Math Refresher

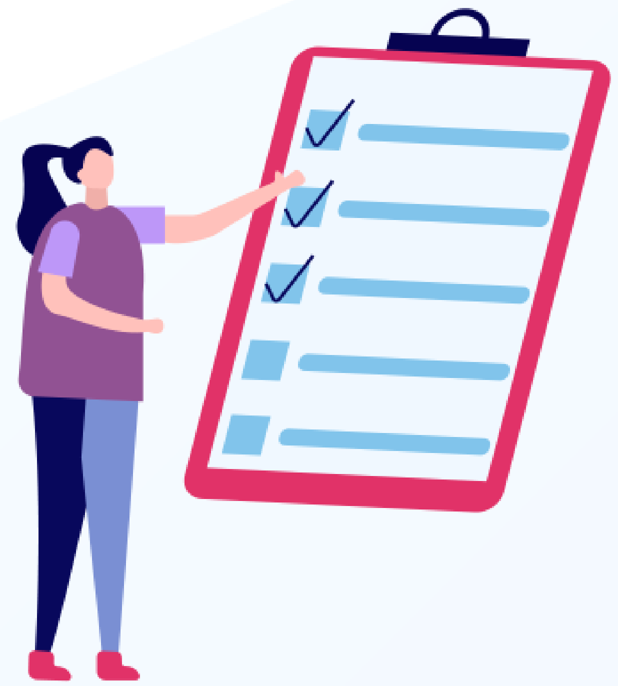
Eigenvalues, Eigenvectors, and Eigendecomposition

Eigenvalues

Learning Objectives

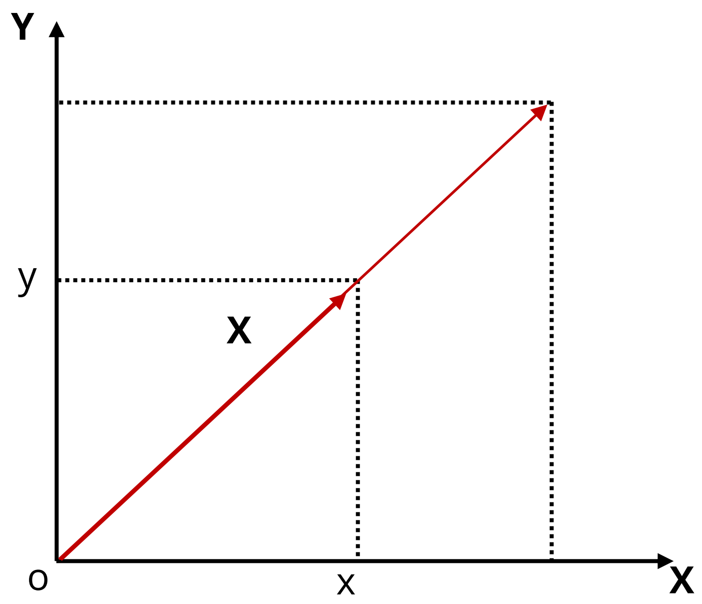
By the end of this lesson, you will be able to:

- Define eigenvalues and eigenvectors
- Describe eigenvalues of a square matrix
- List the properties of eigenvalues
- Illustrate eigendecomposition and its applications



Eigenvalues Definition

Eigenvalues, a fundamental concept in linear algebra, have various important mathematical and scientific applications.

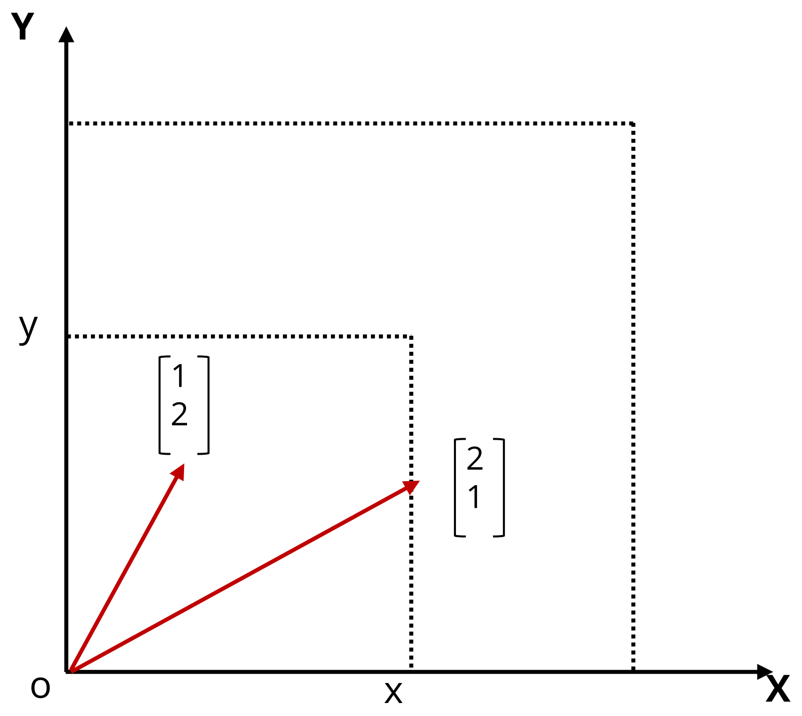


Eigenvalues are used to understand the behavior of linear transformations, particularly those represented by matrices.

The term Eigen in German implies appropriate or characteristic.

Eigenvalues Definition

Eigenvalues are scalar values that are associated with a square matrix.



- They provide information about how the matrix transforms vectors.
- An eigenvalue represents a scalar factor by which an eigenvector (corresponding to that eigenvalue) is stretched or compressed when multiplied by the matrix.

Other names for an eigenvalue include characteristic value, characteristic root, appropriate value, and latent root.

Eigenvalues Definition

An eigenvalue of a square matrix A is a scalar λ for which there exists a non-zero vector v such that when A operates on v, the result is a scalar multiple of v, i.e., $Av = \lambda v$.

Formula

$$Av = \lambda v$$

The number or scalar value λ is an eigenvalue of A.

