Rank of a Matrix

Sandeep Kumar Singh

Definition:The rank of a matrix $\boldsymbol{A}$, denoted by $\boldsymbol{r}\ \boldsymbol{or}\ \boldsymbol{r}(\boldsymbol{A})$ is the order of largest non-zero minor of $\boldsymbol{A}$. Therefore, the rank of a matrix is the largest value of $\boldsymbol{r}$, for which there exists at least one $\boldsymbol{r} \times \boldsymbol{r}$ submatrix of $\boldsymbol{A}$ whose determinant is not zero.

Remark:

1) For an $m \times n$ matrix $r \leq \min(m, n)$

2)For a square matrix of order n ,the rank= n ,if $|A| \neq 0$,otherwise $r < n$.

3) The rank of a null matrix is zero.

Examples

$$A=\begin{bmatrix}1 & 2 & 3\\ 1 & 4 & 2\\ 2 & 6 & 5\end{bmatrix} \qquad \begin{bmatrix}2 & 3 & 4\\ 3 & 5 & 7\end{bmatrix} \qquad \begin{bmatrix}1 & 2 & 0 & 3\\ 2 & 1 & 1 & 4\\ 4 & 5 & 1 & 10\\ 1 & -1 & 1 & 1\end{bmatrix}$$

$$|A| = 1(20-12)-2(5-4)+3(6-8)$$

$$= 8-2-6=0$$

$$\begin{bmatrix}1 & 2\\ 1 & 4\end{bmatrix}, \quad 1\times 4 - 2\times 1 = 4-2 \neq 0$$

2x2 notzero rank $A=2$

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Examples

$$\begin{bmatrix} 1 & 2 & 3 \\ 1 & 4 & 2 \\ 2 & 6 & 5 \end{bmatrix} \quad B = \begin{bmatrix} 2 & 3 & 4 \\ 3 & 5 & 7 \end{bmatrix} \quad \begin{bmatrix} 1 & 2 & 0 & 3 \\ 2 & 1 & 1 & 4 \\ 4 & 5 & 1 & 10 \\ 1 & -1 & 1 & 1 \end{bmatrix}$$

2×3

$\min(2, 3) = 2$, $\operatorname{rank} B \le 2$ 1 or 2

$\begin{bmatrix} 2 & 3 \\ 3 & 5 \end{bmatrix}$ $\det = 2 \times 5 - 3 \times 3 = 10 - 9 = 1 \neq 0$

$\boxed{r = 2}$

$\begin{bmatrix} 2 & 4 \\ 3 & 7 \end{bmatrix}$ $\det = 14 - 12 = 2 \neq 0$

Examples

$$\begin{bmatrix}1 & 2 & 3\\ 1 & 4 & 2\\ 2 & 6 & 5\end{bmatrix} \qquad \begin{bmatrix}2 & 3 & 4\\ 3 & 5 & 7\end{bmatrix} \qquad \begin{bmatrix}1 & 2 & 0 & 3\\ 2 & 1 & 1 & 4\\ 4 & 5 & 1 & 10\\ 1 & -1 & 1 & 1\end{bmatrix}$$

$1\times 1 - 2\times 2 = 1-4 = -3 \neq 0$

$1(1-10) - 1(5+10) + 4(\underline{5+1})$
$= -9 - 15 + 24$
$= 0$

$$|C| = 1\begin{vmatrix}1 & 1 & 4\\ 5 & 1 & 10\\ -1 & 1 & 1\end{vmatrix} - 2\begin{vmatrix}2 & 1 & 4\\ 4 & 1 & 10\\ 1 & 1 & 1\end{vmatrix} - 3\begin{vmatrix}2 & 1 & 1\\ 4 & 5 & 1\\ 1 & -1 & 1\end{vmatrix}$$

$= 1[1(\underline{1-10}) - 1(5+10) + 4(\underline{5+1})]$
$-2[2(1-10) - 1(4-10) + 4(4-1)]$
$-3[2(5+1) - 1(4-1) + 1(-4-5)]$

$= [-9-15+24] - 2[-18+\underline{6+12}] - 3[12-3-9]$

$= 0 - 2\times 0 - 3\times 0 = \underline{0}$

16 submatrics of size 3x3 are possible

$r \neq 4$

$r \neq 3$

rank $C = 2$

Examples

$$A = \begin{bmatrix} 1 & 3 & -4 \\ -1 & -3 & 4 \\ 2 & 6 & -8 \end{bmatrix} \quad \begin{bmatrix} 1 & 2 & -3 \\ 2 & 1 & -1 \\ 1 & -1 & 2 \\ 5 & 4 & -5 \end{bmatrix} \rightarrow \text{Ex}$$

$|A| = 1(24-24) - 3(8-8) - 4(-6+6)$
$= 0$

$r \neq 3$

all 9 submatrices of size 2x2 will have zero determinant

$r \neq 2$

$\boxed{r = 1}$

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Elementary row and column operations

1) $R_i \leftrightarrow R_j$

2) $R_i \to kR_i$

3) $R_i \to R_i + kR_j$

Similarly for Columns also

1) $C_i \leftrightarrow C_j$

2) $C_i \to kC_i$

3) $C_i \to C_i + kC_j$

Echelon form of matrix

An $m \times n$ matrix is called a row echelon matrix or in row echelon form if the following conditions are satisfied

1) If ith row contains all zeros ,then it is true for all subsequent rows.
2) If the ith and $(i+1)th$ rows are both non-zero rows, then the initial non-zero entry of $(i+1)th$ row appears in the later column than that of ith row.
3) Rows containing all zeros occur only after all non-zero rows.

1st 2nd 1st

$$\begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & 4 & 2 & 1 \\ 0 & 0 & 0 & 9 \end{bmatrix} \begin{bmatrix} 1 & 3 & 3 & 4 \\ 0 & 0 & 2 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 2 & 3 & 4 & 2 \\ 0 & 0 & 1 & 2 & 3 \\ 0 & 4 & 3 & 0 & 4 \\ 0 & 0 & 0 & 0 & 5 \\ 1 & 0 & 0 & 0 & 0 \end{bmatrix}$$

2nd

row Echelon form

Definition: The number of non-zero rows in the row echelon form of a matrix $\boldsymbol{A}$ gives the rank of the matrix

Reduce the matrix to row echelon form and find its rank

$$\rightarrow \begin{bmatrix} 1 & 3 & 5 \\ 2 & -1 & 4 \\ -2 & 8 & 2 \end{bmatrix} \quad \begin{bmatrix} 1 & 2 & -3 \\ 2 & 1 & -1 \\ 1 & -1 & 2 \\ 5 & 4 & -5 \end{bmatrix}$$

$$\Big\downarrow \begin{array}{l} R_2 \to R_2 - 2R_1 \\ R_3 \to R_3 + 2R_1 \end{array}$$

$$\rightarrow \begin{bmatrix} \boxed{1} & 3 & 5 \\ 0 & \boxed{-7} & -6 \\ 0 & 14 & 12 \end{bmatrix} \xrightarrow{R_3 \to R_3 + 2R_2} \begin{bmatrix} 1 & 3 & 5 \\ 0 & -7 & -6 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\boxed{r = 2}$$

Definition: The number of non-zero rows in the row echelon form of a matrix $\boldsymbol{A}$ gives the rank of the matrix

Reduce the matrix to row echelon form and find its rank

$$\begin{bmatrix} 1 & 3 & 5 \\ 2 & -1 & 4 \\ -2 & 8 & 2 \end{bmatrix} \quad \begin{bmatrix} 1 & 2 & -3 \\ 2 & 1 & -1 \\ 1 & -1 & 2 \\ 5 & 4 & -5 \end{bmatrix} \xrightarrow[\substack{R_3 \to R_3 - R_1 \\ R_4 \to R_4 - 5R_1}]{R_2 \to R_2 - 2R_1} \begin{bmatrix} 1 & 2 & -3 \\ 0 & \boxed{-3} & 5 \\ 0 & -3 & 5 \\ 0 & -6 & 10 \end{bmatrix}$$

$$\downarrow \begin{matrix} R_3 \to R_3 - R_2 \\ R_4 \to R_4 - 2R_2 \end{matrix}$$

$$\begin{bmatrix} 1 & 2 & -3 \\ 0 & -3 & 5 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\boxed{r = 2}$$

