Unit 5

Solution of Ordinary Differential Equations

5.1 Review

Ordinary Differential Equation (ODE): An ODE is an equation that contains one or several derivatives of an unknown function $y(x)$ having one independent variable x. Solving a differential equation means to find that unknown function $y(x)$ or its relation with x which satisfies the given differential equation. In that case, $y(x)$ is called the solution of the given differential equation.

For example,

$$\frac{dy}{dx} = 6x + 1$$

is an ordinary differential equation. Its solution is $y(x) = 3x^2 + x + c$ where c is a constant.

Order of differential equation: The order of a differential equation is the order of the highest derivative that appears in the equation.

For example,

$$\frac{dy}{dx} = 3x^2 + y$$

is a first-order differential equation whereas

$$\frac{d^4y}{dx^4} = -x^4$$

is a fourth-order differential equation.

Degree of differential equation: The degree of a differential equation is the power of the highest-order derivative that appears in the equation.

For example,

$$\frac{d^4y}{dx^4} = -x^4$$

is a fourth-order, first-degree differential equation whereas

$$\left(\frac{d^3y}{dx^3}\right)^2 + 5\left(\frac{dy}{dx}\right)^5 = 3x^2$$

is a third-order, second-degree differential equation.

Initial-Value Problem (IVP) and Boundary-Value Problem (BVP): A general solution of a differential equation contains as many constants as the order of the differential equation. To eliminate these constants, we need the same number of conditions on the solution of differential equation. When all the conditions are specified at a particular value of the independent variable x, then the problem is called an initial-value problem and the conditions are called initial conditions. For example, solving the differential equation

$$\frac{d^2y}{dx^2} = y, \qquad y(0) = 4, y'(0) = -2$$

is an IVP where $y(0) = 4$ and $y'(0) = -2$ are initial conditions.

However if the conditions on the solution of a differential equation are specified at different values of the independent variable x, then the problem is called a boundary-value problem and the conditions are called boundary conditions. For example, solving the differential equation

$$\frac{d^2y}{dx^2} = -4y, \qquad y(0) = 3, y(\pi/2) = -3$$

is a BVP where $y(0) = 3$ and $y(\pi/2) = -3$ are the boundary conditions.

5.2 Solving 1^{st}-order 1^{st}-degree IVP

First-order first-degree initial value problems are of the form

$$\frac{dy}{dx} = f(x, y), \qquad y(x_0) = y_0;$$

$$\text{or, } y' = f(x, y), \qquad y(x_0) = y_0.$$

The methods used to solve such differential equations numerically are

1. Taylor's series method
2. Picard's method
3. Euler's method
4. Heun's method (Modified Euler's Method)
5. Runge-Kutta method

5.2.1 Taylor's Series Method

Suppose we are given a differential equation of the form

$$y' = f(x, y), \qquad y(x_0) = y_0.$$

If $y(x)$ is a solution of this differential equation, then the Taylor's series expansion of $y(x)$ at $x = x_0$ is given by

$$y(x) = y(x_0) + y'(x_0)(x - x_0) + \frac{y''(x_0)}{2!}(x - x_0)^2 + \frac{y'''(x_0)}{3!}(x - x_0)^3 + \cdots$$

$$\text{or, } y(x) = y(x_0) + y'(x_0)h + \frac{y''(x_0)}{2!}h^2 + \frac{y'''(x_0)}{3!}h^3 + \cdots$$

where $h = x - x_0$.

The coefficients $y(x_0)$, $y'(x_0)$, $y''(x_0), \cdots$ of this series can be calculated by using the initial condition and successively differentiating the equation for the first derivative $y' = f(x, y)$. Evaluation these derivatives at $x = x_0$ gives the various coefficients. Also the more terms of the Taylor's series we use, the more accurate our solution will be.

Examples

1. Use Taylor's series method to solve $y' = x^2 y - 1$ for $x = 0.1$ and $x = 0.2$ given that $y(0) = 1$.

Solution: The Taylor's series expansion of $y(x)$ at $x = 0$ is given by

$$y(x) = y(0) + y'(0)(x - 0) + \frac{y''(0)}{2!}(x - 0)^2 + \frac{y'''(0)}{3!}(x - 0)^3 + \frac{y^{(4)}(0)}{4!}(x - 0)^4 + \cdots$$

$$\text{or, } y(x) = y(0) + y'(0)x + \frac{y''(0)}{2!}x^2 + \frac{y'''(0)}{3!}x^3 + \frac{y^{(4)}(0)}{4!}x^4 + \cdots\cdots (1)$$

