Numerical Method (CSC 207)

(Lecture Note for B.Sc. CSIT Third Semester)

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Note: This is a work in progress. Please send your suggestions and comments at

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Contents

Unit 1

Solution of Nonlinear Equations

1.1 Review of Calculus

Taylor's Theorem: If the function f is $n+1$ times continuously differentiable in a closed interval $I = [a, b]$, then for any c and x in I,

$$f(x) = f(c) + \frac{f'(c)}{1!}(x-c) + \frac{f''(c)}{2!}(x-c)^2 + \cdots + \frac{f^{(n)}(c)}{n!}(x-c)^n + E_{n+1}$$

where the error term E_{n+1} can be written in the form

$$E_{n+1} = \frac{f^{(n+1)}(r)}{(n+1)!}(x-c)^{n+1}$$

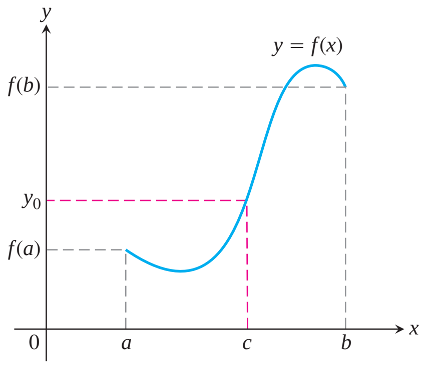
for some point r that lies in between c and x.

For example, let $f(x) = e^x$ and $I = [-1, 1]$. Then $f^{(n)}(x) = e^x$ for all n and so by Taylor's theorem, we can write for $c = 0$ and for $x \in I$,

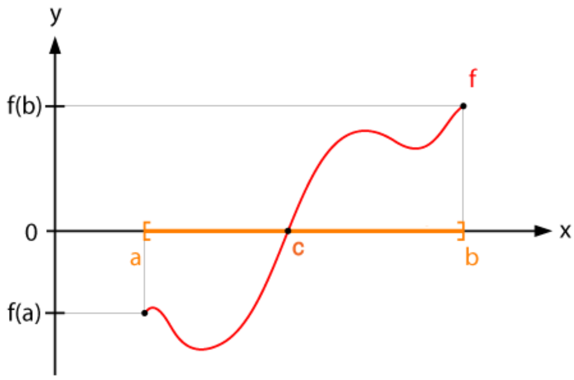
$$\begin{aligned} f(x) &= f(c) + \frac{f'(c)}{1!}(x-c) + \frac{f''(c)}{2!}(x-c)^2 + \cdots + \frac{f^{(n)}(c)}{n!}(x-c)^n + E_{n+1} \\ &= f(0) + f'(0)x + \frac{f''(0)}{2}x^2 + \cdots + \frac{f^{(n)}(0)}{n!}x^n + E_{n+1} \\ &= 1 + x + \frac{x^2}{2} + \cdots + \frac{x^n}{n!} + E_{n+1} \end{aligned}$$

where $E_{n+1} = \dfrac{e^r}{(n+1)!}x^{n+1}$ for some r in between 0 and x.

Intermediate Value Theorem (IVT): If f is a continuous function on the interval $[a, b]$ with $f(a) \leq f(b)$ and $y_0 \in [f(a), f(b)]$, then there exists a point $c \in [a, b]$ such that $f(c) = y_0$.



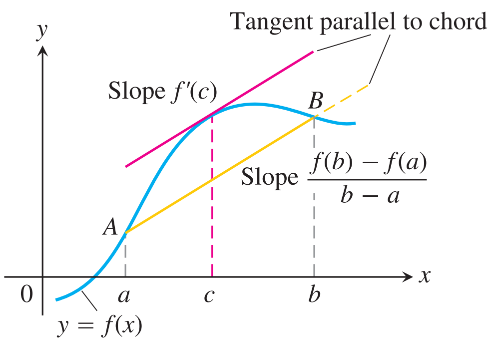
Note: Suppose that f is a continuous function on the internal $[a, b]$ such that $f(a)f(b) < 0$. Then either $f(a) < 0, f(b) > 0$ or $f(a) > 0, f(b) < 0$. Suppose that the first case holds. Then $0 \in (f(a), f(b))$ and so by IVT, there is a $c \in (a, b)$ such that $f(c) = 0$ i.e. f has a root in the interval (a, b). If the second case holds, then $0 \in (f(b), f(a))$ and so again by IVT, there exists $c \in (a, b)$ such that $f(c) = 0$.
Therefore if f is a continuous on $[a, b]$ with $f(a)f(b) < 0$, then there must exist a root of f inside the interval $[a, b]$. We will later use this fact to construct the bisection method for finding roots of a function f.



Mean Value Theorem (MVT): If f is a continuous function on the interval $[a, b]$ and differentiable on the open interval (a, b), then

$$f(b) - f(a) = (b - a)f'(c)$$

for some $c \in (a, b)$.



1.2 Errors in Numerical Calculation

Truncation Error: The error caused by the numerical method used to solve the problem is known as truncation error . For example, we can approximate e^x as

$$e^x \approx 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!}$$

which is obtained by taking only the first four terms of the following Taylor's series expansion of e^x:

$$1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots + \frac{x^n}{n!} + \cdots$$

So the truncation error that occurs in this case due to the use of Taylor's series is

$$\sum_{n=4}^{\infty} \frac{x^n}{n!}.$$

Round-Off Error: Every computer uses a fixed amount of memory to represent numerical quantities such as integers and floating-point numbers. If during some calculation, a number is generated which is larger that the allocated amount of memory, then such numbers are stored by approximating it to the nearest number that the computer can represent. The error that occurs during this process is known as the round-off error. For example, if a computer can represent only 5 digit floating point number, then a number like 362.682 is rounded off to 362.68 with 0.002 as the round-off error.

> **Process of rounding-off a number:** Suppose we have to round-off a number, say 3.22363, up to 3 decimal places. Then we have to discard the last two decimal digits in the above number. In this case, since the first digit to be discarded is 6, which is greater than 5, we increase the digit just left to the first discarded digit by 1, so the number would be 3.224 after round-off. If the number had been 3.22133, then because the first digit to be discarded is 3 which is less than 5, we do not change the value of the left digit. So the number would be just 3.221 after round-off. However if the first digit to be discarded is exactly 5, then the digit just left to the first discarded digit is left as it is if it is even and increased by 1 if it is odd. So 3.22453 would be 3.224 and 3.22153 would be 3.222 after round-off.

Error in Original Data: When real world problems are solved using numerical methods, one must gather data to model the physical situation. But the data gathered during this process are usually not exact due to various reasons such as incorrect measurement or inaccurate measuring instruments. Such kinds of errors are called error in original data.