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$$\begin{bmatrix} 1 & 2 & 3 & 4 \\ 2 & 1 & 4 & 5 \\ 1 & 5 & 5 & 7 \\ 8 & 1 & 14 & 17 \end{bmatrix} \xrightarrow[\substack{R_3 \to R_3 - R_1 \\ R_4 \to R_4 - 8R_1}]{R_2 \to R_2 - 2R_1} \begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & \boxed{-3} & -2 & -3 \\ 0 & 3 & 2 & 3 \\ 0 & -15 & -10 & -15 \end{bmatrix}$$

$$\downarrow \begin{array}{l} R_3 \to R_3 + R_2 \\ R_4 \to R_4 - 5R_2 \end{array}$$

$$\boxed{r = 2} \qquad \longrightarrow \begin{bmatrix} \boxed{1} & 2 & 3 & 4 \\ 0 & -3 & -2 & -3 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Examples

$$\begin{bmatrix} 1 & 2 & 3 \\ 1 & 4 & 2 \\ 2 & 6 & 5 \end{bmatrix} \qquad \begin{bmatrix} 2 & 3 & 4 \\ 3 & 5 & 7 \end{bmatrix} \qquad \begin{bmatrix} 1 & 2 & 0 & 3 \\ 2 & 1 & 1 & 4 \\ 4 & 5 & 1 & 10 \\ 1 & -1 & 1 & 1 \end{bmatrix}$$

Examples

$$\begin{bmatrix} 1 & 3 & -4 \\ -1 & -3 & 4 \\ 2 & 6 & -8 \end{bmatrix} \begin{bmatrix} 1 & 2 & -3 \\ 2 & 1 & -1 \\ 1 & -1 & 2 \\ 5 & 4 & -5 \end{bmatrix}$$

Find the value of μ for which the rank of the matrix is equal to three.

$(-1)^{1+2} = (-1)^3 = 1$

$$A = \begin{bmatrix} \mu & -1 & 0 & 0 \\ 0 & \mu & -1 & 0 \\ 0 & 0 & \mu & -1 \\ -6 & 11 & -6 & 1 \end{bmatrix}$$

$$|A| = \mu \begin{vmatrix} \mu & -1 & 0 \\ 0 & \mu & -1 \\ 11 & -6 & 1 \end{vmatrix} + 1 \begin{vmatrix} 0 & -1 & 0 \\ 0 & \mu & -1 \\ -6 & -6 & 1 \end{vmatrix}$$

$$= \mu[\mu(\mu-6)+1(0+11)] + 1[0-6]$$

$$= \mu[\mu^2 - 6\mu + 11] - 6 = \mu^3 - 6\mu^2 + 11\mu - 6$$

$$|A| = 0 \Rightarrow \mu^3 - 6\mu^2 + 11\mu - 6 = 0$$

$\mu = 1$, $1 - 6 + 11 - 6 = 0$

$(\mu - 1)$ is a factor $\Rightarrow \mu^2(\mu-1) - 5\mu(\mu-1) + 6(\mu-1) = 0$

$\Rightarrow (\mu-1)(\mu^2 - 5\mu + 6) = 0 \Rightarrow (\mu-1)(\mu-2)(\mu-3) = 0$

$\mu = 1, 2, 3$

$\mu^2 - 5\mu + 6$
$= \mu^2 - 3\mu - 2\mu + 6$
$= \mu(\mu-3) - 2(\mu-3)$
$= (\mu-3)(\mu-2)$

Find the rank of the matrix

$$A=\begin{bmatrix} 1 & 1 & 1 \\ p & q & r \\ p^3 & q^3 & r^3 \end{bmatrix}$$

$a^3-b^3=(a-b)(a^2+ab+b^2)$

$|A| = \begin{vmatrix} 1 & 1 & 1 \\ p & q & r \\ p^3 & q^3 & r^3 \end{vmatrix} = \begin{vmatrix} 1 & 1 & 1 \\ 0 & q-p & r-p \\ 0 & q^3-p^3 & r^3-p^3 \end{vmatrix}$

$p=q$

$R_2 \to R_2 - pR_1$

$R_3 \to R_3 - p^3 R_1$

$= 1 \begin{vmatrix} q-p & r-p \\ q^3-p^3 & r^3-p^3 \end{vmatrix} = (q-p)(r-p)\begin{vmatrix} 1 & 1 \\ q^2+qp+p^2 & r^2+rp+p^2 \end{vmatrix}$

$= (q-p)(r-p)\left[r^2+rp+\cancel{p^2}-q^2-qp-\cancel{p^2}\right]$

$p=1, q=2$

$r=-3$

$= (q-p)(r-p)\left[(r+q)(r-q)+p(r-q)\right]$

$= (q-p)(r-p)(r-q)(r+q+p) = (p-q)(q-r)(r-p)(p+q+r)$

Case I: $|A| \neq 0$, $p \neq q \neq r$, $p+q+r \neq 0$, $r=3$

Case II: $p \neq q \neq r$, $p+q+r=0$, $r=2$

Case III: Exactly two values of p, q and r are identical

$r=2$

Case IV: $p=q=r$, $r=1$

Linear Dependence and Independence of Vectors

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Let $\{v_1, v_2, v_3, \dots, v_n\}$ be a set of vectors. This set of vector is said to be linearly independent if
$a_1v_1 + a_2v_2 + a_3v_3 + \dots + a_nv_n = 0$ *implies* $a_1 = a_2 = a_3 = \dots = a_n = 0$
,otherwise it is said to be dependent.

We can use the concept of rank of a matrix to decide whether the given set of vectors are linearly independent (L.I) or linearly dependent(L.D).

We form the matrix with each vector as its row and reduce it to the row echelon form. The given vectors are linearly independent ,if the row echelon form has no zero row. $r =$ no of vectors

In case if the matrix formed is a square matrix then we can decide the linear dependence and independence by evaluating their determinant.

If Determinant of $A = 0$, then vectors are dependent

If Determinant of $A \neq 0$, then vectors are independent

Examine whether the following set of vectors is linearly independent.

1) $\{(1,0,2),(3,2,4),(1,-1,-1)\}$ $\longrightarrow$ Linearly independent

2) $\{(1,1,0,1),(1,1,1,1),(-1,1,1,1),(1,0,0,1)\}$

L.I

Sol: $A=\begin{bmatrix}1 & 0 & 2\\ 3 & 2 & 4\\ 1 & -1 & -1\end{bmatrix}$, $|A| = 1(-2+4)+2(-3-2)$
$= 2-10 = -8 \neq 0$

$-3+\frac{1}{2}(-2)$

$R_2 \to R_2 - 3R_1$
$R_3 \to R_3 - R_1$

$\begin{bmatrix}\boxed{1} & 0 & 2\\ 0 & \boxed{2} & -2\\ 0 & -1 & -3\end{bmatrix} \xrightarrow{R_3 \to R_3 + \frac{1}{2}R_2} \begin{bmatrix}1 & 0 & 2\\ 0 & 2 & -2\\ 0 & 0 & -4\end{bmatrix}$

$r = 3$, L.I

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Examine whether the following set of vectors is linearly independent.

L.D

1) $\{(1,0,2),(3,2,4),(1,-1,-1)\}$
2) $\{(1,1,0,1),(1,1,1,1),(-1,1,1,1),(1,0,0,1)\}$

$$(3,2,4) = \alpha(1,0,2) + \beta(1,-1,-1)$$

$$\Rightarrow (3,2,4) = (\alpha+\beta, -\beta, 2\alpha-\beta)$$

$$\begin{cases} \alpha+\beta = 3 \Rightarrow \alpha = 3-\beta = 3+2 = 5 \\ -\beta = 2 \quad \beta = -2 \\ 2\alpha - \beta = 4 \end{cases}$$

$$2 \cdot 5 - (-2) = 12 \neq 4$$

We can not find α and β such that

Examine whether the following set of vectors is linearly independent.