Now, as given

$$y(0) = 1,$$

$$\text{and } y'(x) = x^2 y(x) - 1 \Rightarrow y'(0) = 0^2 y(0) - 1 = -1.$$

Successively differentiating $y'(x)$ and substituting $x = 0$, we get

$$y''(x) = x^2 y'(x) + 2xy(x) \Rightarrow y''(0) = 0^2 y'(0) + 2 \cdot 0 \cdot y(0) = 0,$$

$$y'''(x) = x^2 y''(x) + 2xy'(x) + 2xy'(x) + 2y(x) = x^2 y''(x) + 4xy'(x) + 2y(x)$$

$$\Rightarrow y'''(0) = 0^2 y''(0) + 4 \cdot 0 \cdot y'(0) + 2y(0) = 0 + 0 + 2 \times 1 = 2$$

$$y^{(4)}(x) = x^2 y'''(x) + 2xy''(x) + 4xy''(x) + 4y'(x) + 2y'(x) = x^2 y'''(x) + 6xy''(x) + 6y'(x)$$

$$\Rightarrow y^{(4)}(0) = 0 + 0 + 6y'(0) = 6 \cdot (-1) = -6$$

Therefore from (1), ignoring the terms containing x^5 and higher powers of x, we get,

$$y(x) = 1 + (-1)x + \frac{0}{2!}x^2 + \frac{2}{3!}x^3 + \frac{(-6)}{4!}x^4$$

$$\text{or, } y(x) = 1 - x + \frac{x^3}{3} - \frac{x^4}{4}$$

When $x = 0.1$, then

$$y(0.1) = 1 - 0.1 + \frac{(0.1)^3}{3} - \frac{(0.1)^4}{4} = 0.9 + 0.00033 - 0.000025 = 0.900305$$

When $x = 0.2$, then

$$y(0.2) = 1 - 0.2 + \frac{(0.2)^3}{3} - \frac{(0.2)^4}{4} = 0.8 + 0.00267 - 0.0004 = 0.80227$$

2. Use Taylor's series method to solve $y' = -2x - y$ at $x = 0.1$ and $x = 0.3$ given that $y(0) = -1$.

Solution: The Taylor's series expansion of $y(x)$ at $x = 0$ is given by

$$y(x) = y(0) + y'(0)(x-0) + \frac{y''(0)}{2!}(x-0)^2 + \frac{y'''(0)}{3!}(x-0)^3 + \frac{y^{(4)}(0)}{4!}(x-0)^4 + \cdots$$

$$\text{or, } y(x) = y(0) + y'(0)x + \frac{y''(0)}{2!}x^2 + \frac{y'''(0)}{3!}x^3 + \frac{y^{(4)}(0)}{4!}x^4 + \cdots\cdots (1)$$

Now, as given

$$y(0) = -1,$$

$$\text{and } y'(x) = -2x - y(x) \Rightarrow y'(0) = -2 \cdot 0 - y(0) = -(-1) = 1.$$

Successively differentiating $y'(x)$ and substituting $x = 0$, we get

$$y''(x) = -2 - y'(x) \Rightarrow y''(0) = -2 - y'(0) = -2 - 1 = -3$$

$$y'''(x) = -y''(x) \Rightarrow y'''(0) = -y''(0) = -(-3) = 3$$

$$y^{(4)}(x) = -y'''(x) \Rightarrow y^{(4)}(0) = -y'''(0) = -3$$

Therefore from (1), ignoring the terms containing x^5 and higher powers of x, we get,

$$y(x) = -1 + 1 \cdot x + \frac{(-3)}{2!}x^2 + \frac{3}{3!}x^3 + \frac{(-3)}{4!}x^4$$

$$\text{or, } y(x) = -1 + x - \frac{3}{2}x^2 + \frac{x^3}{2} - \frac{x^4}{8}$$

When $x = 0.1$, then

$$y(0.1) = -1+0.1-\frac{3}{2}(0.1)^2+\frac{(0.1)^3}{2}-\frac{(0.1)^4}{8} = -0.9-0.015+0.0005-0.0000125 = -0.9145125$$

When $x = 0.3$, then

$$y(0.3) = -1+0.3-\frac{3}{2}(0.3)^2+\frac{(0.3)^3}{2}-\frac{(0.3)^4}{8} = -0.7-0.135+0.0135-0.0010125 = -0.8225125$$

Problems

Use Taylor's series method to solve the following first-order differential equations:

1. $y' = x^2 + y^2$ for $x = 0.25$ and $x = 0.5$ given that $y(0) = 1$.
2. $\dfrac{dy}{dx} = x + y + xy$, $y(0) = 1$ for $x = 0.25, 0.5$.
3. $\dfrac{dy}{dx} = y(x^2 - 1)$, $y(0) = 1$ for $x = 1.0, 1.5, 2.0$.
4. $\dfrac{dy}{dx} = x + y$, $y(0) = 1$ for $x = 0.1, 0.5$.
5. $\dfrac{dy}{dx} = \dfrac{2x}{y} - xy$, $y(0) = 1$ for $x = 0.25, 0.5$.
6. $\dfrac{dy}{dx} = x^2y^2$, $y(1) = 0$ for $x = 2.0, 3.0$.