Eigenvalues Definition

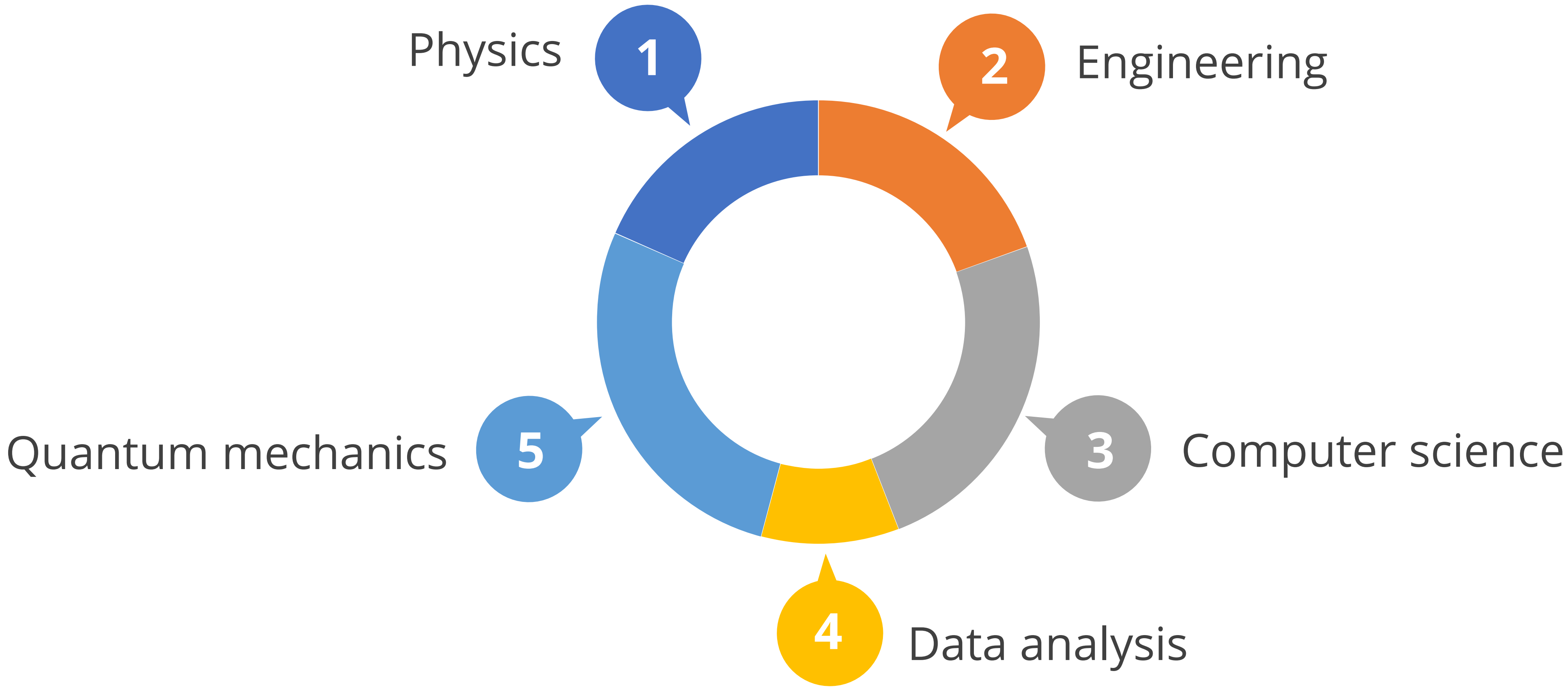
Eigenvalues provide the following insights:

They reveal information about the structure, behavior, and properties of matrices and linear transformations.

They serve as a powerful tool for solving complex problems and analyzing system characteristics described by matrices.

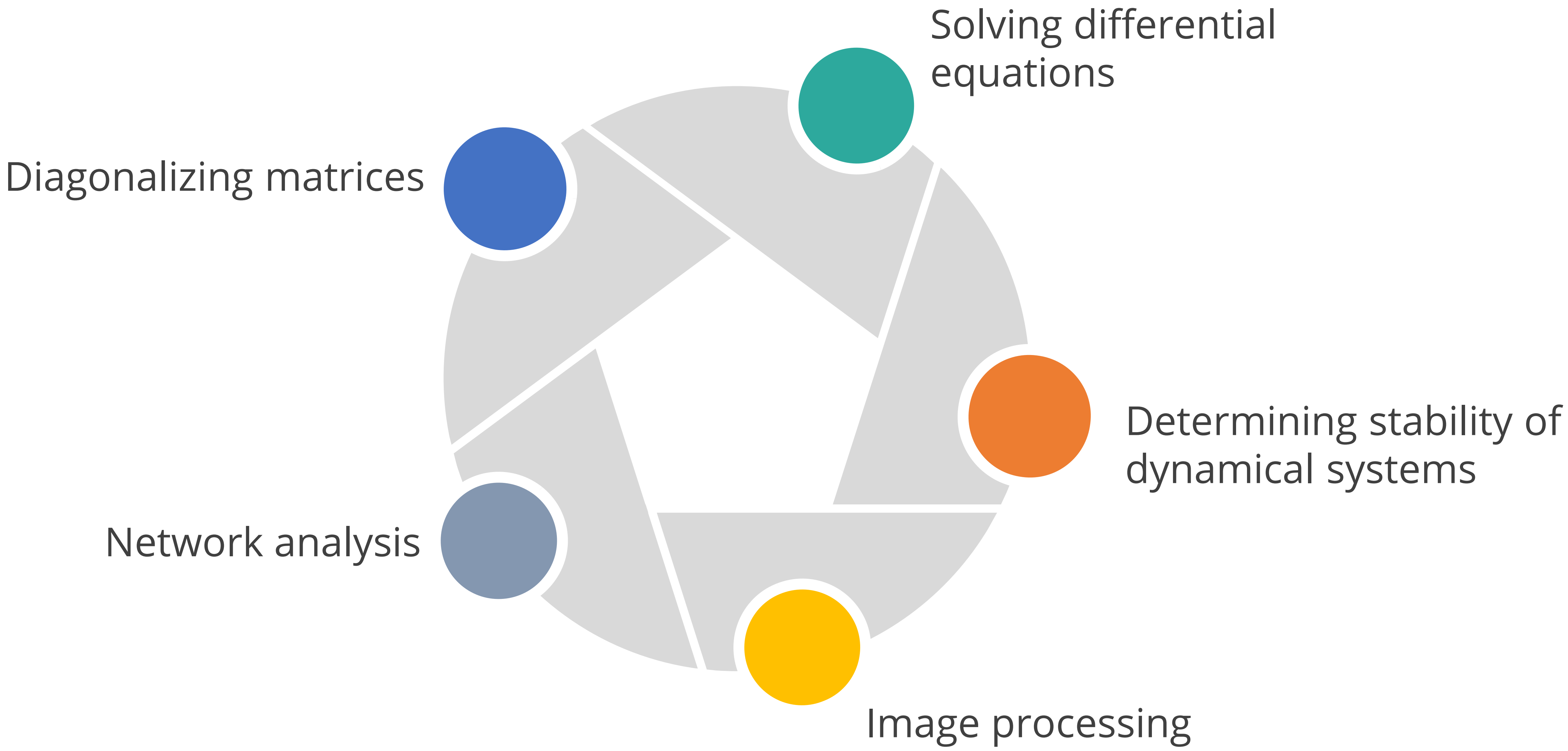
Eigenvalues Applications

Eigenvalues are used in a wide range of fields, such as:



Eigenvalues Applications

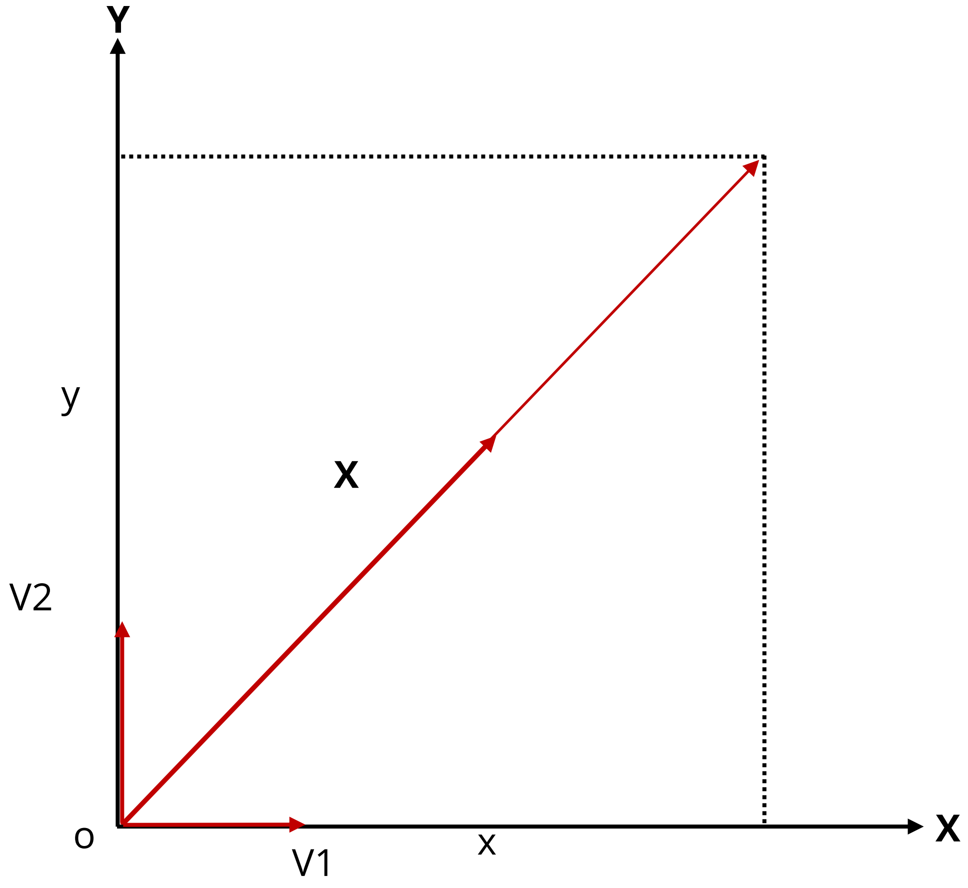
Eigenvalues are used in a wide range of fields, such as:



Eigenvectors

Eigenvectors Definition

In mathematics, an eigenvector corresponds to the real non-zero eigenvalues that point in the direction in which the transformation stretches.



The eigenvalue associated with an eigenvector represents the scaling factor by which the eigenvector is stretched or compressed when transformed by a linear operator or matrix.

What Are Eigenvectors?

Given a square matrix, an eigenvector is a non-zero vector that, when multiplied by the matrix, results in a scaled version of itself.

Formula

$$Av = \lambda v$$

Where A is the square matrix
v is the eigenvector
λ is the eigenvalue associated with v

An eigenvector of a square matrix A is a non-zero vector v such that when A operates on v, the result is a scalar multiple of v.

Eigenvectors

- A matrix scales an eigenvector by its corresponding eigenvalue while keeping the eigenvector's direction unchanged.
- Eigenvectors are associated with eigenvalues, indicating that each eigenvalue has one or more linearly independent eigenvectors.
- If a matrix has distinct eigenvalues, then the corresponding eigenvectors are linearly independent.
- The set of all eigenvectors corresponding to a specific eigenvalue forms a subspace known as the eigenspace.
- The vector zero is not an eigenvector.

Eigenvalues of a Square Matrix

If **$A_{n \times n}$** is a square matrix, then **[A- λI]** is referred to as an eigen or characteristic matrix, and it is an indeterminate scalar.

The eigen matrix determinant and eigen equation can be expressed as:

Determinant

$|A - \lambda I|$

Characteristic Equation

$|A - \lambda I| = 0$

Where:
I is the identity matrix

Steps to Obtain Eigenvalues and Eigenvectors

Start with a square matrix A for which you want to find the eigenvalues and eigenvectors

Compute the eigenvalues of the matrix A by solving the characteristic equation, $\det(A - \lambda I) = 0$. Here, λ represents the eigenvalue, det denotes the determinant, A is the matrix, and I is the identity matrix of the same size as A

Solve the characteristic equation to find the eigenvalues. This can involve algebraic manipulations, factorizations, or numerical methods depending on the size and properties of the matrix

Steps to Obtain Eigenvalues and Eigenvectors

Once you have determined the eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_n$, substitute each eigenvalue back into the equation $(A - \lambda I)v = 0$, where v is the eigenvector you want to find

For each eigenvalue, solve the equation $(A - \lambda I)v = 0$ to find the eigenvectors using Gaussian elimination or other numerical methods to find the null space or kernel of the matrix $(A - \lambda I)$

The non-zero solutions to the equation $(A - \lambda I)v = 0$ will give you the eigenvectors corresponding to each eigenvalue. Remember that eigenvectors are not unique; any scalar multiple of an eigenvector is also an eigenvector.