$(1,0,0,1) = \alpha(\quad)$

1) $\{(1,0,2), (3,2,4), (1,-1,-1)\}$ v_3 v_4 $v_4 = \alpha v_1 + \beta v_2 + \gamma v_3$

2) $\{\underbrace{(1,1,0,1)}_{v_1}, \underbrace{(1,1,1,1)}_{v_2}, (-1,1,1,1), (1,0,0,1)\}$

$$\begin{bmatrix} 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 1 \\ -1 & 1 & 1 & 1 \\ 1 & 0 & 0 & 1 \end{bmatrix} \xrightarrow[\substack{R_3 \to R_3 + R_1 \\ R_4 \to R_4 - R_1}]{R_2 \to R_2 - R_1} \begin{bmatrix} \boxed{1} & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 2 & 1 & 2 \\ 0 & -1 & 0 & 0 \end{bmatrix}$$

$R_2 \leftrightarrow R_3$

$$\begin{bmatrix} 1 & 1 & 0 & 1 \\ 0 & 2 & 1 & 2 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \end{bmatrix} \xrightarrow{R_4 \to R_4 + \frac{1}{2} R_2} \begin{bmatrix} \boxed{1} & 1 & 0 & 1 \\ 0 & \boxed{2} & 1 & 2 \\ 0 & 0 & \boxed{1} & 0 \\ 0 & 0 & 1/2 & 1 \end{bmatrix} \xrightarrow{R_4 \to R_4 - \frac{1}{2} R_3} \begin{bmatrix} 1 & 1 & 0 & 1 \\ 0 & 2 & 1 & 2 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

L.I

$r = 4$

$\{(2,3,6,-3,4),(4,2,12,-3,6),(4,10,12,-9,10)\}$

$$A = \begin{bmatrix} \boxed{2} & 3 & 6 & -3 & 4 \\ 4 & 2 & 12 & -3 & 6 \\ 4 & 10 & 12 & -9 & 10 \end{bmatrix} \xrightarrow[R_3 \to R_3 - 2R_1]{R_2 \to R_2 - 2R_1}$$

$\underline{3 \times 5}$

$$\begin{bmatrix} 2 & 3 & 6 & -3 & 4 \\ 0 & \boxed{-4} & 0 & 3 & -2 \\ 0 & 4 & 0 & -3 & 2 \end{bmatrix} \xrightarrow{R_3 \to R_3 + R_2} \begin{bmatrix} 2 & 3 & 6 & -3 & 4 \\ 0 & -4 & 0 & 3 & -2 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$r = 2 < 3$

~~LI~~ linearly depen

$\{(2,3,6,-3,4),(4,2,12,-3,6),(4,10,12,-9,10)\}$, L.D

$(4,10,12,-9,10) = \alpha(2,3,6,-3,4) + \beta(4,2,12,-3,6)$

$$\Rightarrow \begin{cases} 2\alpha + 4\beta = 4 \\ 3\alpha + 2\beta = 10 \\ 6\alpha + 12\beta = 12 \\ -3\alpha - 3\beta = -9 \\ 4\alpha + 6\beta = 10 \end{cases}$$

$2\alpha + 4\beta = 4$

$6\alpha + 4\beta = 20$

$-4\alpha = -16, \alpha = 4$

$4\beta = 4 - 2\alpha = 4 - 8 = -4$

$\alpha = 4, \beta = -1$

$6 \cdot 4 - 12 \cdot 1 = 12$

$-12 + 3 = -9$

$16 - 6 = 10$

$4(2,3,6,-3,4) - 1(4,2,12,-3,6)$

$= (4,10,12,-9,10)$

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$\{(1,1,0,1),(1,1,1,1),(4,4,1,1),(1,0,0,1)\}$

$$\begin{bmatrix} 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 1 \\ 4 & 4 & 1 & 1 \\ 1 & 0 & 0 & 1 \end{bmatrix} \xrightarrow[\substack{R_3 \to R_3 - 4R_1 \\ R_4 \to R_4 - R_1}]{R_2 \to R_2 - R_1} \begin{bmatrix} 1 & 1 & 0 & 1 \\ 0 & \boxed{0} & 1 & 0 \\ 0 & 0 & 1 & -3 \\ 0 & -1 & 0 & 0 \end{bmatrix}$$

$R_2 \longleftrightarrow R_4$

$$\begin{bmatrix} \boxed{1} & 1 & 0 & 1 \\ 0 & \boxed{-1} & 0 & 0 \\ 0 & 0 & \boxed{1} & -3 \\ 0 & 0 & 1 & 0 \end{bmatrix} \xrightarrow{R_4 \to R_4 - R_3} \begin{bmatrix} 1 & 1 & 0 & 1 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & -3 \\ 0 & 0 & 0 & 3 \end{bmatrix}$$

$r = 4 = n$

Linearly independent

Gauss Elimination Method for solving system of Linear equations

Sandeep Kumar Singh

Process for Non-homogeneous system of equations

Step1: Write the system in matrix form $AX = B$.

Step 2: Write the augmented matrix $[A{:}\, B]$

Step 3: Reduce the augmented matrix to row Echelon form

Let $r =$ rank of coefficient matrix, $r' =$ rank of $[A, B]$, $n =$ number of variables

Case 1: $r = r' = n$, the system of equations are consistent and there is a unique solution

Case 2: $r = r' < n$,the system of equations are consistent and there are infinite number of solutions.(Giving arbitrary values to $(n - r)$ of unknowns we may express the other r unknowns in terms of them).

Case 3: If $r \neq r'$,the equations are inconsistent and the system of equation has no solution

Solve the system of equations if possible

$2x + z = 3, x - y + z = 1, 4x - 2y + 3z = 3$

$$\underset{A}{\begin{bmatrix}2&0&1\\1&-1&1\\4&-2&3\end{bmatrix}}\underset{X}{\begin{bmatrix}x\\y\\z\end{bmatrix}}=\underset{B}{\begin{bmatrix}3\\1\\3\end{bmatrix}}$$

$$[A:B]=\left[\begin{array}{ccc|c}\boxed{2}&0&1&3\\1&-1&1&1\\4&-2&3&3\end{array}\right]\xrightarrow[R_3\to R_3-2R_1]{R_2\to R_2-\frac{1}{2}R_1}\left[\begin{array}{ccc|c}2&0&1&3\\0&\boxed{-1}&1/2&-1/2\\0&-2&1&-3\end{array}\right]$$

$$\xrightarrow{R_3\to R_3-2R_2}\left[\begin{array}{ccc|c}2&0&1&3\\0&-1&1/2&-1/2\\0&0&0&-2\end{array}\right]$$

$r=2,\ r'=3,\ n=3$

$r\neq r'$

Inconsistent and it has no solution

Solve the system of equations if possible

$2x + y - z = 4, x - y + 2z = -2, -x + 2y - z = 2$

$$\begin{bmatrix} 2 & 1 & -1 \\ 1 & -1 & 2 \\ -1 & 2 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 4 \\ -2 \\ 2 \end{bmatrix}$$

A X B

Here the system is consistent and it has a unique sol

$$[A:B] = \left[\begin{array}{ccc|c} \boxed{2} & 1 & -1 & 4 \\ 1 & -1 & 2 & -2 \\ -1 & 2 & -1 & 2 \end{array}\right] \xrightarrow[R_3 \to R_3 + \frac{1}{2}R_1]{R_2 \to R_2 - \frac{1}{2}R_1} \left[\begin{array}{ccc|c} 2 & 1 & -1 & 4 \\ 0 & \boxed{-3/2} & 5/2 & -4 \\ 0 & 5/2 & -3/2 & 4 \end{array}\right]$$

$R_3 \to R_3 + \frac{5}{3} R_2$

$$\left[\begin{array}{ccc|c} \boxed{2} & 1 & -1 & 4 \\ 0 & \boxed{-3/2} & 5/2 & -4 \\ 0 & 0 & \boxed{8/3} & -8/3 \end{array}\right]$$

$r = 3, r' = 3, n = 3, r = r' = n,$ —

$2x + y - z = 4 \Rightarrow 2x = 4 - y + z = 2$

$-\frac{3}{2} y + \frac{5}{2} z = -4 \Rightarrow -\frac{3}{2} y = -4 + \frac{5}{2} = -3/2$

$y = 1$

$\frac{8}{3} z = -\frac{8}{3} \Rightarrow z = -1$

$x = 2, y = 1, z = -1$

Solve the system of equations if possible

$x - y + z = 1, 2x + y - z = 2, 5x - 2y + 2z = 5$

We assign that variable an arbitrary value which corresponds to the column not containing the pivot element.

$$\begin{bmatrix} 1 & -1 & 1 \\ 2 & 1 & -1 \\ 5 & -2 & 2 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 5 \end{bmatrix}$$

$$[A:B] = \left[\begin{array}{ccc|c} 1 & -1 & 1 & 1 \\ 2 & 1 & -1 & 2 \\ 5 & -2 & 2 & 5 \end{array}\right] \xrightarrow[R_3 \to R_3 - 5R_1]{R_2 \to R_2 - 2R_1} \left[\begin{array}{ccc|c} 1 & -1 & 1 & 1 \\ 0 & \boxed{3} & -3 & 0 \\ 0 & 3 & -3 & 0 \end{array}\right]$$

$R_3 \to R_3 - R_2$

$$\begin{array}{cccc} \left[\begin{array}{ccc|c} \boxed{1} & -1 & 1 & 1 \\ 0 & \boxed{3} & -3 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \\ x \quad\; y \quad\; z \end{array}$$

$r = 2, r' = 2, n = 3, n - r = 3 - 2 = 1$

$r = r' < n$, Inconsistent and has infinite solution.

