LINEAR DIFFERENTIAL EQUATION #L1

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Topics

1) Identify the Linear Differential equations and its types
2) Finding the interval where the differential equation is normal
3) Linear dependence and independence
4) Wronskian
5) Fundamental solution(Basis) of the Differential equation.
6) Abel's Formula

Definition

A linear ordinary differential equation of order n, is written as

$$a_0(x)\frac{d^n y}{dx^n} + a_1(x)\frac{d^{n-1}y}{dx^{n-1}} + \cdots + a_{n-1}(x)\frac{dy}{dx} + a_n(x)y = r(x).$$

$a_0(x), a_1(x), a_2(x), \cdots a_n(x), r(x)$ function of x only

or

$$a_0(x)y^n(x) + a_1(x)y^{n-1}(x) + \cdots + a_{n-1}(x)y'(x) + a_n(x)y = r(x).$$

Where y is the dependent variable and x is the independent variable and $a_0(x) \neq 0$.

$a_0(x) \neq 0$

If $r(x) = 0$,then it is called a Homogeneous equation, otherwise it is called a non-homogeneous equation.

$\frac{dy}{dx}, \frac{d^2y}{dx^2}, \frac{d^3y}{dx^3}, \cdots \frac{d^ny}{dx^n}$

$r(x) = 0$, $r(x) \neq 0$

Examples

$y'' + 4y' + 3y = x^2 e^{3x}$

$y'' + 2y' + y = \sin x$

$x^2 y'' + xy' + (x^2 - 4)y = 0$

$(1 - x^2)y'' - 2xy' + 20y = 0$

$y' = 3y/x \longrightarrow xy' = 3y$

$\boxed{xy' - 3y = 0}$

$x, -3$

$a_0(x) = x^2$

$a_1(x) = x$

$a_2(x) = x^2 - 4$

$r(x) = 0$

$y'' \equiv \frac{d^2y}{dx^2}$

$y''' \equiv \frac{d^3y}{dx^3}$

$y^{iv} = \frac{d^4y}{dx^4}$

$a_0(x) = 1 \quad a_1(x) = 4$

$a_2(x) = 3, \quad r(x) = x^2 e^{3x}$

$x^2 y'' + 2xy\,y' + x^2 y = \sin x$

$x^2 \cancel{y}\, y'' + 7xy' + 3y = 0$

$y'' + 7xy' + y = x^2 y$ X

.

Theorem: If the functions $a_0(x), a_1(x), \dots a_n(x)$ and $r(x)$ are continuous over I and $a_0(x) \neq 0$ on I, then there exists a unique solution to the initial value problem

$$a_0(x)y^n(x) + a_1(x)y^{n-1}(x) + \cdots + a_{n-1}(x)y'(x) + a_n(x)y = r(x).$$

$y(x_0) = c_1, y'(x_0) = c_2, \dots, y^{n-1}(x_0) = c_n$ where $x_0 \in I$,and

$c_1, c_2, \dots c_n$ *are n known constants.* $a_0(x) \neq 0$

This theorem does not give us a procedure to find the solution but guarantees that there exists a unique solution if the conditions stated in the theorem are satisfied.

If the conditions of the theorem are satisfied then the differential equation is said to be normal on I(these conditions are both necessary and sufficient for the differential equation to be normal).

A point $x_0 \in I, for$ which $a_0(x) \neq 0$, is called an ordinary point or a regular point of the differential equation.

Imp Ex) Find the intervals on which the following differential equations are normal.

(a) $(1-x^2)y'' - 2xy' + n(n+1)y = 0, n \text{ is an integer}.$

(b) $x^2y'' + xy' + (n^2 - x^2)y = 0, n \text{ is real}.$

$1 - x^2 = -x^2 + 1$

Sol:- a) $a_0(x) = 1 - x^2$, $a_1(x) = -2x$, $a_2(x) = n(n+1)$

$r(x) = 0$

Polynomial function and constant functions are everywhere continuous.

$(2, 73)$

$(0, 23)$

Not normal

$(-\infty, \infty)$

$a_0(x) \neq 0$, $\Rightarrow 1 - x^2 \neq 0$

$\Rightarrow x \neq -1, 1$

$-\infty \quad -1 \quad 1 \quad \infty$

$\mathbb{R} - \{-1, 1\}$

$(-\infty, -1) \cup (-1, 1) \cup (1, \infty)$

Ex) Find the intervals on which the following differential equations are normal.

(a) $(1-x^2)y'' - 2xy' + n(n+1)y = 0, n\ is\ an\ integer.$

(b) $x^2y'' + xy' + (n^2 - x^2)y = 0, n\ is\ real.$

$a_0(x) = x^2, \quad a_1(x) = x, \quad a_2(x) = n^2 - x^2, \quad r(x) = 0$

$(-\infty,\infty) \qquad (-\infty,\infty) \qquad (-\infty,\infty) \qquad (-\infty,\infty)$

$a_0(x) \neq 0 \Rightarrow x^2 \neq 0 \Rightarrow x \neq 0$

$(-\infty, 0) \cup (0, \infty)$

Not normal

a) $(2, 23)$ b) $(-\infty, -1)$

c) $(-\infty, 0]$ d) $(5, 23)$

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Ex) Find the intervals on which the following differential equations are normal.

(c) $\sqrt{x}y'' + 6xy' + 15y = \ln(x^4 - 256)$.

(d) $(1 + x^2)y'' + 2xy' + y = 0$

$\log x,\ x > 0$

(c) $a_0(x) = \sqrt{x}$, $a_1(x) = 6x$, $a_2(x) = 15$, $r(x) = \ln(x^4 - 256)$

$x > 0$ $\quad$ $(-\infty, \infty)$ $\quad$ $(-\infty, \infty)$

$x^4 - 256 > 0 \rightarrow$ Inequality

$\Rightarrow (x^2 + 16)(x^2 - 16) > 0 \Rightarrow x^2 - 16 > 0$

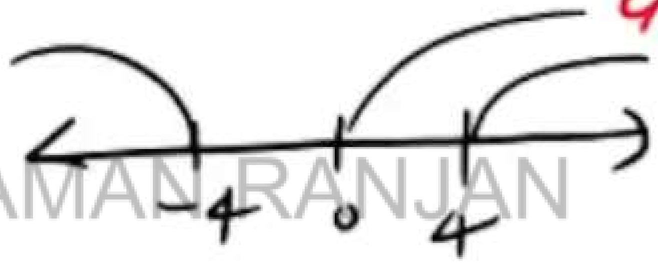
$\Rightarrow (x - 4)(x + 4) > 0$

$+ \quad - \quad +$

$-4 \quad 4$

$(-\infty, -4), (4, \infty)$

$a_0(x) \neq 0 \Rightarrow \sqrt{x} \neq 0 \Rightarrow x \neq 0$



$(4, \infty)$

Ex) Find the intervals on which the following differential equations are normal.

(c) $\sqrt{x}y'' + 6xy' + 15y = \ln(x^4 - 256)$.