5.2.2 Picard's Method

Suppose we are given a differential equation of the form

$$y' = f(x, y), \qquad y(x_0) = y_0.$$

Then $dy = f(x, y)dx$. Integrating both sides of above equation in the interval $[x_0, x]$, we get,

$$\int_{x_0}^{x} dy = \int_{x_0}^{x} f(t, y(t))\ dt$$

$$\text{or, } y(x) - y(x_0) = \int_{x_0}^{x} f(t, y(t))\ dt$$

$$\text{or, } y(x) = y_0 + \int_{x_0}^{x} f(t, y(t))\ dt \cdots\cdots (1)$$

Now to solve (1), we use the method of iteration as follows:
We replace $y(t)$ on the right of (1) by y_0 and calculate the first approximation $y_1(x)$ of $y(x)$ as

$$y_1(x) = y_0 + \int_{x_0}^{x} f(t, y_0)\ dt.$$

The second approximation $y_2(x)$ of $y(x)$ is calculated by substituting $y(t)$ by $y_1(t)$ on the right of (1) as

$$y_2(x) = y_0 + \int_{x_0}^{x} f(t, y_1(t))\ dt.$$

Proceeding similarly, the n^{th} approximation of $y(x)$ is given by the iteration

$$y_n(x) = y_0 + \int_{x_0}^{x} f(t, y_{n-1}(t))\ dt.$$

This iterative method of solving the differential equation is known as Picard's method.

Examples

1. Use Picard's method to solve $y' = x + y$ for $x = 0.25$ and $x = 0.5$ given $y(0) = 1$.

Solution: Here $x_0 = 0$, $y_0 = 1$ and $f(x, y) = x + y$. Then Picard's iteration is given by

$$y_n(x) = y_0 + \int_{x_0}^{x} f(t, y_{n-1}(t))dt$$

$$\text{or, } y_n(x) = 1 + \int_0^x (t + y_{n-1}(t))dt.$$

When $n = 1$, we get

$$y_1(x) = 1 + \int_0^x (t + y_0(t))dt = 1 + \int_0^x (t + 1)dt = 1 + \left[\frac{t^2}{2} + t\right]_0^x = 1 + x + \frac{x^2}{2}.$$

When $n = 2$, we get

$$y_2(x) = 1 + \int_0^x (t + y_1(t))dt = 1 + \int_0^x \left(t + 1 + t + \frac{t^2}{2}\right) dt = 1 + \int_0^x \left(1 + 2t + \frac{t^2}{2}\right) dt$$

$$= 1 + \left[t + t^2 + \frac{t^3}{6}\right]_0^x = 1 + x + x^2 + \frac{x^3}{6}.$$

When $n = 3$, we get

$$y_3(x) = 1+\int_0^x (t+y_2(t))dt = 1+\int_0^x \left(t + 1 + t + t^2 + \frac{t^3}{6}\right) dt = 1+\int_0^x \left(1 + 2t + t^2 + \frac{t^3}{6}\right) dt$$

$$= 1 + \left[t + t^2 + \frac{t^3}{3} + \frac{t^4}{24}\right]_0^x = 1 + x + x^2 + \frac{x^3}{3} + \frac{x^4}{24}.$$

Now when $x = 0.25$, we have

$$y_3(0.25) = 1 + 0.25 + (0.25)^2 + \frac{(0.25)^3}{3} + \frac{(0.25)^4}{24} = 1.31787.$$

And when $x = 0.5$, we get

$$y_3(0.5) = 1 + 0.5 + (0.5)^2 + \frac{(0.5)^3}{3} + \frac{(0.5)^4}{24} = 1.79427.$$

2. Obtain a solution up to the fifth approximation of the equation $\dfrac{dy}{dx} = x + y$ such that $y = 1$ when $x = 0$ using Picard's method of successive approximations.

Solution: Here $x_0 = 0$, $y_0 = 1$ and $f(x, y) = x + y$. Then Picard's iteration is given by

$$y_n(x) = y_0 + \int_{x_0}^{x} f(t, y_{n-1}(t))dt$$

$$\text{or, } y_n(x) = 1 + \int_0^x (t + y_{n-1}(t))dt.$$

When $n = 1$, we get

$$y_1(x) = 1 + \int_0^x (t + y_0(t))dt = 1 + \int_0^x (t+1)dt = 1 + \left[\frac{t^2}{2} + t\right]_0^x = 1 + x + \frac{x^2}{2}.$$

When $n = 2$, we get

$$y_2(x) = 1 + \int_0^x (t + y_1(t))dt = 1 + \int_0^x \left(t + 1 + t + \frac{t^2}{2}\right) dt = 1 + \int_0^x \left(1 + 2t + \frac{t^2}{2}\right) dt$$

$$= 1 + \left[t + t^2 + \frac{t^3}{6}\right]_0^x = 1 + x + x^2 + \frac{x^3}{6}.$$