$x - y + z = 1 \Rightarrow x = 1 + y - z = 1 + c - c = 1$

$3y - 3z = 0, z = c, y = c.$

$(1, c, c), c \in \mathbb{R}$

Solve the system of equations if possible

$4x - 3y - 9z + 6w = 0, 2x + 3y + 3z + 6w = 6, 4x - 21y - 39z - 6w = -24$

$$\begin{bmatrix} 4 & -3 & -9 & 6 \\ 2 & 3 & 3 & 6 \\ 4 & -21 & -39 & -6 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 6 \\ -24 \end{bmatrix}$$

$$\frac{9}{2}y = 6 - \frac{15}{2}c_1 - 3c_2 = \frac{12 - 15c_1 - 6c_2}{2}$$

$$[A:B] = \left[\begin{array}{cccc|c} 4 & -3 & -9 & 6 & 0 \\ 2 & 3 & 3 & 6 & 6 \\ 4 & -21 & -39 & -6 & -24 \end{array}\right] \xrightarrow[R_3 \to R_3 - R_1]{R_2 \to R_2 - \frac{1}{2}R_1} \left[\begin{array}{cccc|c} 4 & -3 & -9 & 6 & 0 \\ 0 & \boxed{9/2} & 15/2 & 3 & 6 \\ 0 & -18 & -30 & -12 & -24 \end{array}\right]$$

$R_3 \to R_3 + 4R_2$

$$\begin{array}{ccccc} & & & & \\ \left[\begin{array}{cccc|c} \boxed{4} & -3 & -9 & 6 & 0 \\ 0 & \boxed{9/2} & 15/2 & 3 & 6 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right] \end{array}$$

$x \quad y \quad \uparrow \quad \uparrow$

$r = 2, \ r' = 2, \ n = 4, \ r = r' < n$

consistent, Infinite soln, $n - r = 4 - 2 = 2$

$4x - 3y - 9z + 6w = 0$

$\frac{9}{2}y + \frac{15}{2}z + 3w = 6$

$z = c_1, \ w = c_2$

$y = \frac{1}{9}(12 - 15c_1 - 6c_2)$

$x =$

Investigate for what values of λ and μ the system of equations

$$x + y + z = 6, x + 2y + 3z = 10, x + 2y + \lambda z = \mu \; have$$

(a) No solution (b) an infinite no of solution (c) unique solution

$$\underbrace{\begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 1 & 2 & \lambda \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} x \\ y \\ z \end{bmatrix}}_{X} = \underbrace{\begin{bmatrix} 6 \\ 10 \\ \mu \end{bmatrix}}_{B}$$

$$[A:B] = \left[\begin{array}{ccc|c} 1 & 1 & 1 & 6 \\ 1 & 2 & 3 & 10 \\ 1 & 2 & \lambda & \mu \end{array}\right] \xrightarrow[R_3 \to R_3 - R_1]{R_2 \to R_2 - R_1} \left[\begin{array}{ccc|c} 1 & 1 & 1 & 6 \\ 0 & 1 & 2 & 4 \\ 0 & 1 & \lambda-1 & \mu-6 \end{array}\right]$$

$R_3 \to R_3 - R_2$

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 6 \\ 0 & 1 & 2 & 4 \\ 0 & 0 & \underset{\neq 0}{\lambda-3} & \mu-10 \end{array}\right]$$

a) $r \neq r'$, $\lambda = 3$, $\mu \neq 10$

b) $r = r' < n$, $\lambda = 3$, $\mu = 10$

c) $r = r' = n = 3$, $\lambda \neq 3$, μ may be anything

Homogeneous System of equation $\boxed{AX = 0}$

$x = 0, y = 0, z = 0$

The homogenous system is always consistent since there is always a zero solution.

$$x + 2y - 3z = 0, x + y - z = 0, x - y + z = 0$$

Method of Solution: Reduce the matrix A to row reduced echelon form. Let r be the rank of the matrix A.

Case1: If $rank\ A = r < n$, then the system has a non-trivial solution. In case of square matrix A ,this corresponds to $|A| = 0$. $(n - r)$ variables are assigned the arbitrary values and remaining variables are written in terms of those variables.

Case2: If $rank\ A = r = n$, then the system has only a trivial solution. In case of square matrix this corresponds to $|A| \neq 0$

$AX = 0$

$|A| \neq 0$, A^{-1} exist

$A^{-1}(AX) = A^{-1}0$

$X = 0$

$A \rightarrow$ row reduced Ech. form

$r < n$, $|A| = 0$, Non-zero, Non trivial sol.

$r = n$, $|A| \neq 0$, Trivial sol, zero

Solve the system of equations

$x + 2y - 3z = 0, x + y - z = 0, x - y + z = 0$

m-1

$$\begin{bmatrix} 1 & 2 & -3 \\ 1 & 1 & -1 \\ 1 & -1 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$A \quad X \quad O$

$|A| = 1(1-1)$
$-2(1+1)$
$-3(-1-1)$
$= 0 - 4 + 6 = 2 \neq 0$

It will have a zero sol

$x = y = z = 0$, $r = 3$

$r = 3$

$\boxed{r = n}$ $n = 3$

m-2

$R_2 \to R_2 - R_1$
$R_3 \to R_3 - R_1$

$$\begin{bmatrix} 1 & 2 & -3 \\ 0 & -1 & 2 \\ 0 & -3 & 4 \end{bmatrix} \xrightarrow{R_3 \to R_3 - 3R_2} \begin{bmatrix} 1 & 2 & -3 \\ 0 & -1 & 2 \\ 0 & 0 & -2 \end{bmatrix}$$

$\underline{r = n}$

So it has a zero sol only

$x = y = z = 0$

$x + 2y - 3z = 0 \Rightarrow x = 0$
$-y + 2z = 0 \Rightarrow y = 0$
$-2z = 0 \Rightarrow z = 0$

Solve the system of equations

$x + y - z + w = 0, 2x + 3y + z + 4w = 0, 3x + 2y - 6z + w = 0$

$$\underbrace{\begin{bmatrix}1&1&-1&1\\2&3&1&4\\3&2&-6&1\end{bmatrix}}_{A\ \ 3\times 4}\underbrace{\begin{bmatrix}x\\y\\z\\w\end{bmatrix}}_{X}=\underbrace{\begin{bmatrix}0\\0\\0\end{bmatrix}}_{O}$$

$r \le \min(m,n)$

$r \le 3, \quad n = 4$

$\boxed{r < n}$

$R_2 \to R_2 - 2R_1$

$R_3 \to R_3 - 3R_1$

$$\begin{bmatrix}1&1&-1&1\\0&1&3&2\\0&-1&-3&-2\end{bmatrix}\xrightarrow{R_3 \to R_3 + R_2}\begin{array}{cccc}x&y&&\\ \end{array}\begin{bmatrix}\boxed{1}&1&-1&1\\0&\boxed{1}&3&2\\0&0&0&0\end{bmatrix}$$

$r = 2, n = 4$

$n - r = 4 - 2 = 2$

$x + y - z + w = 0 \Rightarrow x = -y + z - w = 3c_1 + 2c_2 + c_1 - c_2$

$y + 3z + 2w = 0 \Rightarrow y = -3c_1 - 2c_2$

$z = c_1, \quad w = c_2$

$(4c_1 + c_2, -3c_1 - 2c_2, c_1, c_2) \quad x = 4c_1 + c_2$

Determine the values of k for which the system of equations

$$x - ky + z = 0, kx + 3y - kz = 0, 3x + y - z = 0$$

has (i) only trivial solution, (ii) non-trivial solution

$$\underset{A}{\begin{bmatrix} 1 & -k & 1 \\ k & 3 & -k \\ 3 & 1 & -1 \end{bmatrix}} \underset{X}{\begin{bmatrix} x \\ y \\ z \end{bmatrix}} = \underset{O}{\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}}$$

(i) $|A| \neq 0$

$$|A| = 1(-3+k) + k(\underbrace{-k+3k}) + 1(k-9)$$

$$= -3 + \underline{k} + 2k^2 \underline{+k} - 9 = \underbrace{2k^2 + 2k - 12}$$

$$|A| \neq 0 \Rightarrow 2(k^2 + k - 6) \neq 0 \Rightarrow k^2 + k - 6 \neq 0$$

$$\Rightarrow k^2 + 3k - 2k - 6 \neq 0 \Rightarrow k(k+3) - 2(k+3) \neq 0$$

$$\Rightarrow (k+3)(k-2) \neq 0$$

$$k \neq -3, 2$$

(ii) $|A| = 0$

$$\Rightarrow 2k^2 + 2k - 12 = 0$$

$$k = -3, 2$$

Eigen Values and Eigen Vectors

Sandeep Kumar Singh

Let A is a square matrix , X is a non-zero vector and λ is a scalar such that $AX = \lambda X$, then X is called the eigen vector corresponding to the eigen value λ.