Blunders: Blunders are the mistakes that involves humans. Such human errors can take many forms such as lack of problem understanding, wrong assumptions, making mistakes in computer programs, various mistakes in data inputs etc. Blunders can be avoided through a good understanding of problems and the numerical solution methods used as well as use of good programming tools and techniques.

Propagated Error: Errors that occur in the succeeding steps of a numerical process due to an occurrence of an earlier error is called a propagated error. Controlling propagated error is very important in numerical computation. If errors increase continuously as the method continues, then it may grow out of control. Such numerical method is called unstable. If the errors made at early stages cancel out as the method continues, then such numerical method is said to be stable. For example, the process of rounding off numbers introduced earlier, more-or-less cancels the round-off errors made in earlier stages of the process.

Floating-Point Arithmetic: In decimal number system, a nonzero real number x can be represented as

$$x = \pm 0.d_1 d_2 \dots d_p \times 10^n$$

where $1 \leq d_1 \leq 9$, $0 \leq d_i \leq 9$ for $2 \leq i \leq p$ and n is an integer. This is called a normalized floating-point representation of x. It consists of three parts: a symbol $+$ or $-$ which is called the sign, the fraction part $d_1 d_2 \dots d_p$ which is also called the mantissa or significand where the number of digits p is called the precision and the exponent part n which is also called the characteristic. For example, the normalized floating-point representation of some numbers is given below:

$$\begin{aligned} 25.1236 &= 0.251236 \times 10^2 \\ 0.001236584 &= 0.1236584 \times 10^{-2} \\ -11566.12036 &= -0.1156612036 \times 10^5 \end{aligned}$$

Absolute and Relative Error: If x_t denotes the true value of a numerical quantity and x_0 denotes its approximate value, then we define absolute error e_a and relative error e_r as follows:

$$e_a = |x_t - x_0|$$

$$e_r = \frac{\text{absolute error}}{|\text{true value}|} = \left| \frac{x_t - x_0}{x_t} \right|$$

Relative error is also expressed in terms of percentages by multiplying e_r by 100. Among these two measurements of error, the relative error gives a truer reflection of the actual error made. For example, suppose a numerical quantity with true value 10 is approximated as 10.1 and another numerical quantity with true value 100 is approximated as 100.1. The absolute error in both the cases is 0.1 so we might falsely conclude that both the approximations are equally good. But it is clear that the first approximation has more error in it than the second approximation. This is reflected in the fact that the relative error in the first case is 0.01 and in the second case it is 0.001.

1.3 Trial and Error Method

As the name itself suggests, trial and error method of finding a root of a nonlinear function $f(x)$ consists of taking a guess c at the root of $f(x)$ and then checking whether $f(c)$ is zero or not. If $f(c) = 0$, then c is the root. If not, then we take another guess at the root. We continue this process of 'trial and error' until we find a root or a sufficiently close approximation of it.

Although it is a very simple method of root-finding, it is also the most inefficient. It is impossible to tell as to how many guesses one has to make before finding a satisfactory approximation of the root.

1.4 Bisection Method (Half-Interval Method)

Suppose that $f(x)$ is a continuous function on the interval $[a_0, b_0]$ and $f(a_0)f(b_0) < 0$. Then by intermediate value theorem, there exists a root of $f(x)$ in the interval (a_0, b_0). Bisection method calculates the first approximation c_0 of this root as the midpoint of the interval (a_0, b_0):

$$c_0 = \frac{a_0 + b_0}{2}.$$

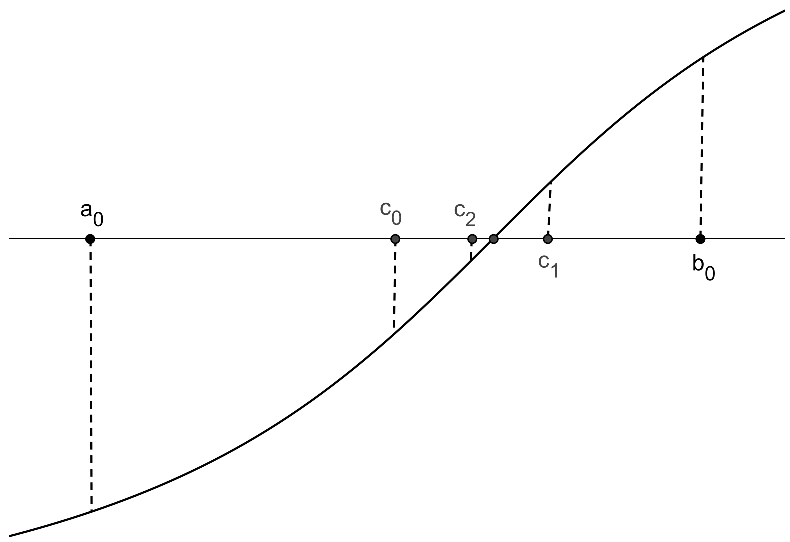
If $f(c_0) = 0$, then c_0 is the root of $f(x)$. If not, then we bisect the interval $[a_0, b_0]$ into two equal-length subintervals $[a_0, c_0]$ and $[c_0, b_0]$. From these two subintervals, we keep the one that is guaranteed to contain a root and discard the other. So we obtain an interval $[a_1, b_1]$ by setting $a_1 = a_0$, $b_1 = c_0$ if $f(a_0)f(c_0) < 0$ (this guarantees that there is a root inside $[a_0, c_0]$) and $a_1 = c_0$, $b_1 = b_0$ if $f(c_0)f(b_0) < 0$ (this guarantees that there is a root inside $[c_0, b_0]$). The second approximation of the root is now calculated as

$$c_1 = \frac{a_1 + b_1}{2}.$$

If $f(c_1) = 0$, then c_1 is the root of $f(x)$. If not, then we again bisect the interval $[a_1, b_1]$ into two equal-length subintervals $[a_1, c_1]$ and $[c_1, b_1]$ and set $a_2 = a_1$, $b_2 = c_1$ if $f(a_1)f(c_1) < 0$ and $a_2 = c_1$, $b_2 = b_1$ if $f(c_1)f(b_1) < 0$ and then calculate the third approximation as

$$c_2 = \frac{a_2 + b_2}{2},$$

thus continuing the above process. This process of calculating the approximations $c_0, c_1, c_2, \cdots$ is repeated until we find a root of $f(x)$ or a satisfactory approximation of it.



Note: Among many numerical methods of root approximation that needs two points to start with, some have the property that all the approximations generated by that method always lie inside the interval formed by those two starting points. Such methods are called **bracketing method** . For example, bisection method is a bracketing method. Method of false position (we do not study this) is another such method. If the approximations generated by the method may lie outside of the initial interval, then such methods of root approximation are called **non-bracketing method** . Secant method (to be studied below) is an example of non-bracketing method.