When $n = 3$, we get

$$y_3(x) = 1 + \int_0^x (t + y_2(t))dt = 1 + \int_0^x \left(t + 1 + t + t^2 + \frac{t^3}{6}\right) dt = 1 + \int_0^x \left(1 + 2t + t^2 + \frac{t^3}{6}\right) dt$$

$$= 1 + \left[t + t^2 + \frac{t^3}{3} + \frac{t^4}{24}\right]_0^x = 1 + x + x^2 + \frac{x^3}{3} + \frac{x^4}{24}.$$

When $n = 4$, we get

$$y_4(x) = 1 + \int_0^x (t + y_3(t))dt = 1 + \int_0^x \left(t + 1 + t + t^2 + \frac{t^3}{3} + \frac{t^4}{24}\right) dt$$

$$= 1 + \int_0^x \left(1 + 2t + t^2 + \frac{t^3}{3} + \frac{t^4}{24}\right) dt = 1 + \left[t + t^2 + \frac{t^3}{3} + \frac{t^4}{12} + \frac{t^5}{120}\right]_0^x = 1 + x + x^2 + \frac{x^3}{3} + \frac{x^4}{12} + \frac{x^5}{120}.$$

When $n = 5$, we get

$$y_5(x) = 1 + \int_0^x (t + y_4(t))dt = 1 + \int_0^x \left(t + 1 + t + t^2 + \frac{t^3}{3} + \frac{t^4}{12} + \frac{t^5}{120}\right) dt$$

$$= 1 + \int_0^x \left(1 + 2t + t^2 + \frac{t^3}{3} + \frac{t^4}{12} + \frac{t^5}{120}\right) dt = 1 + \left[t + t^2 + \frac{t^3}{3} + \frac{t^4}{12} + \frac{t^5}{60} + \frac{t^6}{720}\right]_0^x$$

$$= 1 + x + x^2 + \frac{x^3}{3} + \frac{x^4}{12} + \frac{x^5}{60} + \frac{x^6}{720}.$$

Therefore the fifth approximation is

$$y_5(x) = 1 + x + x^2 + \frac{x^3}{3} + \frac{x^4}{12} + \frac{x^5}{60} + \frac{x^6}{720}.$$

3. Use Picard's method to solve $y' = x^2 y - y$ for $x = 0.25$ and $x = 0.5$ given that $y(0) = 1$.

Solution: Here $x_0 = 0$, $y_0 = 1$ and $f(x, y) = x^2 y - y$. Then Picard's iteration is given by

$$y_n(x) = y_0 + \int_{x_0}^x f(t, y_{n-1}(t))dt$$

$$\text{or, } y_n(x) = 1 + \int_0^x (t^2 y_{n-1}(t) - y_{n-1}(t))dt.$$

When $n = 1$, we get

$$y_1(x) = 1 + \int_0^x (t^2 y_0(t) - y_0(t))dt = 1 + \int_0^x (t^2 \cdot 1 - 1)dt$$

$$= 1 + \int_0^x (t^2 - 1)dt = 1 + \left[\frac{t^3}{3} - t\right]_0^x = 1 - x + \frac{x^3}{3}.$$

When $n = 2$, we get

$$y_2(x) = 1 + \int_0^x (t^2 y_1(t) - y_1(t))dt = 1 + \int_0^x \left[t^2\left(1 - t + \frac{t^3}{3}\right) - \left(1 - t + \frac{t^3}{3}\right)\right] dt$$

$$= 1 + \int_0^x \left(-1 + t - \frac{t^3}{3} + t^2 - t^3 + \frac{t^5}{3}\right) dt = 1 + \int_0^x \left(-1 + t + t^2 - \frac{4t^3}{3} + \frac{t^5}{3}\right) dt$$

$$= 1 + \left[-t + \frac{t^2}{2} + \frac{t^3}{3} - \frac{t^4}{3} + \frac{t^6}{18}\right]_0^x = 1 - x + \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{3} + \frac{x^6}{18}.$$

Now when $x = 0.25$, we get,

$$y_2(0.25) = 1 - 0.25 + \frac{(0.25)^2}{2} + \frac{(0.25)^3}{3} - \frac{(0.25)^4}{3} + \frac{(0.25)^6}{18} = 0.78517$$

When $x = 0.5$, we get,

$$y_2(0.5) = 1 - 0.5 + \frac{(0.5)^2}{2} + \frac{(0.5)^3}{3} - \frac{(0.5)^4}{3} + \frac{(0.5)^6}{18} = 0.64671$$

Problems

Solve the following equations by Picard's iteration method and estimate $y(0.1)$ and $y(0.2)$:

1. $y' = x^2 + y^2$, $y(0) = 0$
2. $y' = xe^y$, $y(0) = 0$
3. $\dfrac{dy}{dx} = x + (x+1)y$, $y(0) = 1$
4. $\dfrac{dy}{dx} = x^2 y^2$, $y(1) = 0$
5. $\dfrac{dy}{dx} = \dfrac{x}{y}$, $y(0) = 1$