(d) $(1 + x^2)y'' + 2xy' + y = 0$

$a_0(x) = \underset{(-\infty,\infty)}{\underline{1+x^2}}$, $a_1(x) = \underset{(-\infty,\infty)}{\underline{2x}}$, $a_2(x) = \underset{(-\infty,\infty)}{\underline{1}}$, $r(x) = \underset{(-\infty,\infty)}{\underline{0}}$

$a_0(x) \neq 0 \Rightarrow 1 + x^2 \neq 0$

For no value of x $1+x^2 = 0$

$(-\infty, \infty)$

Ex) Find the intervals on which the following differential equations are normal.

(e) $x(1-x)y'' - 3xy' - y = 0.$ ✓

$a_0(x) = x(1-x), \quad a_1(x) = -3x, \quad a_2(x) = -1, \quad r(x) = 0$

$(-\infty, \infty)$

$a_0(x) \neq 0$

$\Rightarrow x(1-x) \neq 0 \Rightarrow x \neq 0, \; x \neq 1$

$(-\infty, 0) \cup (0,1) \cup (1, \infty)$

Linear Combination of solutions

$c_1 f_1(x) + c_2 f_2(x) + c_3 f_3(x)$

If $y_1(x), y_2(x), \ldots y_m(x)$ are m solutions of the linear homogeneous equation $a_0(x)y^n(x) + a_1(x)y^{n-1}(x) + \cdots + a_{n-1}(x)y'(x) + a_n(x)y = 0$ on I, then a linear combination of the solutions $c_1y_1(x) + c_2y_2(x) + \cdots. + c_ny_n(x)$, where $c_1, c_2 \ldots, c_m$ are constants is also a solution of the above equation on I.

1) Show that e^{-x}, e^x and their linear combination $c_1e^{-x} + c_2e^x$ are solutions of the homogeneous equation $y'' - y = 0$. , $\frac{d^2y}{dx^2} - y = 0$

$y = e^{-x}$, $y' = -e^{-x}$, $y'' = e^{-x}$

$y'' - y = e^{-x} - e^{-x} = 0$

$y = e^x$, $y' = e^x$, $y'' = e^x$, $e^x - e^x = 0$

$y = c_1e^{-x} + c_2e^x$, $y' = -c_1e^{-x} + c_2e^x$

$y'' = c_1e^{-x} + c_2e^x$

$y'' - y = 0$

$y_3 = c_1 e^{-x}\cos 2x + c_2 e^{-x}\sin 2x$

2) Show that $e^{-x}\cos 2x$, $e^{-x}\sin 2x$ and their linear combination are solutions of the homogeneous equation $y'' + 2y' + 5y = 0.$

$y_1 = e^{-x}\cos 2x$,

$y_2 = e^{-x}\sin 2x$

$y_1' = e^{-x}(-2\sin 2x) + \cos 2x(-e^{-x})$

$y_1'' = -2[e^{-x}2\cos 2x + \sin 2x(-e^{-x})]$

$- [e^{-x}(-2\sin 2x) + \cos 2x(-e^{-x})]$

$= -4e^{-x}\cos 2x + 2e^{-x}\sin 2x + 2e^{-x}\sin 2x$

$+ e^{-x}\cos 2x = -3e^{-x}\cos 2x + 4e^{-x}\sin 2x$

$-3\cancel{e^{-x}}\cos 2x + 4\cancel{e^{-x}}\sin 2x - 4\cancel{e^{-x}}\sin 2x - 2\cancel{e^{-x}}\cos 2x$

$+ 5\cancel{e^{-x}}\cos 2x = 0$

Linear Independence and Dependence

Let $f_1(x), f_2(x), \ldots f_n(x)$ be n functions. These functions are said to be linearly independent on some interval I(where they are defined), if the equation

$$c_1 f_1(x) + c_2 f_2(x) + \cdots + c_n f_n(x) = 0 \text{ implies } c_1 = 0, c_2 = 0, \ldots c_n = 0.$$

L.I

L.D

* Wronskian

Let $f_1(x), f_2(x), \dots, f_n(x)$ be n functions. The Wronskian of these functions is denoted by $W(f_1, f_2, \dots, f_n)$ and is defined by

$$W(f_1, f_2, \dots f_n) = w(x) = \begin{vmatrix} f_1 & f_2 & f_3 & \cdots & f_n \\ f_1' & f_2' & f_3' & & f_n' \\ f_1'' & f_2'' & f_3'' & \cdots & f_n'' \\ \vdots & & & & \vdots \\ f_1^{(n-1)} & f_2^{(n-1)} & f_3^{(n-1)} & \cdots & f_n^{(n-1)} \end{vmatrix}_{n \times n}$$

The Wronskian of the n functions exists if all the functions $f_1, f_2, \dots, f_n$ are differentiable $n-1$ times on interval I. If any one or more functions are not differentiable then the Wronskian does not exist.

Important

If Wronskian = $W(x) = 0$ then the functions are linearly dependent.

If Wronskian = $W(x) \neq 0$ then the functions are linearly independent.

Ex)Show that the functions x, x^2, x^3 are linearly independent on any interval I, not containing zero

Sol:

$$W(x) = \begin{vmatrix} x & x^2 & x^3 \\ 1 & 2x & 3x^2 \\ 0 & 2 & 6x \end{vmatrix}$$

$$= x(12x^2 - 6x^2) - 1(6x^3 - 2x^3)$$

$$= 6x^3 - 4x^3 = 2x^3 \neq 0$$ on an interval not containing zero

$2x^3 = 0$ at $x = 0$

L.I

Ex) Show that the functions 1, $\sin x, \cos x$ are linearly independent.

Sol:

$$W(x) = \begin{vmatrix} 1 & \sin x & \cos x \\ 0 & \cos x & -\sin x \\ 0 & -\sin x & -\cos x \end{vmatrix}$$

$$= 1(-\cos^2 x - \sin^2 x) = -(\cos^2 x + \sin^2 x)$$

$$= -1 \neq 0$$

L.I

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Ex) Examine whether the following functions are linearly independent for $x \in (0, \infty)$

$2x, 6x+3, 3x+2$

Sol:

$$W(x) = \begin{vmatrix} 2x & 6x+3 & 3x+2 \\ 2 & 6 & 3 \\ 0 & 0 & 0 \end{vmatrix}$$

$= \underline{0}$

not linearly independent

$\underline{\text{L.D}}$ Ans

b) $x^2 - x, 3x^2 + x + 1, 9x^2 - x + 2$

Sol:

$$W(x) = \begin{vmatrix} x^2 - x & 3x^2 + x + 1 & 9x^2 - x + 2 \\ 2x - 1 & 6x + 1 & 18x - 1 \\ 2 & 6 & 18 \end{vmatrix}$$

$$= (x^2 - x)[\cancel{108x} + 18 - \cancel{108x} + 6] - (3x^2 + x + 1)[\cancel{36x} - 18 - \cancel{36x} + 2]$$

$$+ (9x^2 - x + 2)[\cancel{12x} - 6 - \cancel{12x} - 2]$$

$$= (x^2 - x) \cdot 24 + (3x^2 + x + 1)16 + (9x^2 - x + 2)(-8)$$

$$= \cancel{24x^2} - \cancel{24x} + \cancel{48x^2} + \cancel{16x} + \cancel{16} - \cancel{72x^2} + \cancel{8x} - \cancel{16}$$

$$= 0$$

$W(x) = 0$, Linearly dependent

Fundamental solution (Basis) of the Differential equation

Q) Show that the set of functions $\left\{x, \frac{1}{x}\right\}$ forms a basis of the equation $x^2y'' + xy' - y = 0$.