Algorithm (Bisection Method):

INPUT: A continuous function $f(x)$ defined on $[a, b]$ such that $f(a)f(b) < 0$ and termination parameter ε.

PROCESS:

DO {

SET $c = \dfrac{a+b}{2}$
IF $f(c) = 0$, STOP (c is the solution)
IF $f(a)f(c) < 0$ SET $b = c$
ELSE SET $a = c$

} WHILE $\left|\dfrac{b-a}{b}\right| > \varepsilon$

OUTPUT: Root of $f(x)$ or its approximation c.

Convergence of Bisection Method:

If $f(x)$ is a continuous function defined on the interval $[a_0, b_0]$ such that $f(a_0)f(b_0) < 0$, then there exists a root of $f(x)$ on the interval (a_0, b_0). Let that root be r i.e. $f(r) = 0$. Using the bisection method, the first approximation of r is the value $c_0 = \dfrac{a_0 + b_0}{2}$ and hence the error of approximating r by c_0 is $|r - c_0|$. Obviously

$$|r - c_0| \leq \frac{b_0 - a_0}{2}.$$

The next approximation of r is $c_1 = \dfrac{a_1 + b_1}{2}$ where $[a_1, b_1]$ is an interval of length $b_1 - a_1 = \dfrac{b_0 - a_0}{2}$. Hence the error of approximating r by c_1 is

$$|r - c_1| \leq \frac{b_1 - a_1}{2} = \frac{b_0 - a_0}{2^2}.$$

Continuing similarly, if c_n is the $(n+1)^{st}$ approximation of r, then

$$|r - c_n| \leq \frac{b_0 - a_0}{2^{n+1}}.$$

As $n \to \infty$, we have

$$\frac{b_0 - a_0}{2^{n+1}} \to 0$$

and hence

$$|r - c_n| \to 0$$

as $n \to \infty$. This means that the approximations $c_0, c_1, c_2, \cdots, c_n, \cdots$ of the root r obtained by using the bisection method converges to the actual root r as $n \to \infty$.

Advantages:

(1) This method is guaranteed to work for any continuous function $f(x)$ on the interval $[a, b]$ with $f(a)f(b) < 0$.

(2) The minimum number of iterations required to achieve a specified degree of accuracy is known in advance. For instance, if we require that the error $|r - c_n|$ is less than ε for some $\varepsilon > 0$ then we can calculate the required number of iterations n as follows. Since

$$|r - c_n| \leq \frac{b_0 - a_0}{2^{n+1}}$$

so if

$$\frac{b_0 - a_0}{2^{n+1}} < \varepsilon \cdots (A)$$

then we obviously have $|r - c_n| < \varepsilon$ as well. Now taking $\log$ on both sides of (A), we have

$$\log\left(\frac{b_0 - a_0}{2^{n+1}}\right) < \log \varepsilon$$

$$\text{or, } \log(b_0 - a_0) - \log 2^{n+1} < \log \varepsilon$$

$$\text{or, } (n+1)\log 2 > \log(b_0 - a_0) - \log \varepsilon$$

$$\text{or, } n > \frac{1}{\log 2} \log\left(\frac{b_0 - a_0}{\varepsilon}\right) - 1$$

Disadvantages:

(1) The method converges slowly, i.e. it requires more iterations to achieve the same accuracy when compared with some other methods. This problem becomes more severe when the actual root lies very close to one of the endpoints of the initial interval.

(2) This method is not applicable for functions $f(x)$ for which either $f(x) \geq 0$ or $f(x) \leq 0$ for all x in the domain of f.

1.4.1 Exercise

1. Find the root of $x^3 - 3x + 1$ in the interval $[0, 1]$ correct up to 3 decimal places using the bisection method.

   Solution: We have $f(x) = x^3 - 3x + 1$ and $f(0) = 1$, $f(1) = -1$. Hence we can take the initial values as $a_0 = 0$ and $b_0 = 1$. Since we require approximation to be correct up to 3 decimal places, we take at least 4 decimals in our calculations.

n	a	$f(a)$	b	$f(b)$	$c = \frac{a+b}{2}$	$f(c)$
1	0	$+ve$	1	$-ve$	0.5	$-ve$
2	0	$+ve$	0.5	$-ve$	0.25	$+ve$
3	0.25	$+ve$	0.5	$-ve$	0.375	$-ve$
4	0.25	$+ve$	0.375	$-ve$	0.3125	$+ve$
5	0.3125	$+ve$	0.375	$-ve$	0.3438	$+ve$
6	0.3438	$+ve$	0.375	$-ve$	0.3594	$-ve$
7	0.3438	$+ve$	0.3594	$-ve$	0.3516	$-ve$
8	0.3438	$+ve$	0.3516	$-ve$	0.3477	$-ve$
9	0.3438	$+ve$	0.3477	$-ve$	0.3458	$+ve$
10	0.3458	$+ve$	0.3477	$-ve$	0.3468	$+ve$
11	0.3468	$+ve$	0.3477	$-ve$	0.3472	$+ve$
12	0.3472	$+ve$	0.3477	$-ve$	0.3474	

   Since the two consecutive values of c agree up to 3 decimal places, so we terminate the process. Hence the required approximation of root of $x^3 - 3x + 1$ correct up to 3 decimal places is 0.3474.

2. Find the solution of $x^3 - 2\sin x$ in the interval $[0.5, 2]$ correct up to 5 significant digits.

   Solution: The given function is $f(x) = x^3 - 2\sin x$ with $f(0.5) = -0.83385$ and $f(2) = 6.1814$. Hence we can take the initial values as $a_0 = 0.5$, $b_0 = 2$.

n	a	$f(a)$	b	$f(b)$	$c = \dfrac{a+b}{2}$	$f(c)$
1	0.5	$-ve$	2	$+ve$	1.25	$+ve$
2	0.5	$-ve$	1.25	$+ve$	0.875	$-ve$
3	0.875	$-ve$	1.25	$+ve$	1.0625	$-ve$
4	1.0625	$-ve$	1.25	$+ve$	1.15625	$-ve$
5	1.15625	$-ve$	1.25	$+ve$	1.20312	$-ve$
6	1.20312	$-ve$	1.25	$+ve$	1.22656	$-ve$
7	1.22656	$-ve$	1.25	$+ve$	1.23828	$+ve$
8	1.22656	$-ve$	1.23828	$+ve$	1.23242	$-ve$
9	1.23242	$-ve$	1.23828	$+ve$	1.23535	$-ve$
10	1.23535	$-ve$	1.23828	$+ve$	1.23682	$+ve$
11	1.23535	$-ve$	1.23682	$+ve$	1.23608	$-ve$
12	1.23608	$-ve$	1.23682	$+ve$	1.23645	$+ve$
13	1.23608	$-ve$	1.23645	$+ve$	1.23626	$+ve$
14	1.23608	$-ve$	1.23626	$+ve$	1.23617	$-ve$
15	1.23617	$-ve$	1.23626	$+ve$	1.23622	$+ve$
16	1.23617	$-ve$	1.23622	$+ve$	1.23620	

Hence the required root of $x^3 - 2\sin x$ correct up to 5 significant digits is 1.23620.