5.2.3 Euler's Method

Suppose we are given a differential equation of the form

$$y' = f(x, y), \qquad y(x_0) = y_0.$$

If $y(x)$ is a solution of this differential equation, then Taylor's series expansion of $y(x)$ at $x = x_0$ is given by

$$y(x) = y(x_0) + y'(x_0)(x - x_0) + \frac{y''(x_0)}{2!}(x - x_0)^2 + \cdots$$

Euler's method considers only the first two terms in the above Taylor's series expansion of $y(x)$ i.e.,

$$y(x) = y(x_0) + y'(x_0)(x - x_0) \cdots\cdots (1)$$

Then the value of $y(x)$ at $x = x_1$ at a distance h from x_0 is

$$y(x_1) = y(x_0) + y'(x_0)(x_1 - x_0)$$

$$\text{or, } y(x_1) = y(x_0) + f(x_0, y_0)h$$

Now the value of $y(x)$ at $x = x_2$ at a distance h from x_1 is approximated as

$$y(x_2) = y(x_1) + f(x_1, y(x_1))h.$$

In general, we obtain a recursive relation

$$y(x_{n+1}) = y(x_n) + hf(x_n, y(x_n)), \qquad n = 0, 1, 2, \cdots$$

where x_i and x_{i+1} are at a distance h apart for all i. This method of evaluating approximations of $y(x)$ at $x = x_0, x_1, x_2, \cdots$ each at a distance h apart from the previous point is called the Euler's method.

Algorithm (Euler's Method):

INPUT: A differential equation $y' = f(x, y)$ with initial condition $y(x_0) = y_0$, step-size h and point x_a at which $y(x_a)$ is to be approximated.

PROCESS:

SET $n = \dfrac{x_a - x_0}{h}$

FOR $i = 1$ to n {

SET $x_i = x_{i-1} + h$

SET $y_i = y_{i-1} + h * f(x_{i-1}, y_{i-1})$

}

OUTPUT: The approximation y_n of $y(x_a)$.

Examples

1. Given the equation $y' = 3x^2 + 1$ with $y(1) = 2$, estimate $y(2)$ by using Euler's method using $h = 0.25$.

Solution: Here $x_0 = 1$, $y(x_0) = 2$, $f(x, y) = 3x^2 + 1$ and $h = 0.25$. The formula for Euler's iterative method is given by

$$y(x_{n+1}) = y(x_n) + hf(x_n, y(x_n))$$

$$\text{or, } y(x_{n+1}) = y(x_n) + 0.25(3x_n^2 + 1)$$

for $n = 0, 1, 2, \cdots$.

Now we calculate $y(2)$ iteratively using the above formula:

$$y(x_0) = y(1) = 2$$

$$y(x_1) = y(x_0 + h) = y(1.25) = y(1) + 0.25(3 \times 1^2 + 1) = 2 + 1 = 3$$

$$y(x_2) = y(x_1 + h) = y(1.5) = y(1.25) + 0.25(3 \times 1.25^2 + 1) = 3 + 1.42188 = 4.42188$$

$$y(x_3) = y(x_2 + h) = y(1.75) = y(1.5) + 0.25(3 \times 1.5^2 + 1) = 4.42188 + 1.9375 = 6.35938$$

$$y(x_4) = y(x_3 + h) = y(2) = y(1.75) + 0.25(3 \times 1.75^2 + 1) = 6.35938 + 2.54688 = 8.90626$$

Therefore the estimate of $y(2)$ is 8.90626.

2. Given the equation $y' = x^2 + y^2$ with $y(0) = 2$, estimate $y(1)$ by using $h = 0.1$ in Euler's iteration.

Solution: Here $x_0 = 1$, $y(x_0) = 2$, $f(x, y) = x^2 + y^2$ and $h = 0.1$. The formula for Euler's iterative method is given by

$$y(x_{n+1}) = y(x_n) + hf(x_n, y(x_n))$$

$$\text{or, } y(x_{n+1}) = y(x_n) + 0.1(x_n^2 + y(x_n)^2)$$

for $n = 0, 1, 2, \cdots$.

Now we calculate $y(1)$ iteratively using the above formula:

$$y(x_0) = y(0) = 2$$

$$y(x_1) = y(x_0 + h) = y(0.1) = y(0) + 0.1(0^2 + 2^2) = 2 + 0.4 = 2.4$$

$$y(x_2) = y(x_1 + h) = y(0.2) = y(0.1) + 0.1(0.1^2 + 2.4^2) = 2.4 + 0.577 = 2.977$$

$$y(x_3) = y(x_2 + h) = y(0.3) = y(0.2) + 0.1(0.2^2 + 2.977^2) = 2.977 + 0.89025 = 3.86725$$

$$y(x_4) = y(x_3 + h) = y(0.4) = y(0.3) + 0.1(0.3^2 + 3.86725^2) = 3.86725 + 1.50456 = 5.37181$$

$$y(x_5) = y(x_4 + h) = y(0.5) = y(0.4) + 0.1(0.4^2 + 5.37181^2) = 5.37181 + 2.90163 = 8.27344$$

$$y(x_6) = y(x_5 + h) = y(0.6) = y(0.5) + 0.1(0.5^2 + 8.27344^2) = 8.27344 + 6.86998 = 15.14342$$

$$y(x_7) = y(x_6 + h) = y(0.7) = y(0.6) + 0.1(0.6^2 + 15.14342^2) = 15.14342 + 22.96831 = 38.11174$$