Steps to Obtain Eigenvalues and Eigenvectors

Step 1

Consider the following matrix A

$$\boldsymbol{A} = \begin{bmatrix} \mathbf{2} & \mathbf{1} \\ \mathbf{1} & \mathbf{3} \end{bmatrix}$$

Step 2

Compute the eigenvalues of the matrix A by solving the characteristic equation, det(A - λI)

$$\boldsymbol{det(A - \lambda I) = 0}$$

Steps to Obtain Eigenvalues and Eigenvectors

Step 3

Calculate the determinant

$$\boldsymbol{det}\begin{bmatrix} \mathbf{2-\lambda} & \mathbf{1} \\ \mathbf{1} & \mathbf{3-\lambda} \end{bmatrix} = \mathbf{0},$$

$$(\mathbf{2}-\boldsymbol{\lambda})(\mathbf{3}-\boldsymbol{\lambda})-(\mathbf{1})(\mathbf{1})=\mathbf{0}$$

$$\left(\mathbf{6}-\mathbf{5}\boldsymbol{\lambda}+\boldsymbol{\lambda}^{\mathbf{2}}\right)-\mathbf{1}=\mathbf{0}$$

$$\boldsymbol{\lambda}^{\mathbf{2}}-\mathbf{5}\boldsymbol{\lambda}+\mathbf{5}=\mathbf{0}$$

Steps to Obtain Eigenvalues and Eigenvectors

Step 4

Solve the quadratic equation to find the eigenvalues

The eigenvalues would be $\lambda_1 = \frac{5 + \sqrt{5}}{2}$ and $\lambda_2 = \frac{5 - \sqrt{5}}{2}$

Step 5

Now, let's find the eigenvectors corresponding to each eigenvalue:

For $\lambda_1 = \frac{5 + \sqrt{5}}{2}$

Steps to Obtain Eigenvalues and Eigenvectors

Step 6

Substituting λ into the equation (A - λI)v = 0 will give both the eigenvectors:

Eigenvector corresponding to $\lambda_1 = \frac{5+\sqrt{5}}{2}$

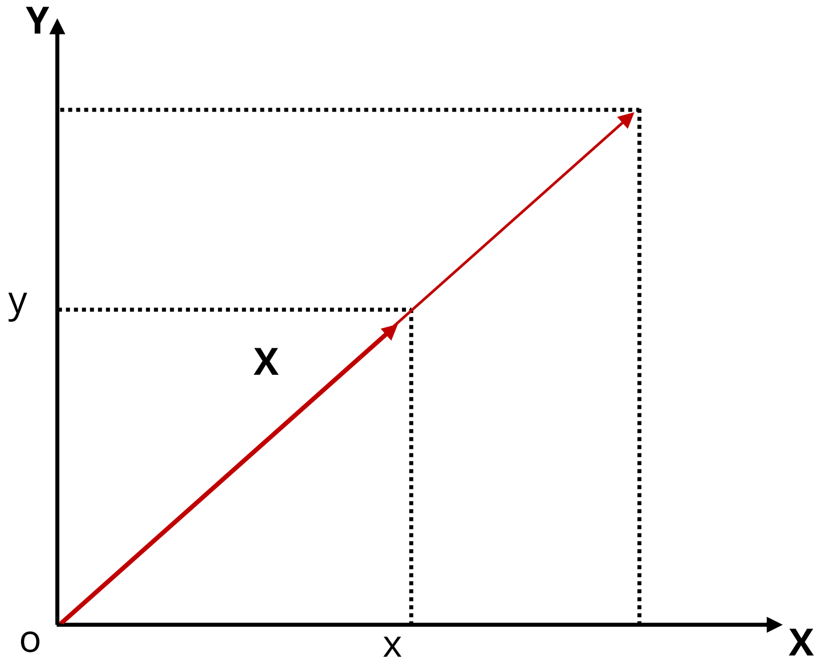
$$v_1 = -4 / (1 - \sqrt{5})$$
$$v_2 = 2$$

Eigenvector corresponding to λ_2 = (5 - sqrt(5))/2:

$$v_1 = -4 / (1 + \sqrt{5})$$
$$v_2 = 2$$

Properties of Eigenvalues

Eigenvalues are scalar quantities.



- They do not have direction or associated vectors.
- They represent the scaling factor by which the corresponding eigenvectors are stretched or compressed.

Properties of Eigenvalues

Eigenvalues determine matrix properties.

The determinant of a matrix is equal to the product of its eigenvalues.
$\det(A) = \lambda_1 * \lambda_2 * \lambda_3 * \ldots * \lambda_n$

The trace (sum of diagonal elements) of a matrix is equal to the sum of its eigenvalues.

$tr(A) = \lambda_1 + \lambda_2 + \lambda_3 + \ldots + \lambda_n$

Different eigenvectors can correspond to the same eigenvalue. However, the eigenvectors associated with the same eigenvalue are linearly dependent.

Properties of Eigenvalues

Eigenvalues possess the following properties under matrix operations:

Addition
When two matrices are added, the eigenvalues of the resulting matrix are equal to the sums of the eigenvalues of the individual matrices.

Scalar Multiplication
When a matrix is multiplied by a scalar, the eigenvalues of the resulting matrix are equal to the eigenvalues of the original matrix multiplied by the scalar.

Properties of Eigenvalues

A matrix can be diagonalized if and only if it has a complete set of linearly independent eigenvectors.

In the context of differential equations or dynamic systems, the eigenvalues of a matrix provide information about the system's stability.

For symmetric matrices, all eigenvalues are real.

Properties of Eigenvalues

Eigenvectors with different eigenvalues are linearly independent.

The eigenvalues of singular matrices are zero.

If A is a square matrix, then **λ = 0** is not one of its eigenvalues.

For a matrix's scalar multiple, if A is a square matrix and λ is one of its eigenvalues, then aλ is an eigenvalue of A.

Properties of Eigenvalues

If A is a square matrix, λ is one of its eigenvalues, and **$n \geq 0$** is an integer, then λ^n is one of A's eigenvalues.

If A is a square matrix, and $p(x)$ is a polynomial in the variable x, then $p(\lambda)$ is the eigenvalue of the matrix $p(A)$.

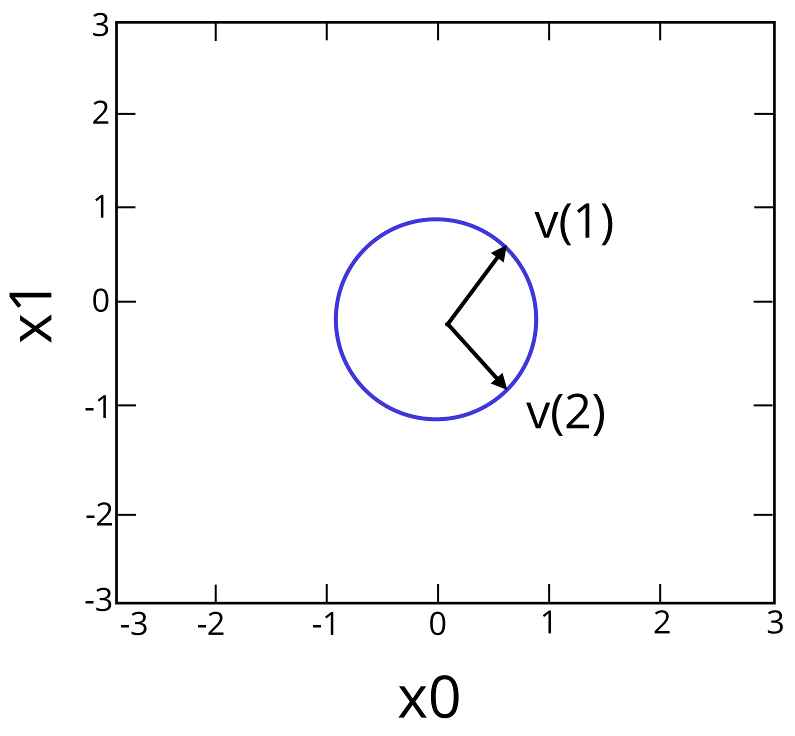
The eigenvalue of A^{-1} is λ^{-1} if the eigenvalue of the square matrix A is λ.