$Ax - \lambda I x = 0$ $\quad$ $IA = A$

$$AX = \lambda X \Rightarrow AX = \lambda IX \Rightarrow (A - \lambda I)X = 0 \ldots (1)$$

$Ix = x$

For a non-zero solution of (1) $\quad$ ↳ a homogenous system of linear eqns

$|A - \lambda I| = 0$, This equation is known as Characteristic equation.

$Ax = \lambda x$

$$\begin{bmatrix} & \\ & \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \lambda \begin{bmatrix} x \\ y \end{bmatrix}$$

$Ax = 0$, $|A| = 0$

$$\begin{bmatrix} 5 & 4 \\ 1 & 2 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 5-\lambda & 4 \\ 1 & 2-\lambda \end{bmatrix}$$

$Ax = \lambda x$

$$\begin{bmatrix} 5-\lambda & 4 \\ 1 & 2-\lambda \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = 0$$

Ex: Find the characteristic equation of the matrix $\begin{bmatrix} 5 & 4 \\ 1 & 2 \end{bmatrix}$. $= A$, $|A| = 5 \times 2 - 4 \times 1$

2×2 $\quad = 10 - 4 = 6$

Sol: $|A - \lambda I| = 0$

$\Rightarrow \begin{vmatrix} 5-\lambda & 4 \\ 1 & 2-\lambda \end{vmatrix} = 0 \Rightarrow (5-\lambda)(2-\lambda) - 4 = 0$

$\Rightarrow 10 - 2\lambda - 5\lambda + \lambda^2 - 4 = 0 \Rightarrow \boxed{\lambda^2 - 7\lambda + 6 = 0}$

ch eqⁿ

Alternate Method

$$\lambda^2 - (\text{Trace of } A)\lambda + \det(A) = 0 \quad \text{---} (*)$$

Trace of A = Sum of the diagonal elements

$$\lambda^2 - (5+2)\lambda + 6 = 0 \Rightarrow \lambda^2 - 7\lambda + 6 = 0$$

Characteristic root or Eigen value

The roots of the characteristic equation are said to be characteristic root or Eigen value of the matrix.

Q) Find the characteristic roots of the matrix $\begin{bmatrix} 5 & 4 \\ 1 & 2 \end{bmatrix}$. , $|A| = 10 - 4 = 6$

Sol. $\lambda^2 - 7\lambda + 6 = 0 \Rightarrow \lambda^2 - 6\lambda - \lambda + 6 = 0$

$\Rightarrow \lambda(\lambda - 6) - 1(\lambda - 6) = 0 \Rightarrow (\lambda - 6)(\lambda - 1) = 0$

$\boxed{\lambda = 6, 1}$

$|A - \lambda I| = 0$

Trace of $A = 5 + 2 = 7$

$6 + 1 = 7$

Product of eigen values $= 6 \times 1 = 6$

Finding Eigen vectors

Find the Eigen vectors of the matrix $\begin{bmatrix} 5 & 4 \\ 1 & 2 \end{bmatrix}$. $- 6\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

$(A - \lambda I)X = 0$ — (1) $|A - \lambda I| = 0$

Sol:- $\lambda^2 - 7\lambda + 6 = 0 \Rightarrow \lambda = 6, 1$

$\lambda = 6$, $(A - 6I)X = 0$

$\Rightarrow \begin{bmatrix} -1 & 4 \\ 1 & -4 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = 0$

$\begin{bmatrix} 8 \\ 2 \end{bmatrix}$

$\downarrow R_2 \to R_2 + R_1$

$\begin{bmatrix} -4 \\ -1 \end{bmatrix}$

$\Rightarrow \begin{bmatrix} \boxed{-1} & 4 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = 0$

$-x + 4y = 0$

$y = 1, x = 4$

$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 4 \\ 1 \end{bmatrix}$ — Eigen vector

$y = c, x = 4c$

$\begin{bmatrix} 4c \\ c \end{bmatrix} = c\begin{bmatrix} 4 \\ 1 \end{bmatrix}$

$\lambda = 1$, $\begin{bmatrix} 5 & 4 \\ 1 & 2 \end{bmatrix}$ ✓, $(A - \lambda I)X = 0 \Rightarrow (A - I)X = 0$

$\Rightarrow \begin{bmatrix} 4 & 4 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = 0 \Rightarrow \begin{bmatrix} \boxed{4} & 4 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = 0$

$R_2 \to R_2 - \frac{1}{4} R_1$

$4x + 4y = 0$ $\begin{bmatrix} -1 \\ 1 \end{bmatrix}$ ✓ $y = c$, $x = -c$ $\begin{bmatrix} -c \\ c \end{bmatrix} = c\begin{bmatrix} -1 \\ 1 \end{bmatrix}$

$y = 1, x = -1$

$\lambda = 6$, $\begin{bmatrix} 4 \\ 1 \end{bmatrix}$ / $\lambda = 1$, $\begin{bmatrix} -1 \\ 1 \end{bmatrix}$, $AX = \lambda X$

$\begin{bmatrix} 5 & 4 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} 4 \\ 1 \end{bmatrix} = \begin{bmatrix} 24 \\ 6 \end{bmatrix} = 6 \begin{bmatrix} 4 \\ 1 \end{bmatrix}$

Find the eigen values and Eigen vectors of $\begin{bmatrix} 1 & 1 & 3 \\ 1 & 5 & 1 \\ 3 & 1 & 1 \end{bmatrix} = A$

Sol: The ch eqn is $|A - \lambda I| = 0$

$$\Rightarrow \begin{vmatrix} 1-\lambda & 1 & 3 \\ 1 & 5-\lambda & 1 \\ 3 & 1 & 1-\lambda \end{vmatrix} = 0 \Rightarrow (1-\lambda)[(5-\lambda)(1-\lambda) - 1] - 1[(1-\lambda) - 3] + 3[1 - 3(5-\lambda)] = 0$$

$$\Rightarrow (1-\lambda)[5 - \lambda - 5\lambda + \lambda^2 - 1] - 1[1 - \lambda - 3] + 3[1 - 15 + 3\lambda] = 0$$

$$\Rightarrow (1-\lambda)(\lambda^2 - 6\lambda + 4) - 1(-\lambda - 2) + 3(-14 + 3\lambda) = 0$$

$$\Rightarrow \lambda^2 - \cancel{6\lambda} + 4 - \lambda^3 + 6\lambda^2 - \cancel{4\lambda} + \cancel{\lambda} + 2 - 42 + \cancel{9\lambda} = 0$$

$$\Rightarrow -\lambda^3 + 7\lambda^2 - 36 = 0 \Rightarrow \boxed{\lambda^3 - 7\lambda^2 + 36 = 0}$$

$\lambda^3 - 7\lambda^2 + 36 = 0$, $\lambda = -2$, $-8 - 28 + 36 = 0$

$(\lambda+2)$ is a factor

$\lambda^2(\lambda+2) - 9\lambda(\lambda+2) + 18(\lambda+2) = 0$

$\Rightarrow (\lambda+2)(\lambda^2 - 9\lambda + 18) = 0 \Rightarrow (\lambda+2)[\lambda^2 - 6\lambda - 3\lambda + 18] = 0$

$\Rightarrow (\lambda+2)[\lambda(\lambda-6) - 3(\lambda-6)] = 0$ $\qquad -2+6+3 = \underline{7}$

$\Rightarrow (\lambda+2)(\lambda-6)(\lambda-3) = 0 \Rightarrow \lambda = \boxed{-2}, 6, 3$

$\begin{bmatrix} 1 & 1 & 3 \\ 1 & 5 & 1 \\ 3 & 1 & 1 \end{bmatrix}$ $\quad \underline{\lambda = -2}$, $(A - \lambda I)X = 0$