$y_1 = x, \quad y_1' = 1, \quad y_1'' = 0, \qquad x^2 \cdot 0 + x \cdot 1 - x = 0$

$y_2 = \frac{1}{x}, \quad y_2' = \frac{-1}{x^2}, \quad y_2'' = -\left(\frac{-2}{x^3}\right) = \frac{2}{x^3}$

$x^2 \cdot \frac{2}{x^3} + x\left(-\frac{1}{x^2}\right) - \frac{1}{x} = \frac{2}{x} - \frac{1}{x} - \frac{1}{x} = 0$

$W(x) = \begin{vmatrix} x & \frac{1}{x} \\ 1 & -\frac{1}{x^2} \end{vmatrix} = -\frac{1}{x} - \frac{1}{x} = -\frac{2}{x} \neq 0$

$\left\{x, \frac{1}{x}\right\}$ are linearly independent

$\left\{x, \frac{1}{x}\right\}$ forms a basis.

* Q) Show that the set of functions $\{e^{x}, e^{4x}\}$ forms a basis of the equation below and hence solve the IVP $y'' - 5y' + 4y = 0.\ y(0) = 2, y'(0) = 1$

Sol: $y_1 = e^x, \quad y_1' = e^x, \quad y_1'' = e^x$

$e^x - 5e^x + 4e^x = 0$ So $y_1 = e^x$ is a solution.

$y_2 = e^{4x}, \quad y_2' = 4e^{4x}, \quad y_2'' = 16e^{4x}$

$y_2'' - 5y_2' + 4y_2 = 16e^{4x} - 5 \times 4e^{4x} + 4e^{4x}$
$= 0$

$y_2 = e^{4x}$ is a solution

$$W = \begin{vmatrix} e^x & e^{4x} \\ e^x & 4e^{4x} \end{vmatrix} = 4e^{5x} - e^{5x} = 3e^{5x} \neq 0$$

$\{e^x, e^{4x}\}$ forms a Basis of the given diff eqn

The $y = c_1 e^x + c_2 e^{4x}$, $y(0) = 2$, $y'(0) = 1$

General sol

$y(0) = 2 \Rightarrow c_1 e^0 + c_2 e^0 = 2 \Rightarrow c_1 + c_2 = 2$ —(1)

$y' = c_1 e^x + 4c_2 e^{4x}$

$y'(0) = 1 \Rightarrow c_1 e^0 + 4c_2 e^0 = 1 \Rightarrow c_1 + 4c_2 = 1$ —(2)

$$\begin{array}{r} c_1 + c_2 = 2 \\ c_1 + 4c_2 = 1 \\ \hline -3c_2 = 1 \end{array} \Rightarrow c_2 = -\frac{1}{3}$$

$c_1 = 2 - c_2 = 2 + \frac{1}{3} = \frac{7}{3}$

$y = \frac{7}{3} e^x - \frac{1}{3} e^{4x}$

Particular sol

Abel's formula

$\{y_1, y_2\}$

Let $a_0(x)y'' + a_1(x)y' + a_2(x)y = 0$ be a second order differential equation. Let $a_0(x), a_1(x), a_2(x)$ be continuous on I and $a_0(x) \neq 0$.

The Wronskian is given by

$$W(x) = ce^{-\int\left[a_1(x)/a_0(x)\right]dx}$$

$$W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$$

$$W = \begin{vmatrix} x & \frac{1}{x} \\ 1 & -1/x^2 \end{vmatrix} = -\frac{1}{x} - \frac{1}{x} = -2/x$$

$x^2 y'' + x y' - y = 0, \quad \{x, \frac{1}{x}\}$

$$W(x) = ce^{-\int \frac{x}{x^2} dx} = ce^{-\int \frac{1}{x} dx} = ce^{-\log x} = c' \frac{1}{x} = \frac{c}{x}$$

$$y'' - 5y' + 4y = 0 \quad , \quad \{e^{x}, e^{4x}\}$$

$$W = \begin{vmatrix} e^{x} & e^{4x} \\ e^{x} & 4e^{4x} \end{vmatrix} = 4e^{5x} - e^{5x} = 3e^{5x}$$

$$W = c\, e^{-\int \frac{a_1(x)}{a_0(x)} dx}$$

$$= c\, e^{-\int \frac{-5}{1} dx} = c\, e^{\int 5\, dx}$$

$$= c\, e^{5x}$$

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LINEAR DIFFERENTIAL EQUATION #L2

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Solving Homogeneous Linear Differential Equations with constant coefficients

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Method of Solution of Linear Differential Equation

Process

Step:1) Write the Auxiliary Equation

2) Find the roots of the Auxiliary Equation

3) There are three possibilities

(a) Roots are real and distinct

(b) Roots are real and equal

(c) Roots are Complex

Case:1) When roots are real and Distinct

$m^2-4m-12=0$

1.a) $y'' - y' - 6y = 0$ 1.b) $y'' - 4y' - 12y = 0$

Sol:

$\frac{d^2y}{dx^2} - \frac{dy}{dx} - 6y = 0$, $\frac{d}{dx} \equiv D$, $\frac{d^2}{dx^2} \equiv D^2$

$D^2y - Dy - 6y = 0$ $\frac{d^3}{dx^3} \equiv D^3$, ---

A.E

$(D^2 - D - 6)y = 0$

$m^2, m,$

The Auxiliary eqn is given by

$\frac{dy}{dx} \to m$

$m^2 - m - 6 = 0$

$\frac{d^2y}{dx^2} \to m^2$

$\Rightarrow m^2 - 3m + 2m - 6 = 0$

$\Rightarrow m(m-3) + 2(m-3) = 0$

$\frac{d^3y}{dx^3} \to m^3$

$\Rightarrow (m-3)(m+2) = 0 \Rightarrow m = 3, -2$

Case:1) When roots are real and Distinct $m^2 - 4m - 12 = 0$

1.a) $y'' - y' - 6y = 0$ 1.b) $y'' - 4y' - 12y = 0$

Sol:

$\frac{d^2y}{dx^2} - \frac{dy}{dx} - 6y = 0$, $\frac{d}{dx} \equiv D$, $\frac{d^2}{dx^2} \equiv D^2$

$D^2y - Dy - 6y = 0$ $\frac{d^3}{dx^3} \equiv D^3$, - - -

A.E

$(D^2 - D - 6)y = 0$

$m^2, m,$

The Auxiliary eqn is given by

$\frac{dy}{dx} \to m$

$m^2 - m - 6 = 0$

$\frac{d^2y}{dx^2} \to m^2$

$\Rightarrow m^2 - 3m + 2m - 6 = 0$

$\Rightarrow m(m-3) + 2(m-3) = 0$

$\frac{d^3y}{dx^3} \to m^3$

$\Rightarrow (m-3)(m+2) = 0 \Rightarrow m = 3, -2$

$y = c_1 e^{3x} + c_2 e^{-2x}$

Continued

$$m_1, m_2, m_3 \cdots m_n$$

$$y = c_1 e^{m_1 x} + c_2 e^{m_2 x} + \cdots + c_n e^{m_n x}$$

Case:1) When roots are real and Distinct

1.a) $y'' - y' - 6y = 0$ 1.b) $y'' - 4y' - 12y = 0$

Sol:

1b) The A.E is $m^2 - 4m - 12 = 0$

$\Rightarrow m^2 - 6m + 2m - 12 = 0$

$\Rightarrow m(m-6) + 2(m-6) = 0$

$\Rightarrow (m-6)(m+2) = 0 \Rightarrow m = 6, -2$

$y = c_1 e^{6x} + c_2 e^{-2x}$

2) $4y'' - 8y' + 3y = 0, y(0) = 1, y'(0) = 3$ IVP

Sol: The A.E $4m^2 - 8m + 3 = 0 \Rightarrow 4m^2 - 6m - 2m + 3 = 0$

$\Rightarrow 2m(2m-3) - 1(2m-3) = 0 \Rightarrow (2m-3)(2m-1) = 0$

$m = \frac{3}{2}, \frac{1}{2}$

$y = c_1 e^{3/2 x} + c_2 e^{\frac{1}{2}x}$ —— (1)

$y = \frac{5}{2} e^{3/2 x} - \frac{3}{2} e^{\frac{1}{2}x}$ Particular sol

$y(0) = 1 \Rightarrow c_1 e^0 + c_2 e^0 = 1 \Rightarrow c_1 + c_2 = 1$ —(2)

$y' = \frac{dy}{dx} = c_1 \frac{3}{2} e^{3/2 x} + c_2 \cdot \frac{1}{2} e^{1/2 x}, \quad y'(0) = 3$

$\Rightarrow \frac{3c_1}{2} + \frac{c_2}{2} = 3 \Rightarrow 3c_1 + c_2 = 6$ —(3)

$y = \frac{5}{2} e^{3/2 x} - \frac{3}{2} e^{1/2 x}$

$c_1 + c_2 = 1$
$3c_1 + c_2 = 6$

$-2c_1 = -5$
$c_1 = 5/2$

$c_2 = 1 - c_1$
$= 1 - 5/2 = -3/2$

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3) $y''' - 2y'' - 5y' + 6y = 0$

Sol: The A·E is $m^3 - 2m^2 - 5m + 6 = 0$ —— (1)

$m = 1$, $1 - 2 - 5 + 6 = 7 - 7 = 0$, $m = 1$ is a root of (1)

$(m-1)$ is a factor

$m^3 - 2m^2 - 5m + 6 = 0$

$\Rightarrow m^2(m-1) - m(m-1) - 6(m-1) = 0$

$\Rightarrow (m-1)(m^2 - m - 6) = 0$

$\Rightarrow (m-1)[m^2 - 3m + 2m - 6] = 0$

$\Rightarrow (m-1)[m(m-3) + 2(m-3)] = 0$

$\Rightarrow (m-1)(m-3)(m+2) = 0$

$m = 1, -2, 3$

$$
\begin{array}{r|l}
m-1 & m^3 - 2m^2 - 5m + 6 \;/\; m^2 - m - 6 \\
 & m^3 - m^2 \\
\hline
 & -m^2 - 5m + 6 \\
 & -m^2 + m \\
\hline
 & -6m + 6 \\
 & -6m + 6 \\
\hline
 & 0
\end{array}
$$

$y = c_1 e^{x} + c_2 e^{-2x} + c_3 e^{3x}$

4) $y^{iv} - 5y'' + 4y = 0$

Sol: The A.E is $m^4 - 5m^2 + 4 = 0$

$\Rightarrow m^4 - 4m^2 - m^2 + 4 = 0$

$\Rightarrow m^2(m^2 - 4) - 1(m^2 - 4) = 0$

$\Rightarrow (m^2 - 4)(m^2 - 1) = 0$

$\Rightarrow (m+2)(m-2)(m+1)(m-1) = 0$

$\therefore m = -1, 1, -2, 2$

$y = c_1 e^{-x} + c_2 e^{x} + c_3 e^{-2x} + c_4 e^{2x}$

Case:2) When roots are real and equal

$$\underline{m_1} = \underline{m_2} = \underline{m}$$

$$y = c_1 e^{mx} + c_2 e^{mx} = (c_1 + c_2) e^{mx}$$

$$y = c e^{mx}$$

$$y = (c_1 + c_2 x) e^{mx}$$

$$m_1 = m_2 = m_3 = m$$

$$y = (c_1 + c_2 x + c_2 x^2) e^{mx}$$

$$m_1 = m_2 = m_3 = m_4 = m$$

$$y = (c_1 + c_2 x + c_3 x^2 + c_4 x^3) e^{mx}$$

1) $4y'' + 4y' + y = 0$ ✓

Sol: The A.E is $4m^2 + 4m + 1 = 0$

$\Rightarrow (2m+1)^2 = 0$

$\Rightarrow m = -\frac{1}{2}, -\frac{1}{2}$

$4m^2 + 2m + 2m + 1 = 0$

$\Rightarrow 2m(2m+1) + 1(2m+1) = 0$

$\Rightarrow (2m+1)(2m+1) = 0$

$m = -\frac{1}{2}, -\frac{1}{2}$

$e^{-\frac{1}{2}x}$

$xe^{-\frac{1}{2}x}$

$\left\{ e^{-\frac{1}{2}x}, xe^{-\frac{1}{2}x} \right\}$ $y = (c_1 + c_2 x) e^{-\frac{1}{2}x}$

$y = c_1 e^{-\frac{1}{2}x} + c_2 xe^{-\frac{1}{2}x}$

2) $y'' + 6y' + 9y = 0, y(0) = 2, y'(0) = 3$

Sol:

The A.E is $m^2 + 6m + 9 = 0$

$\Rightarrow (m+3)^2 = 0 \Rightarrow m = -3, -3$

$y = (c_1 + c_2 x)e^{-3x}$ —— (1)

Particular sol $\rightarrow$ Comment

3) $y''' - 3y' - 2y = 0$

Sol: The A.E is $m^3 - 3m - 2 = 0$

$m = -1$, $-1 - 3(-1) - 2 = -1 + 3 - 2 = 0$

$(m+1)$ is a factor

$m^3 - 3m - 2 = 0$

$\Rightarrow m^2(m+1) - m(m+1) - 2(m+1) = 0$

$\Rightarrow (m+1)(m^2 - m - 2) = 0$

$\Rightarrow (m+1)[m^2 - 2m + m - 2] = 0$

$\Rightarrow (m+1)[m(m-2) + 1(m-2)] = 0$

$\Rightarrow (m+1)(m+1)(m-2) = 0 \quad \Rightarrow m = -1, -1, 2$

$y = (c_1 + c_2 x)e^{-x} + c_3 e^{2x}$ ✓

4) $8y''' - 12y'' + 6y' - y = 0$

$(a-b)^3 = a^3 - 3a^2b + 3ab^2 - b^3$

Sol: The A.E is $8m^3 - 12m^2 + 6m - 1 = 0$

$\Rightarrow (2m)^3 - 3\cdot(2m)^2\cdot 1 + 3\cdot(2m)\cdot 1^2 - 1^3 = 0$

$\Rightarrow (2m-1)^3 = 0 \Rightarrow m = \frac{1}{2}, \frac{1}{2}, \frac{1}{2}$

$(2m-1)(2m-1)(2m-1)$

$$y = (c_1 + c_2 x + c_3 x^2) e^{\frac{1}{2}x}$$

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Case:3) When roots are Complex