If the square matrix A has the eigenvalue λ, then the eigenvalue of A^T is λ.

Effect of Eigenvectors and Eigenvalues

The matrix A has two orthonormal eigenvectors, v(1) and v(2), with corresponding eigenvalues λ1 and λ2, respectively.

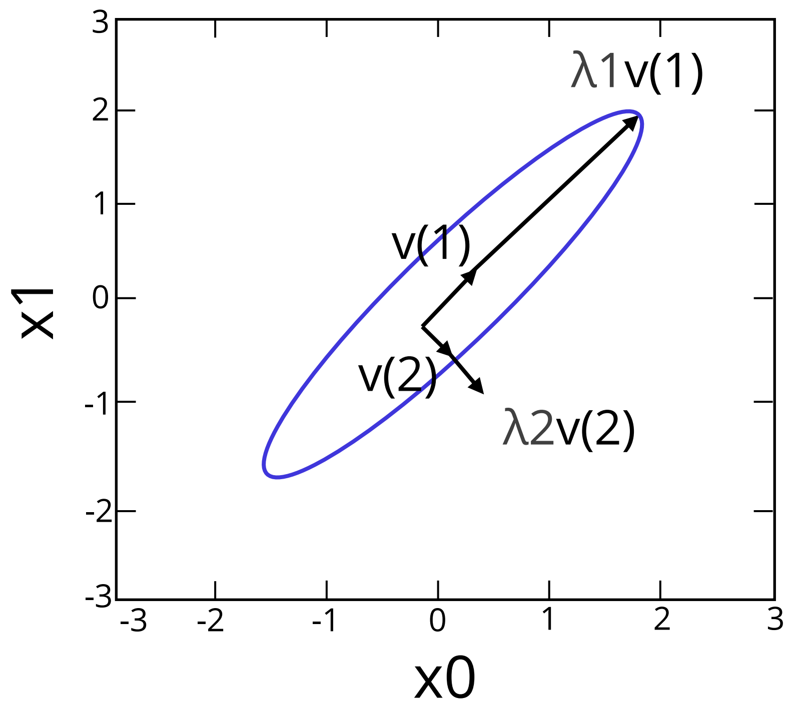
Before multiplication



Effect of Eigenvectors and Eigenvalues

Matrix A distorts the unit circle by scaling the space in the direction of v(i) by a factor of λ(i).

After multiplication



In linear algebra, an eigenvector of a square matrix represents a direction in which the vector does not change its direction but only scales (multiplies) by a scalar factor known as the eigenvalue (λ).

When we multiply a matrix by one of its eigenvectors, the resulting vector is a scaled version of the original eigenvector.

Eigendecomposition

Eigendecomposition

Eigendecomposition

Eigendecomposition is the process of factorizing a matrix into its canonical form, representing the matrix in terms of its eigenvalues and eigenvectors.

Matrix decompositions are useful for breaking down a matrix into its constituent parts, which simplifies various complex operations.

This decomposition is also utilized in machine learning approaches, such as Principal Component Analysis (PCA).

Eigendecomposition

If A contains a set of eigenvectors v1, v2,... represented by matrix V, and associated eigenvalues λ1, λ2,... represented by a vector, then the eigendecomposition of A can be expressed using the following formula:

Formula

$$A = V \operatorname{diag}(\lambda)\, v^{-1}$$

Eigendecomposition

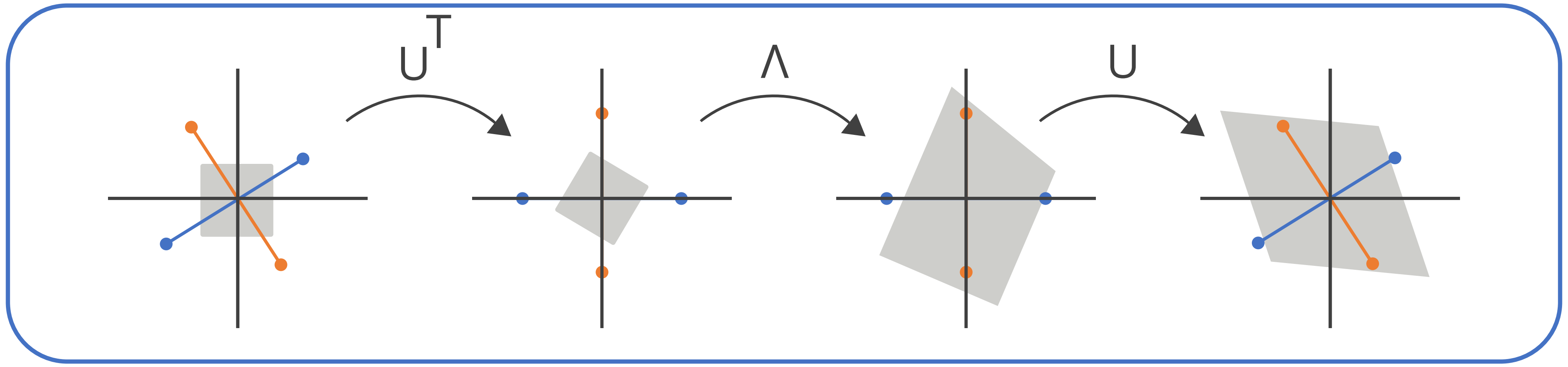
1. Find the eigenvalues of matrix A by solving $\det(A - \lambda I) = 0$
2. Find the corresponding eigenvectors by solving the equation $(A - \lambda I)v = 0$ for each eigenvalue
3. Arrange the eigenvectors as columns of matrix V: The eigenvectors must be normalized to have unit length

Eigendecomposition

4	Form the diagonal matrix Λ by placing the eigenvalues on its diagonal
5	Calculate the inverse of matrix V, V^(-1), if it exists
6	Substitute the eigenvectors, eigenvalues, and their inverses to obtain the decomposed form A = VΛV^(-1)

Eigendecomposition

Consider the following image where U is an orthogonal matrix, U^T is the rotation of matrix U, and Λ is the diagonal matrix.