$y(x_8) = y(x_7+h) = y(0.8) = y(0.7)+0.1(0.7^2+38.11174^2) = 38.11174+145.29947 = 183.41121$

$y(x_9) = y(x_8+h) = y(0.9) = y(0.8)+0.1(0.8^2+183.41121^2) = 183.41121+3364.03119 = 3547.4424$

$y(x_{10}) = y(x_9+h) = y(1) = y(0.9)+0.1(0.9^2+3547.4424^2) = 3547.4424+1258434.8391 = 1261982.2815$

Therefore the estimate of $y(1)$ is 1261982.2815.

3. Given the equation $y' = 3x + \frac{y}{2}$ with $y(0) = 1$, estimate $y(0.2)$ by using $h = 0.05$ in Euler's iteration method.

Solution: Here $x_0 = 1$, $y(x_0) = 1$, $f(x, y) = 3x + \frac{y}{2}$ and $h = 0.05$. The formula for Euler's iterative method is given by

$$y(x_{n+1}) = y(x_n + h) = y(x_n) + hf(x_n, y(x_n))$$

$$\text{or, } y(x_{n+1}) = y(x_n + 0.05) = y(x_n) + 0.05\left(3x_n + \frac{y(x_n)}{2}\right)$$

for $n = 0, 1, 2, \cdots$.

Now we calculate $y(0.2)$ iteratively using the above formula:

$$y(0) = 1$$

$$y(0.05) = y(0 + 0.05) = y(0) + 0.05\left(3 \times 0 + \frac{y(0)}{2}\right)$$

$$= 1 + 0.05\left(3 \times 0 + \frac{1}{2}\right) = 1.025$$

$$y(0.1) = y(0.05 + 0.05) = y(0.05) + 0.05\left(3 \times 0.05 + \frac{y(0.05)}{2}\right)$$

$$= 1.025 + 0.05\left(3 \times 0.05 + \frac{1.025}{2}\right) = 1.058125$$

$$y(0.15) = y(0.1 + 0.05) = y(0.1) + 0.05\left(3 \times 0.1 + \frac{y(0.1)}{2}\right)$$

$$= 1.058125 + 0.05\left(3 \times 0.1 + \frac{1.058125}{2}\right) = 1.09958$$

$$y(0.2) = y(0.15 + 0.05) = y(0.15) + 0.05\left(3 \times 0.15 + \frac{y(0.15)}{2}\right)$$

$$= 1.09958 + 0.05\left(3 \times 0.15 + \frac{1.09958}{2}\right) = 1.1495695$$

Therefore the estimate of $y(0.2)$ is 1.1495695.

Problems

Estimate $y(1)$ using Euler method for the following differential equations taking $h = 0.25$.

1. $y' = 2xy$, $y(0) = 1$
2. $y' = \dfrac{-y}{2y+1}$, $y(0) = 1$
3. $y' = \dfrac{x}{y}$, $y(0) = 1$
4. $y' = x + y + xy$, $y(0) = 1$

5.2.4 Modified Euler's Method (Heun's Method)

Given a differential equation of the form

$$y' = f(x, y), \qquad y(x_0) = y_0,$$

Heun's method or modified Euler's method uses the following recursive relation for approximating its solution:

$$y(x_{n+1}) = y(x_n) + \frac{h}{2}[f(x_n, y(x_n)) + f(x_{n+1}, y^*(x_{n+1}))], \qquad n = 0, 1, 2, \cdots$$

where

$$y^*(x_{n+1}) = y(x_n) + hf(x_n, y(x_n)).$$

Algorithm (Heun's Method):

INPUT: A differential equation $y' = f(x, y)$ with initial condition $y(x_0) = y_0$, step-size h and point x_a at which $y(x_a)$ is to be approximated.

PROCESS:

SET $n = \dfrac{x_a - x_0}{h}$

FOR $i = 1$ to n {

SET $x_i = x_{i-1} + h$

SET $m = y_{i-1} + h * f(x_{i-1}, y_{i-1})$

SET $y_i = y_{i-1} + \dfrac{h}{2}[f(x_{i-1}, y_{i-1}) + f(x_i, m)]$

}

OUTPUT: The approximation y_n of $y(x_a)$.