3. Find a solution of $xe^x - 1$ correct up to 3 decimal places.

   Solution: If we take $a_0 = 0$ and $b_0 = 1$ as the initial points, then $f(a_0) = -1$, $f(b_0) = 1.71828$ so that $f(a_0)f(b_0) < 0$. Therefore there lies a root of $xe^x - 1$ in the interval $[0, 1]$.

n	a	$f(a)$	b	$f(b)$	$c = \dfrac{a+b}{2}$	$f(c)$
1	0	$-ve$	1	$+ve$	0.5	$-ve$
2	0.5	$-ve$	1	$+ve$	0.75	$+ve$
3	0.5	$-ve$	0.75	$+ve$	0.625	$+ve$
4	0.5	$-ve$	0.625	$+ve$	0.5625	$-ve$
5	0.5625	$-ve$	0.625	$+ve$	0.5938	$+ve$
6	0.5625	$-ve$	0.5938	$+ve$	0.5782	$+ve$
7	0.5625	$-ve$	0.5782	$+ve$	0.5704	$+ve$
8	0.5625	$-ve$	0.5704	$+ve$	0.5664	$-ve$
9	0.5664	$-ve$	0.5704	$+ve$	0.5684	$+ve$
10	0.5664	$-ve$	0.5684	$+ve$	0.5674	$+ve$
11	0.5664	$-ve$	0.5674	$+ve$	0.5669	$-ve$
12	0.5669	$-ve$	0.5674	$+ve$	0.5672	$+ve$
13	0.5669	$-ve$	0.5672	$+ve$	0.5670	

Hence 0.5670 is a solution of $xe^x - 1$ correct up to 3 decimal places.

4. Use the Bisection method to find solutions accurate to within 5 decimal places for the following problems: (Burden, Faires)

   (a) $3x - e^x = 0$ for $1 \leq x \leq 2$.

   (b) $2x + 3\cos x - e^x = 0$ for $1 \leq x \leq 2$.

   (c) $x^2 - 4x + 4 - \ln x = 0$ for $1 \leq x \leq 2$ and $2 \leq x \leq 4$.

   (d) $x + 1 - 2\sin \pi x = 0$ for $0 \leq x \leq 0.5$ and $0.5 \leq x \leq 1$.

   (e) $x - 2^{-x} = 0$ for $0 \leq x \leq 1$.

   (f) $e^x - x^2 + 3x - 2 = 0$ for $0 \leq x \leq 1$.

   (g) $2x\cos 2x - (x+1)^2 = 0$ for $-3 \leq x \leq -2$ and $-1 \leq x \leq 0$.

   (h) $x\cos x - 2x^2 + 3x - 1 = 0$ for $0.2 \leq x \leq 0.3$ and $1.2 \leq x \leq 1.3$.

5. Find a solution of each of the following using bisection method:

   (a) $e^x - x - 2 = 0$ (correct up to 4 significant digits)

   (b) $\sin x - 2x + 1 = 0$ (correct up to 2 decimal places)

   (c) $\log x - \cos x = 0$ (correct up to 3 decimal places)

   (d) $x^3 - 2x - 5 = 0$ (correct up to 3 decimal places)

   (e) $4x^3 - 2x - 6 = 0$ (correct up to 4 significant digits)

1.5 Secant Method

Suppose that $f(x)$ is a continuous function whose root is r. Let x_0 and x_1 be two initial points which are sufficiently close to r. We join the two points $P = (x_0, f(x_0))$ and $Q = (x_1, f(x_1))$ by the secant (chord) PQ. Then we take the point of intersection of this secant PQ and the x-axis as the first approximation to the root r. The equation of the secant PQ is

$$y - f(x_0) = \frac{f(x_1) - f(x_0)}{x_1 - x_0}(x - x_0) \cdots\cdots (1)$$

If $(x_2, 0)$ is the point of intersection of this secant and the x-axis, then since the point $(x_2, 0)$ lies on this line, so it must satisfy the above equation (1), i.e.

$$-f(x_0) = \frac{f(x_1) - f(x_0)}{x_1 - x_0}(x_2 - x_0)$$

$$\text{or, } x_2 = \frac{x_0 f(x_1) - x_1 f(x_0)}{f(x_1) - f(x_0)}$$

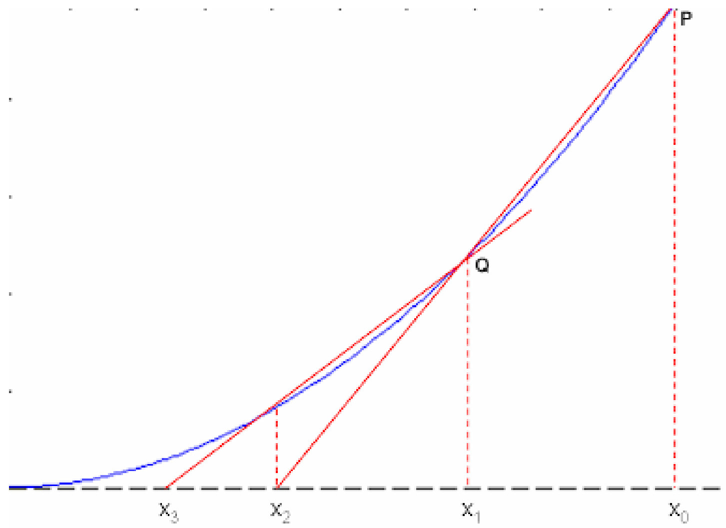
If $f(x_2) = 0$, then x_2 is the root of $f(x)$. If not, then we repeat the above process by joining the points $(x_1, f(x_1))$ and $(x_2, f(x_2))$ by a secant whose intersection with the x-axis is $(x_3, 0)$ where

$$x_3 = \frac{x_1 f(x_2) - x_2 f(x_1)}{f(x_2) - f(x_1)}.$$

This point is taken as the next approximation of the root r. In general, the n^{th} approximation of the root r is given by the formula

$$x_{n+1} = \frac{x_{n-1}f(x_n) - x_n f(x_{n-1})}{f(x_n) - f(x_{n-1})}, \quad n \geq 1.$$

We continue to calculate the approximations $x_2, x_3, \cdots$ using the above formula until we find the root or its satisfactory approximation.