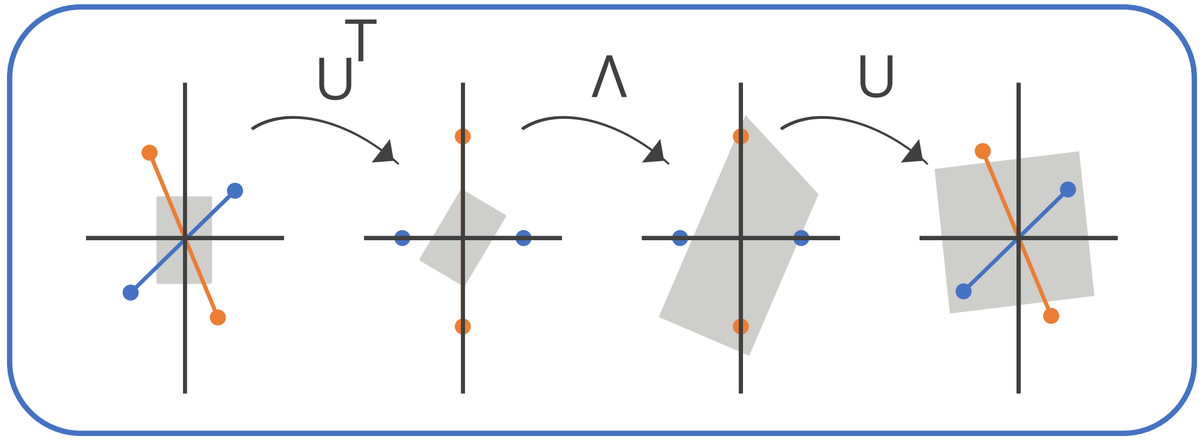
The unit vector U is rotated and transformed to U^T.

U^T is scaled-up by Λ.

Finally, the scaled-up U^T is rotated back to U.

Eigendecomposition

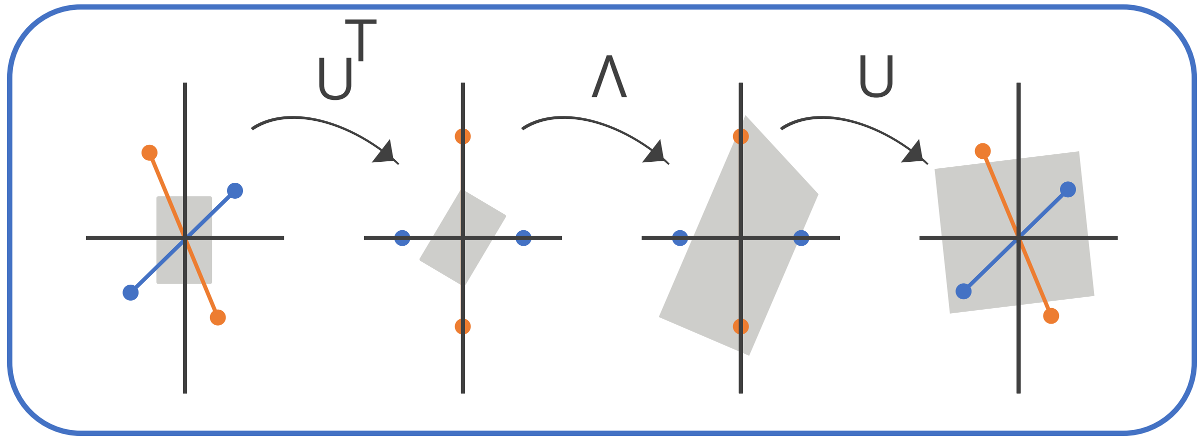
Consider the following image where U is an orthogonal matrix, U^T is the rotation of matrix U, and Λ is the diagonal matrix.



The transformation represented by the orthogonal matrix U can be decomposed into a rotation U^T and a scaling or stretching operation Λ.

Eigendecomposition

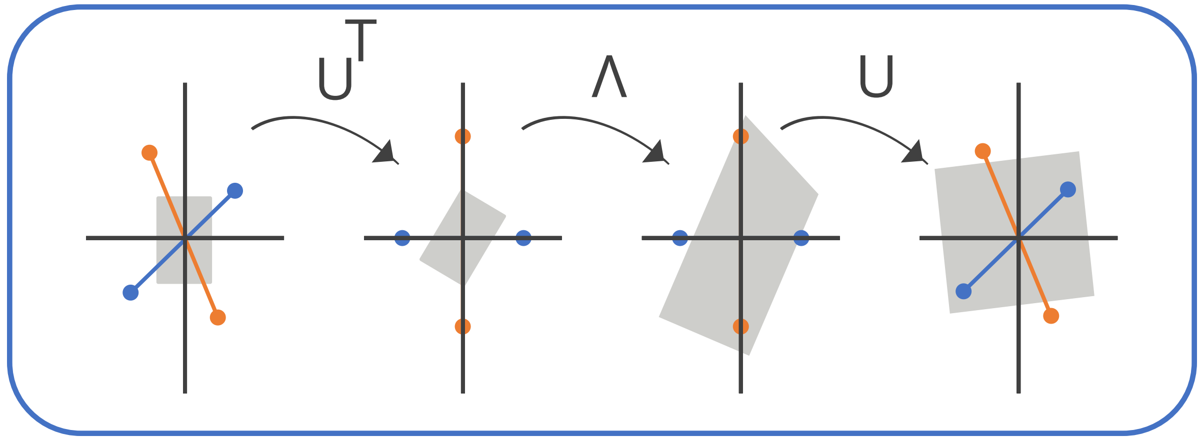
Consider the following image where U is an orthogonal matrix, U^T is the rotation of matrix U, and Λ is the diagonal matrix.



The rotation part (UT) captures the change in orientation or direction of the vectors, while the diagonal matrix (Λ) captures the change in magnitude or scaling of the vectors along each axis.

Eigendecomposition

Consider the following image where U is an orthogonal matrix, U^T is the rotation of matrix U, and Λ is the diagonal matrix.



This decomposition is particularly useful in applications such as computer graphics, image processing, and numerical computations involving orthogonal matrices.