Examples

1. Given the equation $y' = 3x^2 + 1$ with $y(1) = 2$, estimate $y(2)$ by using Heun's method using $h = 0.25$.

Solution: Here $x_0 = 1$, $y(x_0) = 2$, $f(x, y) = 3x^2 + 1$ and $h = 0.25$. The formula for Heun's iterative method is given by

$$y(x_{n+1}) = y(x_n) + \frac{h}{2}[f(x_n, y(x_n)) + f(x_{n+1}, y^*(x_{n+1}))]$$

where

$$y^*(x_{n+1}) = y(x_n) + hf(x_n, y(x_n))$$

for $n = 0, 1, 2, \cdots$.

We can now calculate $y(2)$ iteratively using the above formula:

Calculation of $y(x_1) = y(1.25)$**:**

$$y^*(x_1) = y(x_0) + hf(x_0, y(x_0)) = 2 + 0.25(3 \times 1^2 + 1) = 2 + 1 = 3$$

So

$$y(x_1) = y(x_0) + \frac{h}{2}[f(x_0, y(x_0)) + f(x_1, y^*(x_1))] = 2 + \frac{0.25}{2}[(3 \times 1^2 + 1) + (3 \times 1.25^2 + 1)]$$

$$= 2 + 0.125(4 + 5.6875) = 3.211$$

Calculation of $y(x_2) = y(1.5)$**:**

$$y^*(x_2) = y(x_1) + hf(x_1, y(x_1)) = 3.211 + 0.25(3 \times 1.25^2 + 1) = 3.211 + 1.42187 = 4.63288$$

So

$$y(x_2) = y(x_1) + \frac{h}{2}[f(x_1, y(x_1)) + f(x_2, y^*(x_2))] = 3.211 + \frac{0.25}{2}[(3 \times 1.25^2 + 1) + (3 \times 1.5^2 + 1)]$$

$$= 3.211 + 0.125(5.6875 + 7.75) = 4.89069$$

Calculation of $y(x_3) = y(1.5)$**:**

$$y^*(x_3) = y(x_2) + hf(x_2, y(x_2)) = 4.89069 + 0.25(3 \times 1.5^2 + 1) = 4.89069 + 1.9375 = 6.82819$$

So

$$y(x_3) = y(x_2) + \frac{h}{2}[f(x_2, y(x_2)) + f(x_3, y^*(x_3))] = 4.89069 + \frac{0.25}{2}[(3 \times 1.5^2 + 1) + (3 \times 1.75^2 + 1)]$$

$$= 4.89069 + 0.125(7.75 + 10.1875) = 7.13288$$

Calculation of $y(x_4) = y(2)$**:**

$$y^*(x_4) = y(x_3) + hf(x_3, y(x_3)) = 7.13288 + 0.25(3 \times 1.75^2 + 1) = 7.13288 + 2.54688 = 9.67976$$

So

$$y(x_4) = y(x_3)+\frac{h}{2}[f(x_3, y(x_3))+f(x_4, y^*(x_4))] = 7.13288+\frac{0.25}{2}[(3\times1.75^2+1)+(3\times2^2+1)]$$

$$= 7.13288 + 0.125(10.1875 + 13) = 10.03132$$

Therefore the estimate of $y(2)$ is 10.03132.

2. Given the equation $y' = 2xy$ with $y(0) = 1$, estimate $y(1)$ by using Heun's method using $h = 0.2$.

Solution: Here $x_0 = 1$, $y(x_0) = 1$, $f(x, y) = 2xy$ and $h = 0.2$. The formula for Heun's iterative method is given by

$$y(x_{n+1}) = y(x_n) + \frac{h}{2}[f(x_n, y(x_n)) + f(x_{n+1}, y^*(x_{n+1}))]$$

where

$$y^*(x_{n+1}) = y(x_n) + hf(x_n, y(x_n))$$

for $n = 0, 1, 2, \cdots$.

We can now calculate $y(1)$ iteratively using the above formula:

Calculation of $y(x_1) = y(0.2)$**:**

$$y^*(x_1) = y(x_0) + hf(x_0, y(x_0)) = 1 + 0.2(2 \times 0 \times 1) = 1$$

So

$$y(x_1) = y(x_0) + \frac{h}{2}[f(x_0, y(x_0)) + f(x_1, y^*(x_1))] = 1 + \frac{0.2}{2}[(2 \times 0 \times 1) + (2 \times 0.2 \times 1)]$$

$$= 1 + 0.04 = 1.04$$

Calculation of $y(x_2) = y(0.4)$**:**

$$y^*(x_2) = y(x_1) + hf(x_1, y(x_1)) = 1.04 + 0.2(2 \times 0.2 \times 1.04) = 1.04 + 0.0832 = 1.1232$$