Algorithm (Secant Method):

INPUT: A continuous function $f(x)$ with $x_0, x_1 \in \mathbb{R}$ sufficiently close to the root of $f(x)$ and the termination parameter ε.

PROCESS:

DO {

SET $x_2 = \dfrac{x_0 f(x_1) - x_1 f(x_0)}{f(x_1) - f(x_0)}$

SET $x_0 = x_1$, $x_1 = x_2$

} WHILE $\left|\dfrac{x_1 - x_0}{x_1}\right| > \varepsilon$

OUTPUT: Root of $f(x)$ or its approximation x_2.

Theorem (Convergence of Secant Method): Suppose that $f(x), f'(x), f''(x)$ are continuous in some neighbourhood of a root r of $f(x)$ and $f'(r) \neq 0$. Then there exists a $\delta > 0$ such that if the initial points x_0 and x_1 satisfies $|r - x_0|, |r - x_1| \leq \delta$, then all the subsequent points x_n satisfy the same inequality, converge to r and does so at least linearly i.e.,

$$|r - x_{n+1}| \leq k|r - x_n|$$

where k is some constant.

Notes:

(1) The secant method does not require the initial points x_0 and x_1 to satisfy the condition $f(x_0)f(x_1) < 0$.

(2) If $f(x_{n-1}) = f(x_n)$, then the secant joining $(x_{n-1}, f(x_{n-1}))$ and $(x_n, f(x_n))$ is parallel to the x-axis thus making the calculation of x_{n+1} impossible.

(3) The approximations $x_2, x_3, \cdots$ generated by the secant method converges to the zero r of the function $f(x)$ if the initial points x_0 and x_1 are sufficiently close to r. If x_0 and x_1 are not sufficiently close to r then the secant method may diverge as well.

1.5.1 Exercise

1. Find the root of $x^3 - 3x + 1$ correct up to 3 decimal places using $x_0 = 0$ and $x_1 = 1$ as initial points.

   Solution: The given function is $f(x) = x^3 - 3x + 1$ and the initial points are $x_0 = 0$ and $x_1 = 1$.

n	x_0	$f(x_0)$	x_1	$f(x_1)$	$x_2 = \dfrac{x_0 f(x_1) - x_1 f(x_0)}{f(x_1) - f(x_0)}$	$f(x_2)$
1	0	1	1	-1	0.5	-0.375
2	1	-1	0.5	-0.375	0.2	0.408
3	0.5	-0.375	0.2	0.408	0.3563	-0.0237
4	0.2	0.408	0.3563	-0.0237	0.3477	-0.0011
5	0.3563	-0.0237	0.3477	-0.0011	0.3473	

   Hence the required root of $x^3 - 3x + 1$ correct up to 3 decimal places is 0.3473.

2. Calculate the root of $3x + \sin x - e^x$ up to sixth approximation using secant method with initial points $x_0 = 0$ and $x_1 = 1$.

   Solution: The given function is $3x + \sin x - e^x$ with the initial points $x_0 = 0$ and $x_1 = 1$.

n	x_0	$f(x_0)$	x_1	$f(x_1)$	$x_2 = \dfrac{x_0 f(x_1) - x_1 f(x_0)}{f(x_1) - f(x_0)}$	$f(x_2)$
1	0	-1	1	1.123189	0.470989	0.265157
2	1	1.123189	0.470989	0.265157	0.307509	-0.134821
3	0.470989	0.265157	0.307509	-0.134821	0.362613	0.005478
4	0.307509	-0.134821	0.362613	0.005478	0.360461	0.000098
5	0.362613	0.005478	0.360461	0.000098	0.360422	0.000001
6	0.360461	0.000098	0.360422	0.000001	0.360422	

   Hence the root of $3x + \sin x - e^x$ up to sixth approximation is 0.360422.

3. Find the root of x^3+x^2-3x-3 correct up to 5 significant digits with initial points $x_0 = 1$ and $x_1 = 2$.

4. Find the root of $x^5 + x^3 + 3$ correct up to 4 decimal places with initial points $x_0 = 1$ and $x_1 = -1$.

5. Find the root of $\sin x - 2x + 1$ using secant method correct up to 2 decimal places.

6. Use the secant method to find a root of the following equations correct up to 5 decimal places. (Burden, Faires)

   (a) $e^x + 2^{-x} + 2\cos x - 6 = 0$ with $x_0 = 1$, $x_1 = 2$.
   (b) $\ln(x-1) + \cos(x-1) = 0$ with $x_0 = 1.3$, $x_1 = 2$.
   (c) $2x\cos 2x - (x-2)^2 = 0$ with $x_0 = 2$, $x_1 = 3$.
   (d) $(x-2)^2 - \ln x = 0$ with $x_0 = 1$, $x_1 = 2$.
   (e) $e^x - 3x^2 = 0$ with $x_0 = 0$, $x_1 = 1$.
   (f) $\sin x - e^{-x} = 0$ with $x_0 = 3$, $x_1 = 4$.

1.6 Newton-Raphson Method

Let $f(x)$ be a differentiable function and let x_0 be an initial point which is sufficiently close to a root of $f(x)$. We draw a tangent to this curve at the point $(x_0, f(x_0))$. The equation of this tangent is

$$y - f(x_0) = f'(x_0)(x - x_0) \cdots\cdots (1)$$

Let $(x_1, 0)$ be the point of intersection of the x-axis and this tangent. Newton's method takes this point of intersection as the first approximation for the root of $f(x)$. To calculate this point, we note that $(x_1, 0)$ lies on line (1) and therefore

$$0 - f(x_0) = f'(x_0)(x_1 - x_0)$$
$$\text{or, } x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}$$

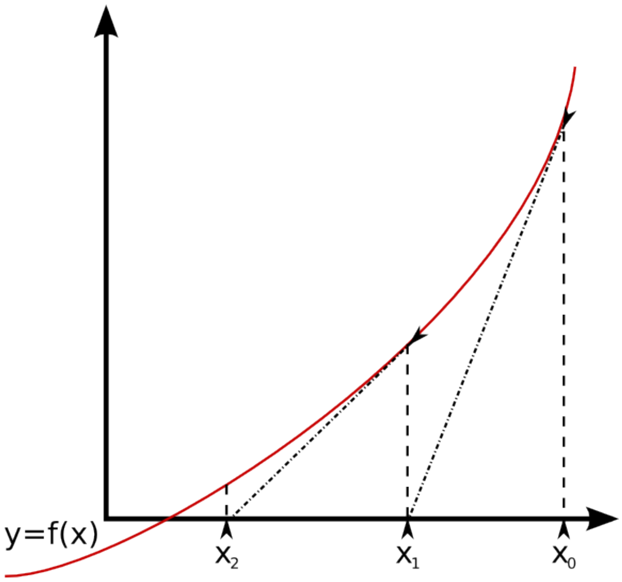
If $f(x_1) = 0$, then x_1 is the required root of $f(x)$. If not, then we take the point of intersection $(x_2, 0)$ of the x-axis and the tangent to $f(x)$ drawn at $(x_1, f(x_1))$ as the next approximation of the root. As above, we have,

$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)}$$

In general, the $(n+1)^{th}$ approximation of the root of $f(x)$ is given by the formula

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}, \quad n \geq 0.$$

We continue to calculate the approximations $x_1, x_2, x_3, \cdots$ using the above formula until we find the root or its satisfactory approximation.