Eigendecomposition

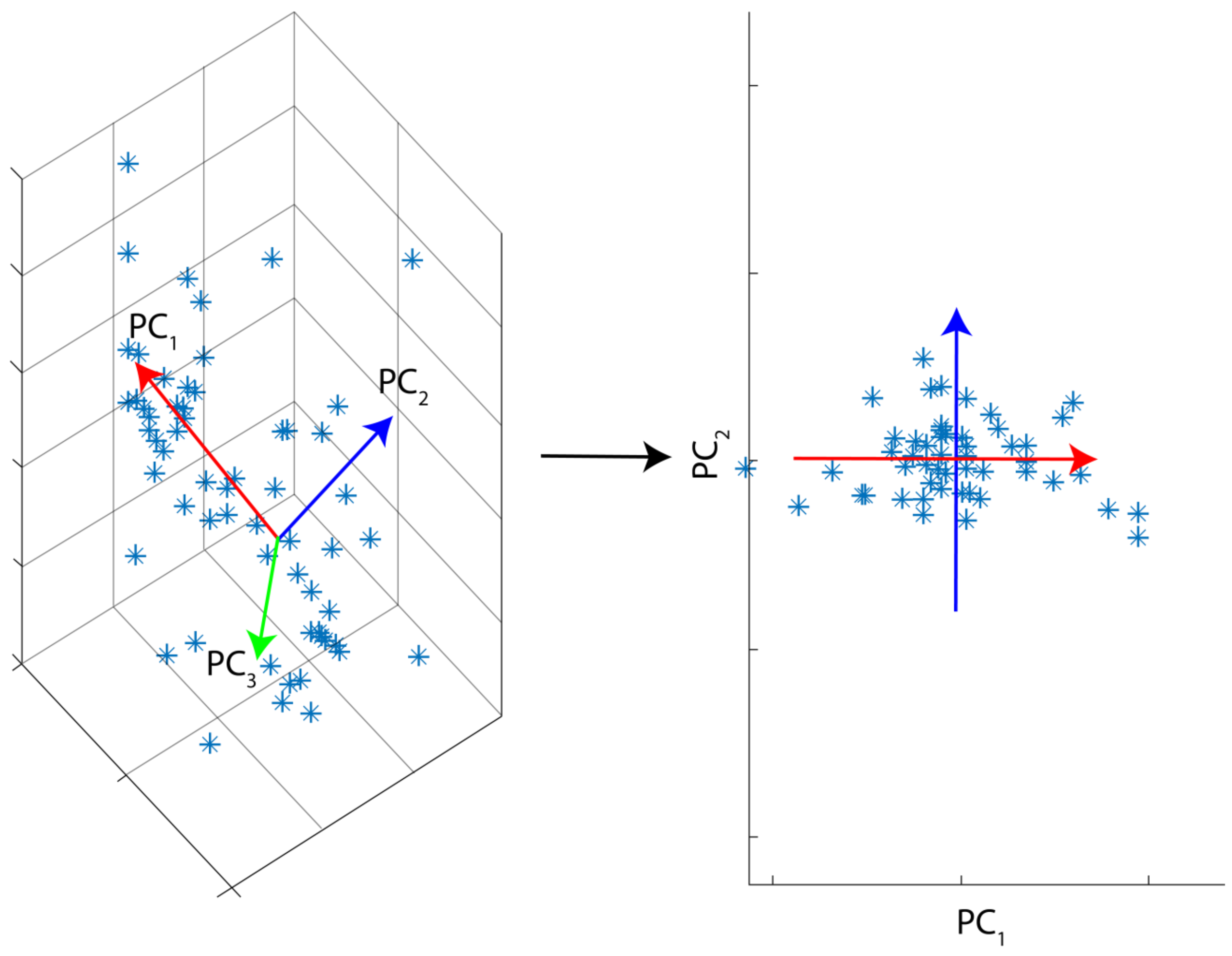
A = QΛQ^{-1} explains how A may be decomposed into a similarity transformation.

$$\underset{\textbf{A}}{\begin{pmatrix} \; & \; & \; \\ \; & \; & \; \\ \; & \; & \; \end{pmatrix}} = \underset{\textbf{Q}}{\begin{pmatrix} | & | & | \\ v_1 & v_2 & v_3 \\ | & | & | \end{pmatrix}} \underset{\boldsymbol{\lambda}}{\begin{pmatrix} \lambda 1 & 0 & 0 \\ 0 & \lambda 2 & 0 \\ 0 & 0 & \lambda 3 \end{pmatrix}} \underset{\textbf{Q}^{-1}}{\begin{pmatrix} | & | & | \\ v_1 & v_2 & v_3 \\ | & | & | \end{pmatrix}^{-1}}$$

Eigenvectors of A | Eigenvalues of A | Eigenvectors of A

The equation A = QAQ^(-1) states that the original matrix A can be decomposed into a similarity transformation using the similarity matrix Q. By applying the transformation, we obtain a new matrix that is similar to the original matrix A, but with possibly different eigenvalues and eigenvectors.

Eigendecomposition: Application

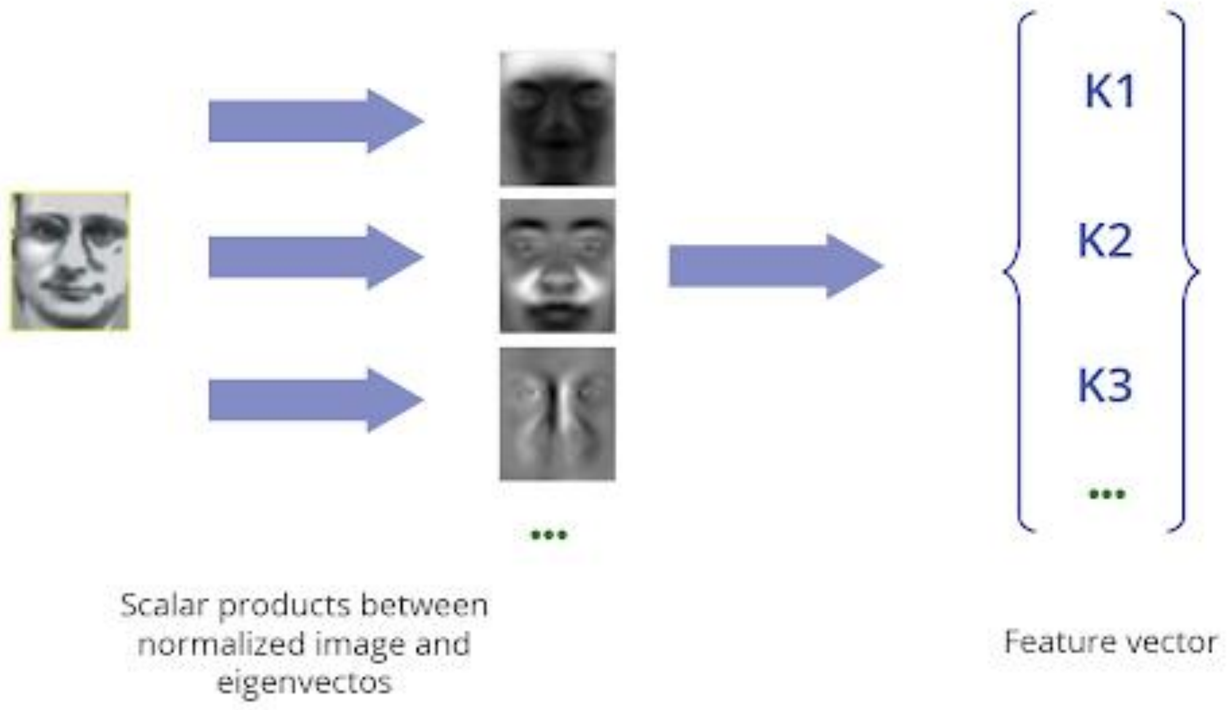


Principal Component Analysis (PCA)

- PCA is a widely used dimensionality reduction technique in data science.
- It relies on eigendecomposition as a fundamental step, which involves finding the eigenvectors and eigenvalues of the covariance matrix of the data.
- It allows us to identify the principal components, which are the directions of maximum variance in the data.