$\alpha + i\beta, \alpha - i\beta \quad , \underline{\alpha \pm i\beta}$

$$y = e^{\alpha x}(c_1 \cos \beta x + c_2 \sin \beta x)$$

$\underline{2} \pm 3i$

$$y = e^{2x}(c_1 \cos 3x + c_2 \sin 3x)$$

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1) $y'' + 2y' + 2y = 0$

Sol: The A·E is $m^2 + 2m + 2 = 0$

$$\frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$\Rightarrow m = \frac{-2 \pm \sqrt{2^2 - 4\cdot 1\cdot 2}}{2\cdot 1}$$

$$= \frac{-2 \pm \sqrt{-4}}{2} = \frac{-2 \pm 2i}{2}$$

$$= -1 \pm i \quad , \quad \alpha = -1, \beta = 1$$

$$\alpha \pm i\beta$$

$$y = e^{-x}(c_1 \cos x + c_2 \sin x)$$

$$y = c_1 e^{-x} \cos x + c_2 e^{-x} \sin x$$

2) $y'' + 4y' + 13y = 0$

Sol: $\therefore$ The A.E is $m^2 + 4m + 13 = 0$

$$\Rightarrow m = \frac{-4 \pm \sqrt{16 - 52}}{2}$$

$$= \frac{-4 \pm \sqrt{-36}}{2} = \frac{-4 \pm 6i}{2}$$

$$m = -2 \pm 3i$$

$$y = e^{-2x}(c_1 \cos 3x + c_2 \sin 3x)$$

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3) $y^{iv} + 5y'' + 4y = 0$

Sol: The A.E is $m^4 + 5m^2 + 4 = 0$

$\Rightarrow m^4 + 4m^2 + m^2 + 4 = 0$

$\Rightarrow m^2(m^2+4) + 1(m^2+4) = 0$

$\Rightarrow (m^2+4)(m^2+1) = 0$ $\qquad \pm 2i = 0 \pm 2i$

$m = \pm 2i \quad m = \pm i \qquad \pm i = 0 \pm i$

$y = e^{0x}(c_1 \cos 2x + c_2 \sin 2x) + e^{0x}(c_3 \cos x + c_4 \sin x)$

$y = c_1 \cos 2x + c_2 \sin 2x + c_3 \cos x + c_4 \sin x$

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4) $y^{iv} + 32y'' + 256y = 0$

Sol: The A.E is $m^4 + 32m^2 + 256 = 0$

$\Rightarrow (m^2)^2 + 2 \cdot m^2 \cdot 16 + (16)^2 = 0$

$\Rightarrow (m^2+16)^2 = 0$

$\Rightarrow (m^2+16)(m^2+16) = 0$

$m = \pm 4i, \pm 4i \qquad \alpha = 0$

$\alpha \pm i\beta, \alpha \pm i\beta$

$y = e^{\alpha x}[(c_1 + c_2 x)\cos\beta x + (c_3 + c_4 x)\sin\beta x]$

$\alpha \pm i\beta, \alpha \pm i\beta, \alpha \pm i\beta$

$y = e^{\alpha x}[(c_1 + c_2 x + c_3 x^2)\cos\beta x + (c_4 + c_5 x + c_6 x^2)\sin\beta x]$

$y = e^{0x}[(c_1 + c_2 x)\cos 4x + (c_3 + c_4 x)\sin 4x]$

$y = (c_1 + c_2 x)\cos 4x + (c_3 + c_4 x)\sin 4x$

Find a differential equation of the form $ay'' + by' + cy = 0$, for which following functions are solutions (a) e^{3x}, e^{-2x} (b) $1, e^{-2x}$ (c) e^{2x}, xe^{2x}.

a) e^{3x}, e^{-2x} , $m = 3, -2$

The A.E is $(m-3)(m+2) = 0 \Rightarrow m^2 - 3m + 2m - 6 = 0$

$\Rightarrow m^2 - m - 6 = 0$

$y'' - y' - 6y = 0$

b) $1, e^{-2x}$, e^{0x}, e^{-2x} $m = 0, -2$

$(m-0)(m+2) = 0 \Rightarrow m(m+2) = 0$

$\Rightarrow m^2 + 2m = 0$

$y'' + 2y' = 0$

Find a differential equation of the form $ay'' + by' + cy = 0$, for which following functions are solutions (a) e^{3x}, e^{-2x} (b) $1, e^{-2x}$ (c) e^{2x}, xe^{2x}.

(c) $e^{2x}, \; xe^{2x}$

$m = 2, 2$

m, m

$y = (c_1 + c_2 x) e^{mx}$

$= c_1 e^{mx} + c_2 x e^{mx}$

The A.E is $(m-2)(m-2) = 0$

$\Rightarrow m^2 - 4m + 4 = 0$

The Diff eqn $y'' - 4y' + 4y = 0$

d) $e^{2x}, xe^{2x}, x^2 e^{2x}$, $2, 2, 2$

$y = (c_1 + c_2 x + c_3 x^2) e^{2x}$

Q) Find a homogeneous linear differential equation with real constant coefficients of lowest order which has the following particular solution.

(a) $5 + e^x + 2e^{3x}$ (b) $e^{-x} + \cos 5x + 3\sin 5x$ (c) $x^2e^{2x} + 2e^{-2x}$

a) $5 + e^x + 2e^{3x}$

$c_1e^{m_1x} + c_2e^{m_2x} + c_3e^{m_3x} + c_4e^{m_4x}$

$m = 0, 1, 3$

$c_1 = 5, \quad c_2 = 1, \quad c_3 = 2$

The A.E $(m-0)(m-1)(m-3) = 0$

$m_1 = 0 \quad m_2 = 1 \quad m_3 = 3$

$\Rightarrow m(m^2 - m - 3m + 3) = 0$

$5 + e^x + 2e^{3x}$. $c_4 = 0$

$\Rightarrow m(m^2 - 4m + 3) = 0$

$\Rightarrow m^3 - 4m^2 + 3m = 0$

$y''' - 4y'' + 3y' = 0$

Q) Find a homogeneous linear differential equation with real constant coefficients of lowest order which has the following particular solution.