Algorithm (Newton-Raphson Method):

INPUT: A differentiable function $f(x)$ with $x_0 \in \mathbb{R}$ sufficiently close to the root of $f(x)$ and the termination parameter ε.

PROCESS:

DO {

SET $x_1 = x_0$

SET $x_0 = x_0 - \dfrac{f(x_0)}{f'(x_0)}$

} WHILE $\left|\dfrac{x_0 - x_1}{x_0}\right| > \varepsilon$

OUTPUT: Root of $f(x)$ or its approximation x_1.

Theorem (Convergence of Newton's Method): Suppose that $f(x), f'(x), f''(x)$ are continuous in some neighbourhood of a root r of $f(x)$ and $f'(r) \neq 0$. Then there exists a $\delta > 0$ such that if the initial point x_0 satisfies $|r - x_0| \leq \delta$, then all the subsequent points x_n satisfy the same inequality, converge to r and does so quadratically i.e.

$$|r - x_{n+1}| \leq c(\delta)|r - x_n|^2$$

where $c(\delta)$ is a number depending upon δ.

Note:

1. Only one initial point x_0 is required to start the process. However we have to be careful that $f'(x_0)$ is nonzero.

2. The function $f(x)$ has to be a differentiable function.

1.6.1 Exercise

1. Find the square root of 3 correct up to 5 decimal places using Newton's method.

   Solution: The square root of 3 is a solution of the equation $f(x) = x^2 - 3$. Therefore $f'(x) = 2x$. Taking $x_0 = 1.5$, we get the following table.

n	x_0	$f(x_0)$	$f'(x_0)$	$x_1 = x_0 - \dfrac{f(x_0)}{f'(x_0)}$
1	1.5	-0.75	3	1.75
2	1.75	0.0625	3.5	1.732143
3	1.732143	0.00032	3.464286	1.732051
4	1.732051	0.0000007	3.464102	1.732051

   Therefore, the root of 3 correct up to 5 decimal places is 1.732051.

2. Find a root of the equation $x \sin x + \cos x$ correct up to 5 decimal places using Newton-Raphson method and initial point $x_0 = \pi$.

   Solution: Here the given equation is $f(x) = x \sin x + \cos x$. So $f'(x) = x \cos x + \sin x - \sin x = x \cos x$. Taking the initial point $x_0 = \pi \approx 3.141593$, we get the following table.

n	x_0	$f(x_0)$	$f'(x_0)$	$x_1 = x_0 - \dfrac{f(x_0)}{f'(x_0)}$
1	3.141593	-1.000001	-3.141593	2.823283
2	2.823283	-0.066187	-2.681458	2.7986
3	2.7986	-0.000564	-2.635588	2.798386
4	2.798386	0.00000012	-2.635185	2.798386

   Hence the root of $x \sin x + \cos x$ correct up to 5 decimal places is 2.798386.

3. Find the root of $x^3 + x^2 - 3x - 3$ in the interval $[1, 2]$ correct up to 4 significant digits.

   Solution: Here $f(x) = x^3 + x^2 - 3x - 3$ and $f'(x) = 3x^2 + 2x - 3$. Let $x_0 = \dfrac{1+2}{2} = 1.5$ be the initial point.

n	x_0	$f(x_0)$	$f'(x_0)$	$x_1 = x_0 - \dfrac{f(x_0)}{f'(x_0)}$
1	1.5	-1.875	6.75	1.7778
2	1.7778	0.446	10.0373	1.7334
3	1.7334	0.0128	9.4808	1.7320
4	1.7320	-0.0005	9.4635	1.7320

   Hence the root of $x^3 + x^2 - 3x - 3$ in the interval $[1, 2]$ correct up to 4 significant digits is 1.7320.

4. Find the root of $xe^x - 1$ using Newton-Raphson method correct up to 5 decimal places with initial point $x_0 = 1$.

5. Find the root of $3x + \sin x - e^x$ using Newton-Raphson method correct up to 7 significant digits with initial point $x_0 = 0$.

6. Use the Newton-Raphson method to find a root of the following equations correct up to 6 decimal places. (from Burden, Faires and Sastry)

   (a) $e^x + 2^{-x} + 2\cos x - 6 = 0$ with $x_0 = 1.5$.
   (b) $\ln(x-1) + \cos(x-1) = 0$ with $x_0 = 1.6$.
   (c) $2x\cos 2x - (x-2)^2 = 0$ with $x_0 = 2.5$.
   (d) $(x-2)^2 - \ln x = 0$ with $x_0 = 1.5$.
   (e) $e^x - 3x^2 = 0$ with $x_0 = 0.5$.
   (f) $\sin x - e^{-x} = 0$ with $x_0 = 3.5$.
   (g) $\sin x = 1 - x$ with $x_0 = 0$.
   (h) $x^3 - 5x + 3 = 0$ with $x_0 = 0.5$.
   (i) $x^4 + x^2 - 80 = 0$ with $x_0 = 3$.
   (j) $x^3 + 3x^2 - 3 = 0$ with $x_0 = -2$.
   (k) $4(x - \sin x) = 1$ with $x_0 = 1$.
   (l) $x - \cos x = 0$ with $x_0 = 0.5$.
   (m) $\sin x = \dfrac{x}{2}$ with $x_0 = 2$.
   (n) $x + \log x = 2$ with $x_0 = 1.5$.

1.7 Newton's Method for Polynomials

Horner's method (Synthetic division method): Let

$$P_n(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0 \cdots\cdots (I)$$

be a polynomial of degree n that has to be evaluated at $x = c$. To evaluate $P_n(c)$ by directly using expression (I) requires n additions and $\dfrac{n(n+1)}{2}$ multiplications. Instead, we can use the Horner's method to evaluate $P_n(c)$ which is based on the following result:

Theorem 1: Let

$$P_n(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0.$$

Define $b_n = a_n$ and

$$b_k = a_k + b_{k+1}c$$

for $k = n-1, n-2, \ldots, 1, 0$. If

$$Q_{n-1}(x) = b_n x^{n-1} + b_{n-1}x^{n-2} + \cdots + b_2 x + b_1,$$

then

$$P_n(x) = (x - c)Q_{n-1}(x) + b_0$$

.