$\Rightarrow \begin{bmatrix} 3 & 1 & 3 \\ 1 & 7 & 1 \\ 3 & 1 & 3 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = 0$ $\quad \xrightarrow[R_3 \to R_3 - R_1]{R_2 \to R_2 - \frac{1}{3}R_1}$

✓ ✓

$\begin{bmatrix} 3 & 1 & 3 \\ 0 & 20/3 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = 0$ $\quad 3x + y + 3z = 0$

$\frac{20}{3} y = 0 \Rightarrow y = 0$

$z = 1$

$x = -1$

$\begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$

$$\begin{bmatrix} 1 & 1 & 3 \\ 1 & 5 & 1 \\ 3 & 1 & 1 \end{bmatrix}, \quad \underline{\lambda = 3}, \quad (A - \lambda I)x = 0$$

$$\Rightarrow (A - 3I)X = 0$$

$$\begin{bmatrix} -2 & 1 & 3 \\ 1 & 2 & 1 \\ 3 & 1 & -2 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = 0 \Rightarrow \begin{bmatrix} -2 & 1 & 3 \\ 0 & 5/2 & 5/2 \\ 0 & 5/2 & 5/2 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = 0$$

$R_2 \to R_2 + \frac{1}{2}R_1$

$R_3 \to R_3 + \frac{3}{2}R_1$

$R_3 \to R_3 - R_2$

$$\begin{bmatrix} -2 & 1 & 3 \\ 0 & 5/2 & 5/2 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = 0 \Rightarrow \begin{aligned} -2x + y + 3z &= 0 \\ \tfrac{5}{2}y + 5/2\, z &= 0 \end{aligned}$$

$z = 1, \; y = -1,$

$$\begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$$

$-2x - 1 + 3 = 0 \Rightarrow -2x + 2 = 0 \Rightarrow x = 1$

$$\begin{bmatrix} 1 & 1 & 3 \\ 1 & 5 & 1 \\ 3 & 1 & 1 \end{bmatrix}, \lambda = 6, \quad (A-\lambda I)X = 0$$

$$\Rightarrow (A-6I)X = 0$$

$$\begin{bmatrix} -5 & 1 & 3 \\ 1 & -1 & 1 \\ 3 & 1 & -5 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = 0 \Rightarrow \begin{bmatrix} -5 & 1 & 3 \\ 0 & -4/5 & 8/5 \\ 0 & 8/5 & -16/5 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = 0$$

$-5+4/5$

$R_2 \to R_2 + \frac{1}{5}R_1$

$R_3 \to R_3 + \frac{3}{5}R_1$

$\downarrow, R_3 \to R_3 + 2R_2$

$$\begin{bmatrix} -5 & 1 & 3 \\ 0 & -4/5 & 8/5 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = 0$$

$\frac{-4}{5}y = -8/5$

$-5x + y + 3z = 0$

$-4/5\, y + 8/5\, z = 0$

$z = 1, \; y = 2, \; x = 1$

$-5x + 2 + 3 = 0$

$-5x + 5 = 0$

$x = 1$

$$\begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$$

Find the eigen values and Eigen vectors of $\begin{bmatrix} 1 & 1 & 3 \\ 1 & 5 & 1 \\ 3 & 1 & 1 \end{bmatrix} A$ $\quad \lambda^3 - 7\lambda^2 + 36 = 0$

Alternate method

$\lambda^3 - (\text{Trace of } A)\lambda^2 + (\text{Sum of minors along the main diagonal})\lambda - \det A = 0$

$\text{Trace } A = 1+5+1 = 7$

$M_{11} = 5\times1 - 1\times1 = 5-1 = 4, \quad M_{22} = 1-9 = -8,$

$M_{33} = 1\times5 - 1\times1 = 5-1 = 4 \qquad M_{11} + M_{22} + M_{33} = 4-8+4 = 0$

$\det A = 1(5-1) - 1(1-3) + 3(1-15)$

$= 4+2-42 = -36$

$\lambda^3 - 7\lambda^2 + 0\lambda + 36 = 0$

$\lambda^3 - 7\lambda^2 + 36 = 0$

Find the eigen values and Eigen vectors of $\begin{bmatrix} -2 & 2 & -3 \\ 2 & 1 & -6 \\ -1 & -2 & 0 \end{bmatrix}$

$\lambda^3 + \lambda^2 - 21\lambda - 45 = 0$, $\lambda = 5, -3, -3$

$\underline{\lambda = -3}$ ✓, $(A - \lambda I)X = 0 \Rightarrow (A + 3I)X = 0$

$\Rightarrow \begin{bmatrix} 1 & 2 & -3 \\ 2 & 4 & -6 \\ -1 & -2 & 3 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = 0 \Rightarrow \begin{bmatrix} 1 & 2 & -3 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = 0$

$R_2 \rightarrow R_2 - 2R_1$

$R_3 \rightarrow R_3 + R_1$

$x + 2y - 3z = 0$

$y = 1, z = 0, x = -2$ ✓ $u_1 \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix}$

$y = 0, z = 1, x = 3$ ✓ $\begin{bmatrix} 3 \\ 0 \\ 1 \end{bmatrix} u_2$

$c\begin{bmatrix} 3 \\ 0 \\ 1 \end{bmatrix} \neq \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix}$ $\begin{bmatrix} 3 \\ 0 \\ 1 \end{bmatrix} \neq c\begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix}$

$c_1 u_1 + c_2 u_2$

Properties of Eigen Values and Eigen Vectors

Sandeep Kumar Singh

Properties

A λ $AX = \lambda X$

If λ be an eigen value of A and X be its corresponding eigen vector.

1) Any square matrix A and its transpose A' have the same eigen values.

A^T

2) The eigen values of the triangular matrices are just the diagonal elements

upper triangular, lower triangular

3) αA has eigen value $\alpha\lambda$ and the corresponding eigen vector is X.

$$A = \begin{bmatrix} 1 & 4 \\ 3 & 2 \end{bmatrix} \quad -2, 5$$

$$A^T = \begin{bmatrix} 1 & 3 \\ 4 & 2 \end{bmatrix}, \quad -2, 5$$

$$A = \begin{bmatrix} 2 & 0 & 0 \\ 3 & 4 & 0 \\ 7 & 3 & 2 \end{bmatrix}$$

2, 4, 2

$$\begin{bmatrix} 1 & 7 & 3 \\ 0 & 6 & 5 \\ 0 & 0 & 2 \end{bmatrix}$$

$\hookrightarrow$ 1, 6, 2

Properties

If λ be an eigen value of A and X be its corresponding eigen vector.

1) Any square matrix A and its transpose A' have the same eigen values.

2) The eigen values of the triangular matrices are just the diagonal elements

3) αA has eigen value $\alpha\lambda$ and the corresponding eigen vector is X.

$A \longrightarrow \lambda, \quad \alpha A \longrightarrow \alpha\lambda$

A

$\begin{bmatrix} 1 & 4 \\ 3 & 2 \end{bmatrix}, \quad -2, 5$

$2\begin{bmatrix} 1 & 4 \\ 3 & 2 \end{bmatrix} = \begin{bmatrix} 2 & 8 \\ 6 & 4 \end{bmatrix}$

$2A$

$2\times4-6\times8$

$= 8-48 = -40 \quad -4, 10$

$\lambda^2 - 6\lambda - 40 = 0$

$\lambda^2 - 10\lambda + 4\lambda - 40 = 0$

$\Rightarrow \lambda(\lambda-10) + 4(\lambda-10) = 0$

$\lambda = 10, \lambda = -4$

-4

Properties

4) $A - kI$ has the eigen value $\lambda - k$, for any scalar k and corresponding eigen vector is X.

5)The sum of eigen values of a matrix is the sum of the elements of the principal diagonal.

6) The product of the eigen values of a matrix is equal to its determinant.

$A = \begin{bmatrix} 1 & 4 \\ 3 & 2 \end{bmatrix} \longrightarrow -2, 5$

$A - 2I$

$-2-2 = -4, \; 5-2 = 3$

$A - 2I = \begin{bmatrix} -1 & 4 \\ 3 & 0 \end{bmatrix} \longrightarrow \lambda^2 + \lambda - 12 = 0$

$k=2$

$\Rightarrow \lambda^2 + 4\lambda - 3\lambda - 12 = 0$

$\Rightarrow \lambda(\lambda+4) - 3(\lambda+4) = 0$

$\lambda = -4, 3$

$A \longrightarrow \lambda_1, \lambda_2, \lambda_3$

$A - 3I, \; \lambda_1 - 3, \; \lambda_2 - 3, \; \lambda_3 - 3$

Properties

4) $A - kI$ has the eigen value $\lambda - k$, for any scalar k and corresponding eigen vector is X.