(a) $5 + e^{x} + 2e^{3x}$ ✓(b) $e^{-x} + \cos 5x + 3\sin 5x$ (c) $x^2 e^{2x} + 2e^{-2x}$

b) $e^{-x} + \cos 5x + 3\sin 5x$

-1 $\quad 0 \pm 5i$

$\alpha + i\beta$

$e^{\alpha x}(c_1 \cos\beta x + c_2 \sin\beta x)$

$c_1 = 1 \quad c_2 = 3$

$\beta = 5$

$\alpha = 0$

$-1, \ \pm 5i$

The A.E $\quad (m+1)(m^2+25) = 0$

$\Rightarrow m^3 + 25m + m^2 + 25 = 0 \quad (c_1 \cos 5x + c_2 \sin 5x)$

$y''' + y'' + 25y' + 25y = 0 \qquad 1 \qquad 3$

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Q) Find a homogeneous linear differential equation with real constant coefficients of lowest order which has the following particular solution.

(a) $5 + e^{x} + 2e^{3x}$ (b) $e^{-x} + \cos 5x + 3\sin 5x$ (c) $x^2 e^{2x} + 2e^{-2x}$

(c) $x^2 e^{2x} + 2e^{-2x}$

$c_1 e^{mx}$

m, m

$(c_1 + c_2 x)e^{mx}$

$2, 2, 2, -2$

m, m, m

$(c_1 + c_2 x + c_2 x^2)e^{mx}$

$(m+2)(m-2)^3 = 0$

$\downarrow$

Diff eqn $\rightarrow$??

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LINEAR DIFFERENTIAL EQUATION #3

Sandeep Kumar Singh

solution of non-homogeneous linear differential equations with constant coefficients

Operator method

Process

The solution is written as

$$y = Complementary\ function + Particular\ Integral$$

The process to find the Complementary function(C.F) remains same as to find the solution of corresponding homogeneous LDE.

Here we will discuss the process of finding the particular Integral.

Type 1.a

$(D^2-2D-3)y=0$

$\frac{1}{f(D)}3e^{2x}$

$$\frac{1}{f(D)}e^{ax} = \frac{1}{f(a)}e^{ax}, provided\ f(a) \neq 0.$$

$D \equiv \frac{d}{dx}$, $D^2 \equiv \frac{d^2}{dx^2}$

Ex-1) Find the general solution of the differential equation

$$y'' - 2y' - 3y = 3e^{2x}. \Rightarrow D^2y - 2Dy - 3y = 3e^{2x}$$

$\Rightarrow (D^2-2D-3)y = 3e^{2x}$

$f(D) = D^2-2D-3$ $\quad \frac{1}{f(D)}$

$f(D)\,y = 3e^{2x}$

The sol is $y = C.F + P.I$

$D \to a$

The A.E is $m^2-2m-3=0 \Rightarrow m^2-3m+m-3=0$

$\Rightarrow m(m-3)+1(m-3)=0 \Rightarrow (m-3)(m+1)=0 \Rightarrow m=3,-1$

$C.F = c_1e^{3x}+c_2e^{-x}$

$P.I = \frac{1}{D^2-2D-3}3e^{2x} = 3\frac{1}{\cancel{2^2}-\cancel{2\cdot 2}-3}e^{2x} = -e^{2x}$

$y = c_1e^{3x}+c_2e^{-x} - e^{2x}$

Ex-2) Find the general solution of the differential equation

$$y''' - 2y'' - 5y' + 6y = 4e^{-x} - e^{2x}$$

Sol: $y = C.F + P.I$ $\quad (D^3 - 2D^2 - 5D + 6)y = 4e^{-x} - e^{2x}$

The A.E is $m^3 - 2m^2 - 5m + 6 = 0 \Rightarrow m = 1, -2, 3$

$C.F = c_1 e^{x} + c_2 e^{-2x} + c_3 e^{3x}$ $\qquad \boxed{a = -1}$ $D \to a$

$$P.I = \frac{1}{D^3 - 2D^2 - 5D + 6}\left(4e^{-x} - e^{2x}\right) \qquad a = 2,\ D \to 2$$

$$= 4\frac{1}{(-1)^3 - 2(-1)^2 - 5(-1) + 6}e^{-x} - \frac{1}{\cancel{2^3} - \cancel{2 \cdot 2^2} - 5 \cdot 2 + 6}e^{2x}$$

$$= 4\frac{1}{-1 - 2 + 5 + 6}e^{-x} - \frac{1}{-4}e^{2x} = \frac{1}{2}e^{-x} + \frac{1}{4}e^{2x}$$

$$y = c_1 e^{x} + c_2 e^{-2x} + c_3 e^{3x} + \frac{1}{2}e^{-x} + \frac{1}{4}e^{2x}$$

$f(D)y = R.H.S$ $\quad$ $f(D)y=0$

C.F.

Type 1.b): When $f(a) = 0$

Ex- $y'' + y' - 6y = 5e^{-3x}$ $\Rightarrow (D^2+D-6)y = 5e^{-3x}$

$y = C.F + P.I$

The A.E is $m^2+m-6=0 \Rightarrow m=-3,2$

$C.F = c_1e^{-3x} + c_2e^{2x}$

$\frac{d}{dD}(D^2+D-6) = 2D+1$

$a=-3$

$D \to -3$, $(-3)^2-3-6$

$9-9=0$

$P.I = \frac{1}{D^2+D-6}5e^{-3x} = 5\frac{1}{D^2+D-6}e^{-3x}$ w.r.t D

$= 5x\frac{1}{2D+1}e^{-3x} = 5x\frac{1}{-6+1}e^{-3x} = -xe^{-3x}$

$y = c_1e^{-3x} + c_2e^{2x} - xe^{-3x}$

Ex- $4y'' - 4y' + y = e^{x/2} \Rightarrow \underbrace{(4D^2 - 4D+1)}y = e^{x/2}$

Sol: $y = C.F + P.I$

The A.E is $4m^2 - 4m + 1 = 0 \Rightarrow (2m-1)^2 = 0 \Rightarrow m = \frac{1}{2}, \frac{1}{2}$

$C.F = (c_1 + c_2 x) e^{\frac{1}{2}x}$ $\qquad D \to 1/2$

$$P.I = \frac{1}{\underbrace{4D^2 - 4D + 1}} e^{x/2} = x \frac{1}{\underbrace{8D-4}} e^{x/2}$$

$$= x.x \frac{1}{8} e^{x/2} = \frac{x^2}{8} e^{x/2}$$

$$y = (c_1 + c_2 x) e^{\frac{1}{2}x} + \frac{x^2}{8} e^{x/2}$$

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Find the particular integral for the differential equation

$9y''' + 3y'' - 5y' + y = 42e^{x} + 64e^{\frac{x}{3}}$ $\Rightarrow (9D^3+3D^2-5D+1)y = 42e^x + 64e^{x/3}$

$y = \underbrace{C.F}_{} + P.I$

Sol:

$P.I = \dfrac{1}{\underbrace{9D^3+3D^2-5D+1}_{}} \underbrace{42e^x}_{} + 64e^{x/3}$

$= 42 \dfrac{1}{9\cdot1^3+3\cdot1^2-5\cdot1+1} e^x + 64x \dfrac{1}{27D^2+6D-5} e^{x/3}$

$= \dfrac{42}{8} e^x + 64 \times x \dfrac{1}{\underbrace{54D+6}_{}} e^{x/3}$

$= \dfrac{21}{4} e^x + \cancel{64}^{\,8} x^2 \dfrac{1}{\cancel{24}_{\,3}} e^{x/3} = \dfrac{21}{4} e^x + \dfrac{8}{3} x^2 e^{x/3}$

$9 \cdot \dfrac{1}{27} + 3\cdot\dfrac{1}{9} - \dfrac{5}{3} + 1$

$= \dfrac{1}{3} + \dfrac{1}{3} - 5/3 + 1$

$= 0$

$27 \cdot \dfrac{1}{9} + 6\cdot\dfrac{1}{3} - 5$

$= 0$

$54 \cdot \dfrac{1}{3} + 6$

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Type:2

$\frac{1}{f(D^2)}\sin ax = \frac{1}{f(-a^2)}\sin ax$,provided $f(-a^2) \neq 0$

$\frac{1}{f(D^2)}\cos ax = \frac{1}{f(-a^2)}\cos ax$, provided $f(-a^2) \neq 0$

Ex- Find the general solution of $y'' + 4y = 6\cos x$.