So

$$y(x_2) = y(x_1)+\frac{h}{2}[f(x_1, y(x_1))+f(x_2, y^*(x_2))] = 1.04+\frac{0.2}{2}[(2\times0.2\times1.04)+(2\times0.4\times1.1232)]$$

$$= 1.04 + 0.1(0.416 + 0.89856) = 1.17146$$

Calculation of $y(x_3) = y(0.6)$**:**

$$y^*(x_3) = y(x_2)+hf(x_2, y(x_2)) = 1.17146+0.2(2\times0.4\times1.17146) = 1.17146+0.18743 = 1.35889$$

So

$$y(x_3) = y(x_2)+\frac{h}{2}[f(x_2, y(x_2))+f(x_3, y^*(x_3))] = 1.17146+\frac{0.2}{2}[(2\times0.4\times1.17146)+(2\times0.6\times1.35889)]$$

$$= 1.17146 + 0.1(0.93717 + 1.63067) = 1.42824$$

Calculation of $y(x_4) = y(0.8)$**:**

$$y^*(x_4) = y(x_3) + hf(x_3, y(x_3)) = 1.42824 + 0.2(2\times0.6\times1.42824) = 1.42824 + 0.34278 = 1.77102$$

So

$$y(x_4) = y(x_3) + \frac{h}{2}[f(x_3, y(x_3)) + f(x_4, y^*(x_4))] = 1.42824 + \frac{0.2}{2}[(2\times0.6\times1.42824) + (2\times0.8\times1.77102)]$$

$$= 1.42824 + 0.1(1.71389 + 2.83363) = 1.88299$$

Calculation of $y(x_5) = y(1)$**:**

$$y^*(x_5) = y(x_4) + hf(x_4, y(x_4)) = 1.88299 + 0.2(2\times0.8\times1.88299) = 1.88299 + 0.60256 = 2.48555$$

So

$$y(x_5) = y(x_4) + \frac{h}{2}[f(x_4, y(x_4)) + f(x_5, y^*(x_5))] = 1.88299 + \frac{0.2}{2}[(2\times0.8\times1.88299) + (2\times1\times2.48555)]$$

$$= 1.88299 + 0.1(3.01278 + 4.9711) = 2.68138$$

Hence the estimate of $y(1)$ is 2.68138.

Problems

3. Use Heun's method to solve $y' = \dfrac{2y}{x}$, $y(1) = 2$ to estimate the value of $y(2)$ with $h = 0.25$.

4. Solve 3(a) to 3(e) page 461 of Balagurusamy's book with $h = 0.25$ and find $y(1)$.

a. $y' = 2xy$, $y(0) = 1$

b. $y' = \dfrac{-y}{2y+1}$, $y(0) = 1$

c. $y' = \dfrac{x}{y}$, $y(0) = 1$

d. $y' = x + y + xy$, $y(0) = 1$

5.2.5 Runge-Kutta Method

Given a differential equation of the form

$$y' = f(x, y), \qquad y(x_0) = y_0,$$

the fourth order Runge-Kutta method uses the following recursive relation for approximating its solution:

$$y(x_{n+1}) = y(x_n) + \frac{h}{6}(m_1 + 2m_2 + 2m_3 + m_4), \qquad n = 0, 1, 2, \cdots$$

where

$$m_1 = f(x_n, y(x_n)),$$
$$m_2 = f\left(x_n + \frac{h}{2}, y(x_n) + \frac{m_1 h}{2}\right),$$
$$m_3 = f\left(x_n + \frac{h}{2}, y(x_n) + \frac{m_2 h}{2}\right),$$
$$m_4 = f\left(x_n + h, y(x_n) + m_3 h\right).$$

Algorithm (Fourth Order Runge-Kutta Method):

INPUT: A differential equation $y' = f(x, y)$ with initial condition $y(x_0) = y_0$, step-size h and point x_a at which $y(x_a)$ is to be approximated.

PROCESS:

SET $n = \dfrac{x_a - x_0}{h}$

FOR $i = 1$ to n {

SET $x_i = x_{i-1} + h$

SET $m_1 = f(x_{i-1}, y_{i-1})$

SET $m_2 = f\left(x_{i-1} + \dfrac{h}{2}, y_{i-1} + \dfrac{m_1 h}{2}\right)$

SET $m_3 = f\left(x_{i-1} + \dfrac{h}{2}, y_{i-1} + \dfrac{m_2 h}{2}\right)$

SET $m_4 = f\left(x_{i-1} + h, y_{i-1} + m_3 h\right)$

SET $y_i = y_{i-1} + \dfrac{h}{6}[m_1 + 2(m_2 + m_3) + m_4]$

}

OUTPUT: The approximation y_n of $y(x_a)$.

Examples

Problems

5.3 Solving Systems of Differential Equations

Suppose we are given a system of m first order differential equations with initial values as below:

$$\frac{dy_1}{dx} = f_1(x, y_1, y_2, \cdots, y_m), \quad y_1(x_0) = y_{10}$$
$$\frac{dy_2}{dx} = f_2(x, y_1, y_2, \cdots, y_m), \quad y_2(x_0) = y_{20}$$