The dimensionality of the data can be reduced while retaining most of the important information.

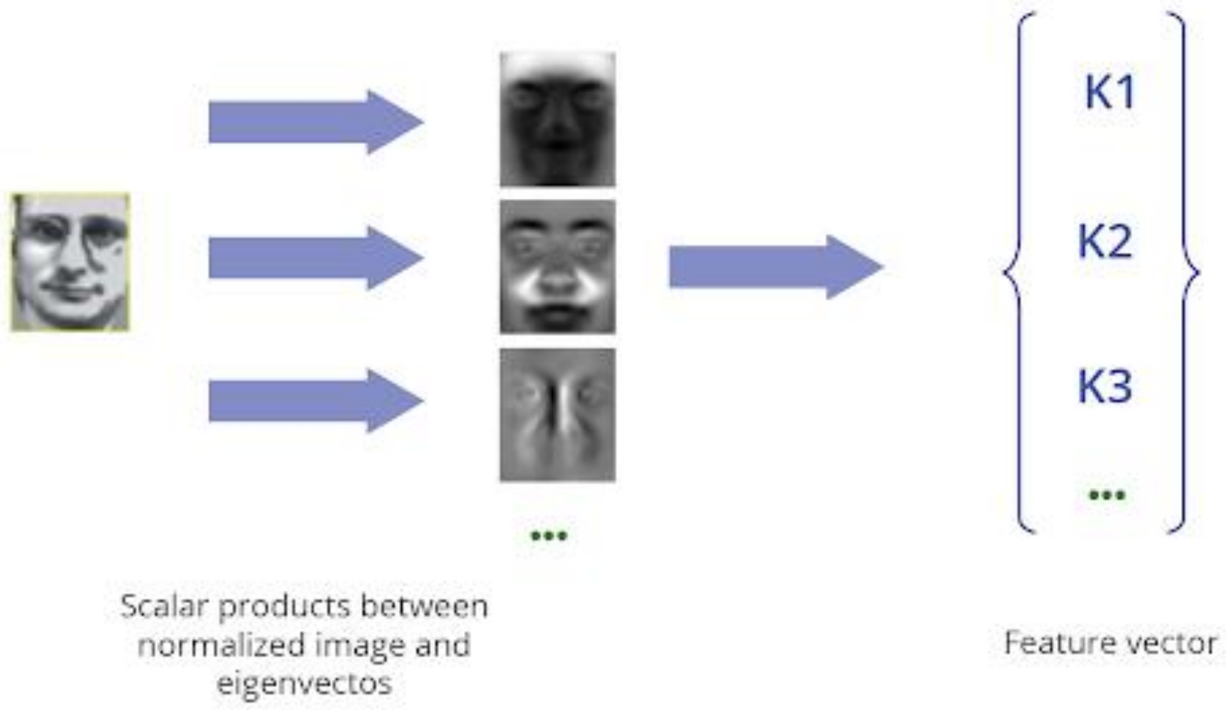
Eigendecomposition: Application



Feature extraction

Eigendecomposition can be used for feature extraction, enabling the transformation of the original features of a dataset into a new set of features that effectively capture the most significant information.

Eigendecomposition: Application



Feature extraction

By performing eigendecomposition on the data covariance or correlation matrix, we can obtain a reduced set of eigenvectors that represent the most significant patterns or features in the data.

These eigenvectors, or principal components, can serve as new features for subsequent analysis or modeling tasks.

Eigendecomposition: Application

Image compression

Original PNG - 12 MB

Compressed JPEG - 2.5 MB

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- Eigendecomposition combined with concepts from signal processing, can be used for image compression.
- Techniques such as Singular Value Decomposition (SVD), which is a generalized form of eigendecomposition, enable the representation of images as a linear combination of eigenvectors with varying weights.

Source: https://techterms.com/definition/lossy

Eigendecomposition: Application

Original PNG - 12 MB

Compressed JPEG - 2.5 MB

Image compression

By selecting a subset of the eigenvectors with the largest eigenvalues, we can reconstruct an image with fewer components, achieving compression while preserving the essential visual information.

Source: https://techterms.com/definition/lossy

Eigendecomposition: Application

Recommender systems

Eigendecomposition methods, such as Singular Value Decomposition (SVD) and Non-Negative Matrix Factorization (NMF), are used in recommender systems to uncover latent factors or patterns in user-item rating matrices.

Eigendecomposition: Application

Recommender systems

These methods decompose the rating matrix into lower-rank approximations represented by eigenvectors and eigenvalues.

The resulting latent factors can be used to make personalized recommendations to users based on their preferences and similarity to other users.

Key Takeaways

- Eigenvalues are a special set of scalar vectors that are associated with the system of linear equations.
- The existence of the eigenvalue for any complex matrices is equal to the fundamental theorem of algebra.
- Eigenvectors are special vectors (non-zero) that do not change direction when a linear transformation is applied.
- Eigenvectors with distinct eigenvalues are linearly independent.
- The eigenvalues of a triangular matrix and a diagonal matrix are identical to the components on their principal diagonals.

Knowledge Check

Knowledge Check

1

What is the determinant of the matrix $\begin{bmatrix} 3 & 4 \\ 2 & 5 \end{bmatrix}$?

A. 7

B. 6

C. 15

D. 23

Knowledge Check

1

What is the determinant of the matrix $\begin{bmatrix} 3 & 4 \\ 2 & 5 \end{bmatrix}$?

A. 7

B. 6

C. 15

D. 23

The correct answer is **A**

Determinant = 15 - 8 = 7

Knowledge Check 2

Suppose the determinant of a 2×2 matrix A is equal to 5. What is the determinant of 2A?

A. 5

B. 10

C. 20

D. 25

Knowledge Check 2

Suppose the determinant of a 2 × 2 matrix A is equal to 5. What is the determinant of 2A?

A. 5

B. 10

C. 20

D. 25

The correct answer is **C**

If det(A) = 5, then det(2A) = (2^2) * 5 = 4 * 5 = 20.