Ex- Find the general solution of

$y'' + 4y = 6\cos x.$ $\Rightarrow (D^2+4)y = 6\cos x$

Sol:

$y = C.F + P.I$

The A.E is $m^2+4=0 \Rightarrow m = \pm 2i$

$C.F = c_1\cos 2x + c_2 \sin 2x$

$P.I = \frac{1}{D^2+4} 6\cos x$

$= 6\frac{1}{-1+4}\cos x = 2\cos x$

$\frac{1}{f(D^2)}\cos ax = \frac{1}{f(-a^2)}\cos ax$

$a=1$

$a^2=1$

$-a^2=-1$

$y = c_1\cos 2x + c_2\sin 2x + 2\cos x$

Find the particular integral for the differential equation

$2y'' + y' - y = 16\cos 2x$ $\Rightarrow (2D^2 + D - 1)y = 16\cos 2x$

Sol: $y = C.F + P.I$ $\quad D^2 \rightarrow -a^2$

$P.I = \frac{1}{2D^2+D-1} 16\cos 2x = 16 \frac{1}{2D^2+D-1}\cos 2x$ $\quad a=2$, $a^2=4$, $-a^2=-4$

$\frac{d}{dx}$

$3D+4$

$(3D-4)$

$= 16 \frac{1}{2(-4)+D-1}\cos 2x = 16\frac{1}{D-9}\cos 2x$

$(D+9)\cos 2x$

$= 16\frac{(D+9)}{(D+9)(D-9)}\cos 2x = 16\frac{(D+9)}{D^2-81}\cos 2x$

$= -2\sin 2x + 9\cos 2x$

$= 16\frac{(D+9)}{-4-81}\cos 2x = -\frac{16}{85}(D+9)\cos 2x$

$= -\frac{16}{85}(-2\sin 2x + 9\cos 2x)$

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Find the particular integral for the differential equation

$y'' - 5y' + 4y = 65\,sin\,2x$ $\Rightarrow (D^2 - 5D + 4)y = 65 \sin 2x$

Sol: $P.I = \frac{1}{D^2 - 5D + 4} 65 \sin 2x$ $y = C.F + P.I$

$= 65 \frac{1}{\cancel{-4} - 5D + \cancel{4}} \sin 2x$ $D \rightarrow \frac{d}{dx}$

$= -13 \frac{1}{D} \sin 2x$ $\frac{1}{D} \rightarrow$ Integral

$= -13 \frac{(-\cos 2x)}{2} = \frac{13}{2} \cos 2x$

Case of Failure $f(-a^2)=0$

Find the particular integral for the differential equation

$y'' + 4y = \cos 2x \Rightarrow (D^2+4)y = \cos 2x$

Sol: $y = C.F + P.I$

The A.E is $m^2+4=0 \Rightarrow m = \pm 2i$

$C.F = C_1 \cos 2x + C_2 \sin 2x$

$D^2 \to -a^2$

$-2^2 = -4$

$P.I = \dfrac{1}{\underbrace{D^2+4}} \cos 2x$

$= x \dfrac{1}{2D} \cos 2x = \dfrac{x}{2} \dfrac{1}{D} \cos 2x = \dfrac{x}{2} \dfrac{\sin 2x}{2}$

$y = C_1 \cos 2x + C_2 \sin 2x + \dfrac{x}{4} \sin 2x$

Find the particular integral for the differential equation
$y'' + y = \sin x$ $\Rightarrow (D^2+1)y = \sin x$

Sol: $P.I = \dfrac{1}{D^2+1}\sin x = x\dfrac{1}{2D}\sin x$

$= \dfrac{x}{2}(-\cos x) = -\dfrac{x}{2}\cos x$

$$D^2 \rightarrow -a^2$$

$$\frac{1}{D^{100}+D-1}\cos x = \frac{1}{(D^2)^{50}+D-1}\cos x$$

$$= \frac{1}{\cancel{(-1)}^{50}+D\cancel{-1}}\cos x = \cancel{\frac{1}{D}\sin x}$$

$$= \cancel{\cos x}$$

$$= \frac{1}{D}\cos x = \sin x$$

More Results

$$\frac{1}{f(D)}e^{ax}V = e^{ax}\frac{1}{f(D+a)}V, where\ V\ is\ any\ function\ of\ x.$$

$$\frac{1}{f(D)}xV = x\frac{1}{f(D)}V + \left\{\frac{d}{dD}\frac{1}{f(D)}\right\}V\ \ , where\ V\ is\ any\ function\ of\ x.$$

$$\frac{1}{D}Q = \int Q\,dx$$

$$\frac{1}{D-\alpha}Q = e^{\alpha x}\int Qe^{-\alpha x}dx$$

$(1-x)^{-1} = 1+x+x^2+x^3+\cdots, \quad |x|<1$

$(1+x)^{-1} = 1-x+x^2-x^3+\cdots, \quad |x|<1$

$D = \frac{d}{dx}$

$$\frac{1}{1+D}x^2 = (1+D)^{-1}x^2$$
$$= (1-D+D^2-D^3+\cdots)x^2$$
$$= x^2-2x+2$$

$2x$

2

$$* \quad \frac{1}{1+D}x^2 = x^2-2x+2$$

$\frac{1}{1\pm X}$

$\frac{1}{D^2\left(1+\frac{3}{4D}\right)}$

$$* \quad \frac{1}{4D^2+3D}\,x^2+3x = \frac{1}{3D\left(1+\frac{4}{3}D\right)}(x^2+3x)$$

$$= \frac{1}{3D}\left(1+\frac{4}{3}D\right)^{-1}(x^2+3x) = \frac{1}{3D}\left(1-\frac{4}{3}D+\frac{16}{9}D^2+\cdots\right)$$

$$= \frac{1}{3D}\left(x^2+3x-\frac{4}{3}(2x+\right.$$
$$\left.+\frac{16}{9}\cdot 2\right)$$