5) The sum of eigen values of a matrix is the sum of the elements of the principal diagonal. (Trace)

6) The product of the eigen values of a matrix is equal to its determinant.

$A = \begin{bmatrix} 1 & 4 \\ 3 & 2 \end{bmatrix} \rightarrow -2, 5$

Sum $= -2 + 5 = 3$ ✓

Trace $= 1 + 2 = 3$ ✓

Product $= (-2)(5) = -10$

$|A|$

$= 1 \times 2 - 3 \times 4 = 2 - 12 = \underline{-10}$

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Properties

$A^2 = A$

7) The eigen values of an Idempotent matrix($A^2 = A$)are either zero or unity.

8) If λ is the eigen value of a matrix A then $\frac{1}{\lambda}$ is the eigen values of A^{-1} and corresponding eigen vector is X.

$A \rightarrow 3, 7, 5, \quad A^{-1} - \frac{1}{3}, \frac{1}{7}, \frac{1}{5}$

9) $(A - kI)^{-1}$ has the eigen value $\frac{1}{(\lambda - k)}$ and corresponding eigen vector is X for any scalar k.

$A = \begin{bmatrix} 1 & 4 \\ 3 & 2 \end{bmatrix}$

$\hookrightarrow (-2, 5)$

$A^{-1} \rightarrow \frac{1}{-2}, \frac{1}{5}$

$AX = \lambda X \Rightarrow A(AX) = \lambda(AX)$

$\Rightarrow A^2X = \lambda \cdot \lambda X$

$\Rightarrow AX = \lambda^2 X \Rightarrow \lambda^2 X = \lambda X$

$\Rightarrow (\lambda^2 - \lambda)X = 0$

$\lambda - \lambda = 0 \Rightarrow \lambda(\lambda - 1) = 0$

$\lambda = 0, 1$

Properties

7) The eigen values of an Idempotent matrix($A^2 = A$)are either zero or unity.

8) If λ is the eigen value of a matrix A then $\frac{1}{\lambda}$ is the eigen values of A^{-1} and corresponding eigen vector is X.

(4, 8)

9) $(A - kI)^{-1}$ has the eigen value $\frac{1}{(\lambda - k)}$ and corresponding eigen vector is X for any scalar k.

$A \longrightarrow 3, 5$ $\qquad (A - 8I)^{-1}$

$(A - 2I)^{-1}$ $\qquad \frac{1}{3-2}, \frac{1}{5-2}$

$k = 2$ $\qquad \frac{1}{1}, \frac{1}{3}$

Properties

P1, P8

10) If λ is an eigen value of an orthogonal matrix$(A^{-1} = A')$then $\frac{1}{\lambda}$ is also its eigen value.

11) If $\lambda_1, \lambda_2, \ldots, \lambda_n$ are the eigen values of a matrix A, then $\lambda_1^m, \lambda_2^m, \ldots, \lambda_n^m$ are the eigen values of A^m(m being a positive integer).

$A \cdot \cdot A^T$ Same

$A \to \lambda, A^{-1} - \frac{1}{\lambda}$

$A^{-1} \quad A^T$

$A^{-1} = A^T$

$A \longrightarrow \lambda, \frac{1}{\lambda}$

$A \longrightarrow \lambda, \frac{1}{\lambda}$

$A^T = A^{-1}$

$A \longrightarrow \lambda$

$\lambda \quad \frac{1}{\lambda}$

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Properties

10) If λ is an eigen value of an orthogonal matrix$(A^{-1} = A')$then $\frac{1}{\lambda}$ is also its eigen value.

$n \times n$

11) If $\lambda_1, \lambda_2, \dots, \lambda_n$ are the eigen values of a matrix A, then $\lambda_1^m, \lambda_2^m, \dots, \lambda_n^m$ are the eigen values of A^m(m being a positive integer).

$A = \begin{bmatrix} 1 & 4 \\ 3 & 2 \end{bmatrix}$ $\quad -2, 5$

$A^7 \quad (-2)^7 \quad 5^{-7}$

$B = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 1 \\ 2 & 0 & 3 \end{bmatrix}$ $\quad 1, 2, 3$

$B^2, \quad 1^2, 2^2, 3^2$

$1, 4, 9$

Properties

12)For a real matrix A if $\alpha + i\beta$ is an eigen value, then its conjugate $\alpha - i\beta$ is also an eigen value(since the characteristic equation has the real coefficients). When A is complex then this property does not hold.

13) If λ is an eigen value of a non-singular matrix A then $\frac{|A|}{\lambda}$ is an eigen value of the matrix $adjA$.

$L\ |A| \neq 0$

$\alpha + i\beta\ ,\ \alpha - i\beta$

$\frac{|A|}{\lambda}$

$A \longrightarrow \lambda$

$adjA$ $\quad \frac{|A|}{\lambda}\ ,\ |A| \neq 0$

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Inverse of a matrix using Gauss Jordan Method

Sandeep Kumar Singh

$$\underbrace{[A \mid I]}_{\downarrow\ I} \xrightarrow[\text{row operations}]{\text{Elementary}} [I \mid A^{-1}]$$

Using Gauss Jordan method find the inverse of the matrix $\begin{bmatrix} 1 & 3 \\ 2 & 5 \end{bmatrix}$. $= A$

$$[A \mid I] = \left[\begin{array}{cc|cc} 1 & 3 & 1 & 0 \\ 2 & 5 & 0 & 1 \end{array}\right] \xrightarrow{R_2 \to R_2 - 2R_1} \left[\begin{array}{cc|cc} 1 & 3 & 1 & 0 \\ 0 & -1 & -2 & 1 \end{array}\right]$$

$$\xrightarrow{R_2 \to (-1)R_2} \left[\begin{array}{cc|cc} 1 & 3 & 1 & 0 \\ 0 & 1 & 2 & -1 \end{array}\right]$$

$$\downarrow R_1 \to R_1 - 3R_2$$

$$\left[\begin{array}{cc|cc} 1 & 0 & -5 & 3 \\ 0 & 1 & 2 & -1 \end{array}\right]$$

$$\underbrace{\qquad}_{I} \qquad \underbrace{\qquad}_{A^{-1}}$$

$$A^{-1} = \begin{bmatrix} -5 & 3 \\ 2 & -1 \end{bmatrix}$$

$$AA^{-1} = I$$

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$$\frac{1}{2}-\frac{7}{2}\times\frac{1}{5}=\frac{1}{2}-\frac{7}{10}$$

Using Gauss Jordan method find the inverse of the matrix $\begin{bmatrix} -1 & 1 & 2 \\ 3 & -1 & 1 \\ -1 & 3 & 4 \end{bmatrix}$.

$$\begin{bmatrix} -7/10 & 2/10 & 3/10 \\ -13/10 & -2/10 & 7/10 \\ 4/5 & 1/5 & -1/5 \end{bmatrix}$$

$$[A \mid I]=\left[\begin{array}{ccc|ccc} -1 & 1 & 2 & 1 & 0 & 0 \\ 3 & -1 & 1 & 0 & 1 & 0 \\ -1 & 3 & 4 & 0 & 0 & 1 \end{array}\right] \xrightarrow{R_1\to(-1)R_1} \left[\begin{array}{ccc|ccc} 1 & -1 & -2 & -1 & 0 & 0 \\ 3 & -1 & 1 & 0 & 1 & 0 \\ -1 & 3 & 4 & 0 & 0 & 1 \end{array}\right]$$

$$\xrightarrow[R_3\to R_3+R_1]{R_2\to R_2-3R_1} \left[\begin{array}{ccc|ccc} \boxed{1} & -1 & -2 & -1 & 0 & 0 \\ 0 & \boxed{2} & 7 & 3 & 1 & 0 \\ 0 & 2 & 2 & -1 & 0 & 1 \end{array}\right] \xrightarrow{R_2\to\frac{1}{2}R_2} \left[\begin{array}{ccc|ccc} 1 & -1 & -2 & -1 & 0 & 0 \\ 0 & 1 & 7/2 & 3/2 & 1/2 & 0 \\ 0 & 2 & 2 & -1 & 0 & 1 \end{array}\right]$$

