t$ : Spot variance

$W_t^{(1)}, W_t^{(2)}$ : BM with correlation  $\rho$

Parameters of the Heston model:  $\theta, k, \rho$

We need to calibrate the model using the prices of the European call options

$\Rightarrow$  find the parameters that minimize the root-mean-squared error of the difference: Heston price - Observed price

In other words, using slides of lecture 4 regarding the calibration problem

- empirical market prices:  $\hat{C}(T_j, K_j)$   $j=1:N$

- option price of the stock parametric model  $S^q$ :  $C^q(T_j, K_j)$

we need to find the parameter  $q$

$$\Rightarrow q^* = \underset{q}{\text{argmin}} \text{dis}((\hat{C}(T_j, K_j))_{j=1:N}, C^q(T_j, K_j))_{j=1:N}$$

We start our problem loading the data and splitting the columns.

Then, we set  $r$  and so

Then, in order to use the function "fminsearchcon.m" to implement the optimization routine, we need to set the initial parameters of the model, the lower and the upper bound. In our case:

INITIAL VALUES	LOWER BOUND	UPPER BOUND
$\theta = 0,04$	$-\infty$	$+\infty$
$k = 1,5$	$0$	$+\infty$
$\sigma = 0,3$	$0$	$+\infty$
$f = -0,6$	$-1$	$1$
$V_0 = 0,0441$	$0$	$+\infty$

even if there is empirical evidence that  $\rho < 0$  (slide 26, first lecture), we cannot conclude that a priori  $\Rightarrow \rho \in [-1, 1]$   
However, we will probably get a negative value

We also need to pass as argument the function to minimize, the distance function.

distance\_prices : function that calculate root-mean-squared error

↳ This function has some restrictions regarding the form it can have:

In order to be passed as argument to fminsearchcon, it has to accept a vector  $x$  as input and return a scalar function value evaluated at  $x$ .  
→ function [error] = distance\_prices ( $x$ )

Regarding the root mean squared error, we can say a bit more =

First, the RMSE is given by

$$RMSE = \sqrt{\frac{\sum_{i=1}^T (\hat{P}_{t,i} - P_{t,i})^2}{T}};$$

where  $P_t$  is the real Heston price and  $\hat{P}_t$  the approximated one and  $T$  represents the number of prices (length(Prices) in the code).

think that

One can think that minimising the root mean square error gives the same result of minimising the square error.

$$\rightarrow SE = \sum_{i=1}^T (\hat{P}_{t,i} - P_{t,i})^2$$

However, this is not entirely correct in our case.

We have tried both and, as expected, even if the results are very similar, the first one (RMSE) is faster: 492 iterations instead of 509.

We also tried the root square error and the results are exactly equivalent

$$RSE = \sqrt{\sum_{i=1}^T (\hat{P}_{t,i} - P_{t,i})^2}$$

to the RMSE (also in terms of number of iterations).

These results make sense: the difference between SE and RMSE

(and between SE and RSE as well) consists also of a root, and therefore is something that changes the 'order' of the function, and this impacts the number of iterations. However, the resulting parameters are quite similar. On the other hand, RMSE and RSE differs only for a factor  $\sqrt{T}$ , a constant, and this has zero impact on the algorithm.

The exercise asks for the RMSE and hence we use that in our code. Please find the other ideas as comment in the code and all the results below.

• Results minimising the squared error :  $\min \left( \sum_{i=1}^T (\hat{P}_{t,i} - P_{t,i})^2 \right)$

Optimal parameters:

Nu = 0.013210

Kappa = 4.644733

Sigma = 0.245656

Rho = -0.672776

V = 0.011532

Optimization routine iterations: 509.000000

Message:

Optimization terminated:

the current x satisfies the termination criteria using OPTIONS.TolX of 1.000000e-04  
and F(X) satisfies the convergence criteria using OPTIONS.TolFun of 1.000000e-04

- Results minimising the root squared error:  $\min \left( \sqrt{\sum_{i=1}^T (\hat{P}_{ti} - P_{ei})^2} \right)$

Optimal parameters:

Nu = 0.013210

Kappa = 4.644926

Sigma = 0.245661

Rho = -0.672759

V = 0.011532

Optimization routine iterations: 492.000000

Message:

Optimization terminated:

the current x satisfies the termination criteria using OPTIONS.TolX of 1.000000e-04

and F(X) satisfies the convergence criteria using OPTIONS.TolFun of 1.000000e-04

- Results minimising the root-mean squared error:  $\min \left( \sqrt{\frac{\sum_{i=1}^T (\hat{P}_{ti} - P_{ei})^2}{T}} \right)$

Optimal parameters:

Nu = 0.013210

Kappa = 4.644926

Sigma = 0.245661

Rho = -0.672759

V = 0.011532

Optimization routine iterations: 492.000000

Message:

Optimization terminated:

the current x satisfies the termination criteria using OPTIONS.TolX of 1.000000e-04

and F(X) satisfies the convergence criteria using OPTIONS.TolFun of 1.000000e-04