Proof: We have

$$\begin{aligned}
&(x - c)Q_{n-1}(x) + b_0 \\
=&(x - c)(b_n x^{n-1} + b_{n-1}x^{n-2} + \cdots + b_2 x + b_1) + b_0 \\
=&(b_n x^n + b_{n-1}x^{n-1} + \cdots + b_2 x^2 + b_1 x) - (b_n c x^{n-1} + \cdots + b_2 c x + b_1 c) + b_0 \\
=&b_n x^n + (b_{n-1} - b_n c)x^{n-1} + \cdots + (b_1 - b_2 c)x + (b_0 - b_1 c) \\
=&a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0 \\
=&P_n(x)
\end{aligned}$$

Hence $P_n(x) = (x - c)Q_{n-1}(x) + b_0$. ■

Theorem 2 (The Remainder Theorem): Let

$$P_n(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$$

be an n degree polynomial. Then $P_n(c) = b_0$ and $P_n'(c) = Q_{n-1}(c)$ where $Q_{n-1}(x)$ is as in Theorem 1 above.

Proof: From Theorem 1 above, we have

$$P_n(x) = (x - c)Q_{n-1}(x) + b_0.$$

Therefore,

$$P_n(c) = (c - c)Q_{n-1}(c) + b_0 = b_0.$$

Also differentiating with respect to x, we get

$$P_n'(x) = (x - c)Q_{n-1}'(x) + Q_{n-1}(x).$$

So

$$P_n'(c) = (c - c)Q_{n-1}'(c) + Q_{n-1}(c) = Q_{n-1}(c).$$

■

Therefore, we see that $P_n(c)$ can be evaluated by calculating b_0 instead. However, calculation of b_0 requires only n additions and n multiplications. So this is a much more efficient method of evaluating $P_n(c)$.

Algorithm (Horner's Method):

INPUT: A polynomial $P_n(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$ and a point $x = c$ at which $P_n(x)$ is to be evaluated.

PROCESS:

SET $b_n = a_n$

SET $d_n = a_n$

FOR $i = n - 1$ TO 1 {

SET $b_i = a_i + cb_{i+1}$
SET $d_i = b_i + cd_{i+1}$
SET $i = i - 1$

}
SET $b_0 = a_0 + cb_1$

OUTPUT: $P_n(c) = b_0$ and $P'_n(c) = d_1$.

Example: Use Horner's method to evaluate $P_4(x) = 2x^4 - 3x^2 + 3x - 4$ at $c = 2$.

Solution: We evaluate $P_4(2)$ using Horner's method by constructing a table as below which is known as the synthetic division method: (See class notes)

Newton's method for polynomials: For a differentiable function $f(x)$, the Newton's method of finding the roots of $f(x)$ is given by

$$x_{k+1} = x_k - \frac{f(x_k)}{f'(x_k)}, k \geq 0.$$

If $f(x)$ is an n-degree polynomial

$$P_n(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0,$$

then $f'(x)$ is an $n-1$-degree polynomial $P'_n(x)$. The evaluation of $P_n(x_k)$ and $P'_n(x_k)$ can be simplified by making use of the synthetic division method and the remainder theorem as above.

1.7.1 Exercise

1. Use Newton's method for polynomials to find the approximate square root of the following polynomials correct up to 4 decimal places.
   (a) $2x^4 - 3x^2 + 3x - 4 = 0$ with $x_0 = -2$.
   (b) $x^3 - 5x + 3 = 0$ with $x_0 = 0.5$.
   (c) $x^4 + x^2 - 80 = 0$ with $x_0 = 3$.
   (d) $x^3 + 3x^2 - 3 = 0$ with $x_0 = -2.5$.

1.8 Fixed-Point Iteration

To find the root of equation $f(x) = 0$ by fixed-point iteration method , we first rearrange this into an equivalent form $x - g(x) = 0$ i.e., $f(x) = 0$ if and only if $x - g(x) = 0$. Then r is a root of $f(x)$ i.e. $f(r) = 0$, if and only if $r - g(r) = 0$ i.e., $g(r) = r$. Such a point r is called the **fixed point** or **root** of $g(x)$. Therefore finding the root of $f(x)$ is equivalent to finding the fixed point of $g(x)$. The fixed point of $g(x)$ is found iteratively as follows: An initial guess x_0 is made which is then used to get the next approximation as

$$x_1 = g(x_0).$$

This point x_1 is then used to obtain the next approximation x_2 as

$$x_2 = g(x_1).$$

This iterative process of finding successive approximations can be expressed in general form as

$$x_{n+1} = g(x_n), \qquad n = 0, 1, 2, \cdots .$$

We continue this process until the fixed point or a sufficiently good approximation of it is found.

Example (Every rearrangement may not converge):

Consider the equation $f(x) = x^2 - 2x - 3 = 0$ whose roots are $x = -1$ and $x = 3$. Then the above equation can be written as

$$\begin{aligned} x &= g_1(x) = \sqrt{2x+3} \cdots\cdots (1) \\ x &= g_2(x) = \frac{3}{x-2} \cdots\cdots (2) \\ x &= g_3(x) = \frac{x^2-3}{2} \cdots\cdots (3) \end{aligned}$$

If we start with $x_0 = 4$ and the iteration $x_{n+1} = \sqrt{2x_n + 3}$ obtained from (1), we get the following values:

$$\begin{aligned} x_0 &= 4 \\ x_1 &= \sqrt{2 \times 4 + 3} = \sqrt{11} = 3.31662 \\ x_2 &= \sqrt{2 \times 3.31662 + 3} = \sqrt{9.63325} = 3.10375 \\ x_3 &= \sqrt{2 \times 3.10375 + 3} = \sqrt{9.20750} = 3.03439 \\ x_4 &= \sqrt{2 \times 3.0439 + 3} = \sqrt{9.06877} = 3.01144 \\ x_5 &= \sqrt{2 \times 3.01144 + 3} = \sqrt{9.02288} = 3.00381 \end{aligned}$$

and so on. Therefore, the x_n's seem to converge to the root $x = 3$ in this case.