$$\xrightarrow[R_3\to R_3-2R_2]{R_1\to R_1+R_2} \left[\begin{array}{ccc|ccc} 1 & 0 & 3/2 & \frac{1}{2} & \frac{1}{2} & 0 \\ 0 & 1 & 7/2 & 3/2 & 1/2 & 0 \\ 0 & 0 & -5 & -4 & -1 & 1 \end{array}\right] \xrightarrow{R_3\to-\frac{1}{5}R_3} \left[\begin{array}{ccc|ccc} 1 & 0 & 3/2 & 1/2 & 1/2 & 0 \\ 0 & 1 & 7/2 & 3/2 & 1/2 & 0 \\ 0 & 0 & 1 & 4/5 & 1/5 & -1/5 \end{array}\right]$$

$$\begin{array}{l} R_2\to R_2-\frac{7}{2}R_3 \\ R_1\to R_1-\frac{3}{2}R_3 \end{array} \quad \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -7/10 & 2/10 & 3/10 \\ 0 & 1 & 0 & -13/10 & -2/10 & 7/10 \\ 0 & 0 & 1 & 4/5 & 1/5 & -1/5 \end{array}\right]$$

$$\frac{1}{2}-\frac{3}{2}\times 4/5=\frac{1}{2}-\frac{12}{10}$$

$$\frac{1}{2}-\frac{3}{2}\times 1/5$$

Cayley-Hamilton Theorem

Sandeep Kumar Singh

Statement: Every square matrix satisfies its characteristic equation.

Illustration: $\begin{bmatrix}1 & 3\\ 2 & 7\end{bmatrix}$. $\longrightarrow$ $\lambda^2 - (\text{Trace of } A)\lambda + \det A = 0$

$\Rightarrow \lambda^2 - 8\lambda + 1 = 0$ —(1)

$\Rightarrow A^2 - 8A + I = 0$ —(2)

$$A^2 = \begin{bmatrix}1 & 3\\ 2 & 7\end{bmatrix}\begin{bmatrix}1 & 3\\ 2 & 7\end{bmatrix} = \begin{bmatrix}7 & 24\\ 16 & 55\end{bmatrix}$$

$$A^2 - 8A + I = \begin{bmatrix}7 & 24\\ 16 & 55\end{bmatrix} - \begin{bmatrix}8 & 24\\ 16 & 56\end{bmatrix} + \begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}$$

$$= \begin{bmatrix}0 & 0\\ 0 & 0\end{bmatrix}$$

$A^2 - 8A + I = 0$

$\Rightarrow A^{-1}(A^2 - 8A + I) = A^{-1}0 = 0$

$\Rightarrow A - 8I + A^{-1} = 0$

$\Rightarrow A^{-1} = 8I - A$

$A^{-1} =$

$\begin{bmatrix}8 & 0\\ 0 & 8\end{bmatrix} - \begin{bmatrix}1 & 3\\ 2 & 7\end{bmatrix}$

$= \begin{bmatrix}7 & -3\\ -2 & 1\end{bmatrix}$

Q) Verify Cayley-Hamilton theorem for the matrix $\begin{bmatrix} 1 & 2 & 0 \\ -1 & 1 & 2 \\ 1 & 2 & 1 \end{bmatrix}$. Find A^{-1}

Sol : $\lambda^3 - 3\lambda^2 + \lambda - 3 = 0$

$\Rightarrow A^3 - 3A^2 + A - 3I = 0$ —— (1)

$A^2 = A \cdot A = \begin{bmatrix} -1 & 4 & 4 \\ 0 & 3 & 4 \\ 0 & 6 & 5 \end{bmatrix}$, $A^3 = A^2 \cdot A = \begin{bmatrix} -1 & 10 & 12 \\ 1 & 11 & 10 \\ -1 & 16 & 17 \end{bmatrix}$

$A^3 - 3A^2 + A - 3I$

$= \begin{bmatrix} -1 & 10 & 12 \\ 1 & 11 & 10 \\ -1 & 16 & 17 \end{bmatrix} - \begin{bmatrix} -3 & 12 & 12 \\ 0 & 9 & 12 \\ 0 & 18 & 15 \end{bmatrix} + \begin{bmatrix} 1 & 2 & 0 \\ -1 & 1 & 2 \\ 1 & 2 & 1 \end{bmatrix} - \begin{bmatrix} 3 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{bmatrix}$

$= \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = 0$

$$A^3 - 3A^2 + A - 3I = 0$$

$$\Rightarrow A^{-1}(A^3 - 3A^2 + A - 3I) = A^{-1}0$$

$$\begin{bmatrix} 1 & 2 & 0 \\ -1 & 1 & 2 \\ 1 & 2 & 1 \end{bmatrix}$$

$$\Rightarrow A^2 - 3A + I - 3A^{-1} = 0$$

$$\Rightarrow 3A^{-1} = A^2 - 3A + I$$

$$\therefore A^{-1} = \frac{1}{3}(A^2 - 3A + I)$$

$$A^2 - 3A + I = \begin{bmatrix} -1 & 4 & 4 \\ 0 & 3 & 4 \\ 0 & 6 & 5 \end{bmatrix} - \begin{bmatrix} 3 & 6 & 0 \\ -3 & 3 & 6 \\ 3 & 6 & 3 \end{bmatrix} + \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} -3 & -2 & 4 \\ 3 & 1 & -2 \\ -3 & 0 & 3 \end{bmatrix}$$

$$A^{-1} = \frac{1}{3}\begin{bmatrix} -3 & -2 & 4 \\ 3 & 1 & -2 \\ -3 & 0 & 3 \end{bmatrix}$$

Q) If $\begin{bmatrix} 1 & 0 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$ then show that $A^n = A^{n-2} + A^2 - I, for\ n \geq 3$ and hence find A^{50}.

Sol:- $\lambda^3 - (\text{Trace ab } A)\lambda^2 + (M_{11} + M_{22} + M_{33})\lambda - \det A = 0$

Trace $A = 1 + 0 + 0 = 1$

$M_{11} = -1, \; M_{22} = 0, \; M_{33} = 0, \; M_{11} + M_{22} + M_{33} = -1$

$\det A = 1(0-1) = -1$

$\lambda^3 - \lambda^2 - \lambda + 1 = 0$ —— (1)

$A^3 - A^2 - A + I = 0$ —— (2)

$\Rightarrow \; A^3 - A^2 = A - I$

$$\cancel{A^3} - A^2 = \cancel{A} - I \quad \text{—— (3)}$$
$$\cancel{A^4} - \cancel{A^3} = \cancel{A^2} - \cancel{A}$$
$$\cancel{A^5} - \cancel{A^4} = \cancel{A^3} - \cancel{A^2}$$
$$\cancel{A^6} - \cancel{A^5} = \cancel{A^4} - \cancel{A^3}$$
$$\vdots$$
$$A^n - \cancel{A^{n-1}} = A^{n-2} - \cancel{A^{n-3}}$$

$$A^n - A^2 = A^{n-2} - I$$
$$\Rightarrow A^n = A^{n-2} + A^2 - I, \; n \geq 3$$

$$A^{n-2} = A^{n-4} + A^2 - I$$

$$A^n = A^{n-2} + A^2 - I$$
$$\Rightarrow A^n = (A^{n-4} + A^2 - I) + (A^2 - I)$$
$$\Rightarrow A^n = A^{n-4} + 2(A^2 - I)$$
$$\Rightarrow A^n = (A^{n-6} + A^2 - I) + 2(A^2 - I)$$
$$\Rightarrow A^n = A^{n-6} + 3(A^2 - I)$$
$$\Rightarrow A^n = (A^{n-8} + A^2 - I) + 3(A^2 - I)$$
$$\Rightarrow A^n = A^{n-8} + 4(A^2 - I)$$
$$A^n = A^{n-(n-2)} + \frac{(n-2)}{2}(A^2 - I)$$
$$\Rightarrow A^n = A^2 + \frac{(n-2)}{2}(A^2 - I)$$
$$A^{50} = A^2 + 24(A^2 - I)$$
$$A^{50} = 25A^2 - 24I$$

$$A^{50} = 25A^2 - 24I$$

$$\therefore \quad A^2 = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}$$

$$A^{50} = \begin{bmatrix} 25 & 0 & 0 \\ 25 & 25 & 0 \\ 25 & 0 & 25 \end{bmatrix} - \begin{bmatrix} 24 & 0 & 0 \\ 0 & 24 & 0 \\ 0 & 0 & 24 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 25 & 1 & 0 \\ 25 & 0 & 1 \end{bmatrix} \checkmark \qquad A^{500}$$