If we start with $x_0 = 4$ again and the iteration $x_{n+1} = \dfrac{3}{x_n - 2}$ obtained from (2), we get the following values:

$$\begin{aligned}
x_0 &= 4 \\
x_1 &= \frac{3}{4-2} = 1.5 \\
x_2 &= \frac{3}{1.5-2} = -6 \\
x_3 &= \frac{3}{-6-2} = -0.375 \\
x_4 &= \frac{3}{-0.375-2} = -1.263158 \\
x_5 &= \frac{3}{-1.263158-2} = -0.919355 \\
x_6 &= \frac{3}{-0.919355-2} = -1.02762 \\
x_7 &= \frac{3}{-1.02762-2} = -0.990876 \\
x_8 &= \frac{3}{-0.990876-2} = -1.00305
\end{aligned}$$

and so on. Therefore, the x_n's seem to converge to the root $x = -1$ in this case.

However if we start with $x_0 = 4$ again and the iteration $x_{n+1} = \dfrac{x_n^2 - 3}{2}$ obtained from (3), we get the following values:

$$\begin{aligned}
x_0 &= 4 \\
x_1 &= \frac{16-3}{2} = 6.5 \\
x_2 &= 19.625 \\
x_3 &= 191.070
\end{aligned}$$

and so on. Clearly x_n's diverge in this case. Therefore different rearrangements of the same equation $f(x) = 0$ may either diverge or converge for the same initial value x_0 and even if it converges, it may do so to different values.

Algorithm (Fixed-point Iterative Method):

INPUT: A function $f(x)$ with $x_0 \in \mathbb{R}$ sufficiently close to the root of $f(x)$ and the termination parameter ε.

PROCESS:

Rearrange the equation to an equivalent form $x = g(x)$

DO {

SET $x_1 = x_0$

SET $x_0 = g(x_0)$

}

WHILE $\left|\dfrac{x_0 - x_1}{x_0}\right| > \varepsilon$

OUTPUT: Root of $f(x)$ or its approximation x_0.

Theorem (Convergence of Fixed Point Iteration): If $g(x)$ and $g'(x)$ are continuous on an interval about a root r of the equation $x = g(x)$ and if $|g'(x)| < K < 1$ for all x in the interval, then $x_{n+1} = g(x_n)$, $n = 0, 1, 2, \cdots$, will converge to the root $x = r$, provided that x_0 is chosen in the interval.

Proof: Suppose that $g(x)$ and $g'(x)$ are continuous on the interval $[r - h, r + h]$, $h > 0$ about the root r and $|g'(x)| < K < 1$ for all x in this interval. Suppose that the initial point x_0 is chosen in this interval, i.e. $|r - x_0| < h$. The iterative process is given by $x_{n+1} = g(x_n)$, $n \geq 0$. We shall show by mathematical induction that $|r - x_n| < h$ for all the points x_n generated by this iteration. Obviously $|r - x_0| < h$. Suppose that $|r - x_n| < h$. Then

$$|r - x_{n+1}| = |r - g(x_n)| = |g(r) - g(x_n)| = |g'(\xi_n)(r - x_n)|$$

by mean value theorem where ξ_n is a point that lies between r and x_n. Therefore

$$|r - x_{n+1}| = |g'(\xi_n)||r - x_n| < Kh < h.$$

Hence $|r - x_n| < h$ for all $n \geq 0$. Also from above,

$$|r - x_{n+1}| = |g'(\xi_n)||r - x_n| < K|r - x_n| < K^2|r - x_{n-1}| < \cdots < K^{n+1}|r - x_0|$$

i.e., $|r - x_{n+1}| < K^{n+1}|r - x_0|$. Now as $n \to \infty$, $K^{n+1} \to 0$ since $0 < K < 1$ and therefore

$$\lim_{n\to\infty} |r - x_{n+1}| \leq \lim_{n\to\infty} K^{n+1}|r - x_0| = 0.$$

That is

$$\lim_{n\to\infty} |r - x_{n+1}| = 0.$$

Thus the sequence of points x_n converge to the root r. ■

1.8.1 Exercise

1. Use the fixed-point iteration method to solve the following:

   (a) $10 - 2x + \sin x = 0$ with $x_0 = 5$, correct up to 4 decimal places,

   Solution: Here the given equation is $10 - 2x + \sin x = 0$. Rearranging the above equation, we get

   $$x = 5 + \frac{1}{2}\sin x.$$

   So the iteration is $x_{n+1} = 5 + \dfrac{1}{2}\sin x_n$, $n \geq 0$ with $x_0 = 5$.

n	x_0	$x_1 = 5 + \frac{1}{2}\sin x_0$
1	5	4.52054
2	4.52054	4.50917
3	4.50917	4.51029
4	4.51029	4.51018
5	4.51018	4.51019

Therefore the required root is 4.51019.

(b) $\cos x - 2x + 3 = 0$ with $x_0 = \frac{\pi}{2}$ correct up to 5 decimal places. Hint: $x = \frac{3 + \cos x}{2}$

(c) $xe^x = 1$ with $x_0 = 1$ correct up to 4 decimal places. Hint: $x = e^{-x}$

(d) $2x - \log_{10} x = 7$ with $x_0 = 3.8$ correct up to 4 decimal places. Hint: $x = \frac{\log_{10} x + 7}{2}$

2. Use the fixed-point iteration method to evaluate a root of the equation $x^2 - x - 1 = 0$ using the following forms of $g(x)$:

   (a) $x = x^2 - 1$

   (b) $x = 1 + 2x - x^2$

   (c) $x = \frac{1}{2}(1 + 3x - x^2)$

   starting with $x_0 = 1$.

3. Solve problem 2 using $x_0 = 2$.

4. Find the square root of 0.75 by writing $f(x) = x^2 - 0.75$ and using the following form of $g(x)$: $x = x^2 + x - 0.75$ with $x_0 = -0.8$ and also $x_0 = 0.8$.

5. Use the fixed-point iteration method to find the root of the following equations correct up to 5 decimal places. (Sastry Page 48)

   (a) $\cos x = 3x - 1$ with $x_0 = 0.5$.

   (b) $x = \frac{1}{(x+1)^2}$ with $x_0 = 0.5$.

   (c) $x = (5 - x)^{1/3}$ with $x_0 = 1.5$.

   (d) $\sin x = 10(x - 1)$ with $x_0 = 1$.

   (e) $e^{-x} = 10x$ with $x_0 = 0$.

   (f) $x \sin x = 1$ with $x_0 = 1$.

   (g) $\sin^2 x = x^2 - 1$ with $x_0 = 1.5$.

   (h) $e^x \tan x = 1$ with $x_0 = 0.5$.

   (i) $1 + x^2 = x^3$ with $x_0 = 1.5$.

   (j) $5x^3 - 20x + 3 = 0$ with $x_0 = 0$.

   (k) $e^{-x} = x^2$ with $x_0 = 0$.

Index