$(a-b)^3 = a^3 - 3a^2b + 3ab^2 - b^3$

Solve $(D^3 + 3D^2 - 4D - 12)y = 12xe^{-2x}$

Sol: $y = C.F + P.I, \quad C.F = ?$

$$P.I = \frac{1}{D^3+3D^2-4D-12} 12xe^{-2x}$$

$$= 12e^{-2x} \frac{1}{(D-2)^3+3(D-2)^2-4(D-2)-12} x$$

$$= 12e^{-2x} \frac{1}{D^3-6D^2+12D-8+3D^2-12D+12-4D+8-12} x$$

$$= 12e^{-2x} \frac{1}{D^3-3D^2-4D} x = 12e^{-2x} \frac{1}{-4D\left(1+\frac{3}{4}D-\frac{D^2}{4}\right)}$$

$$= -3e^{-2x} \frac{1}{D}\left(1+\left(\frac{3}{4}D-\frac{D^2}{4}\right)\right)^{-1} x$$

$$= -3e^{-2x}\frac{1}{D}\left[1+\left(\frac{3}{4}D-\frac{D^2}{4}\right)\right]^{-1}x$$

$$= -3e^{-2x}\frac{1}{D}\left[1-\left(\frac{3}{4}D-\frac{D^2}{4}\right)+\left(\frac{3}{4}D-\frac{D^2}{4}\right)^2-\cdots\right]$$

$$= -3e^{-2x}\frac{1}{D}\left[1-\frac{3}{4}D+\cdots\right]x$$

$$= -3e^{-2x}\frac{1}{D}\left[x-\frac{3}{4}\right] = -3e^{-2x}\left(\frac{x^2}{2}-\frac{3}{4}x\right)$$

Solve $y''+16y=64x^2$ $\Rightarrow (D^2+16)y = 64x^2$

Sol:

$$P.I. = \frac{1}{D^2+16} 64x^2 = 64 \frac{1}{16\left(1+\frac{D^2}{16}\right)} x^2$$

$$= 4\left(1+\frac{D^2}{16}\right)^{-1} x^2$$

$Dx^2 = 2x$

$$= 4\left(1-\frac{D^2}{16}+\frac{D^4}{16^2}-\cdots\right)x^2$$

$$= 4\left(x^2-\frac{1}{16}\cdot 2\right) = 4x^2-\frac{1}{2}$$

More problems

1) Find the particular integral of $(D^2 + 5D + 6)y = e^x$.

2) Find the particular integral of $(D^3 + 1)y = \cos(2x - 1)$

3) Solve the differential equation $\frac{d^3y}{dx^3} + 4\frac{dy}{dx} = \sin 2x$

4) Solve the differential equation $\frac{d^2y}{dx^2} + \frac{dy}{dx} = x^2 + 2x + 4$

5) Solve the differential equation $(D^2 - 2D + 4)y = e^x \cos x$

More Problems
OPERATOR METHOD for Linear Differential Equation.

Sandeep Kumar Singh

1) Solve $(D^2 + 5D + 4)y = 18e^{2x}$

Sol:. $y = C.F + P.I$

The A.E is $m^2 + 5m + 4 = 0 \Rightarrow m^2 + 4m + m + 4 = 0$

$\Rightarrow m(m+4) + 1(m+4) = 0 \Rightarrow (m+4)(m+1) = 0$

$\therefore m = -4, -1$

$C.F = c_1 e^{-4x} + c_2 e^{-x}$

$$\frac{1}{f(D)} e^{ax} = \frac{1}{f(a)} e^{ax}$$

$f(a) \neq 0$

$$P.I = \frac{1}{D^2 + 5D + 4} 18e^{2x}$$

$$= \cancel{18} \frac{1}{(\cancel{2^2 + 5 \cdot 2 + 4})} e^{2x} = e^{2x}$$

$$y = c_1 e^{-4x} + c_2 e^{-x} + e^{2x}$$

2) Solve $(D^2 - 6D + 9)y = 14e^{3x}$

$y = C.F + P.I$

The A.E is $m^2 - 6m + 9 = 0 \Rightarrow (m-3)^2 = 0 \Rightarrow m = 3, 3$

$C.F = (c_1 + c_2 x)e^{3x}$

$P.I = \frac{1}{D^2 - 6D + 9} 14 e^{3x} = 14x \frac{1}{2D - 6} e^{3x}$

$= 7 \cancel{14} x . x \frac{1}{\cancel{2}} e^{3x} = 7x^2 e^{3x}$

$y = (c_1 + c_2 x)e^{3x} + 7x^2 e^{3x}$

3) Solve $(4D^2 + 8D + 3)y = xe^{-x/2}\cos x$.

Sol: $y = C.F + P.I$

The A.E $4m^2 + 8m + 3 = 0 \Rightarrow 4m^2 + 6m + 2m + 3 = 0$

$\Rightarrow 2m(2m+3) + 1(2m+3) = 0 \Rightarrow (2m+3)(2m+1) = 0$

$m = -3/2, -1/2$

$C.F = c_1 e^{-\frac{3}{2}x} + c_2 e^{-\frac{1}{2}x}$

$\frac{1}{f(D)} e^{ax} V = e^{ax} \frac{1}{f(D+a)} V$

$P.I = \frac{1}{4D^2+8D+3} x e^{-x/2} \cos x =$

$= e^{-x/2} \frac{1}{4(D-\frac{1}{2})^2 + 8(D-\frac{1}{2}) + 3} x\cos x = e^{-x/2} \frac{1}{4(D^2 + \frac{1}{4} - D) + 8D - 4 + 3} x\cos x$

$= e^{-x/2} \frac{1}{4D^2 + 4D} x\cos x$

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$$= e^{-x/2}\frac{1}{4D^2+4D}x\cos x \qquad \frac{1}{f(D)}xV = x\frac{1}{f(D)}V + \left\{\frac{d}{dD}\frac{1}{f(D)}\right\}V$$

$$= e^{-x/2}\left[x\frac{1}{4D^2+4D}\cos x + \left\{\frac{d}{dD}\frac{1}{4D^2+4D}\right\}\cos x\right] \qquad \frac{1}{f(D^2)}\cos ax$$

$$= \frac{1}{f(-a^2)}\cos ax$$

$$= e^{-x/2}\left[x\frac{1}{4D-4}\cos x - \frac{1}{(4D^2+4D)^2}(8D+4)\cos x\right]$$

$$a=1$$

$$-a^2=-1$$

$$= e^{-x/2}\left[\frac{x}{4}\frac{(D+1)}{D^2-1}\cos x - \frac{1}{(4D-4)^2}(8D+4)\cos x\right]$$

$$= e^{-x/2}\left[-\frac{x}{8}(-\sin x+\cos x) - \frac{4(2D+1)}{16(D-1)^2}\cos x\right]$$

$$= e^{-x/2}\left[\frac{x}{8}(\sin x-\cos x) - \frac{1}{4}\frac{(2D+1)}{D^2-2D+1}\cos x\right]$$

$$= e^{-x/2}\left[\frac{x}{8}(\sin x-\cos x)-\frac{1}{4}\,\frac{(2D+1)}{D^2-2D+1}\cos x\right]$$

$$= e^{-x/2}\left[\frac{x}{8}(\sin x-\cos x)-\frac{1}{4}\,\frac{(2D+1)}{\cancel{-1}-2D\cancel{+1}}\cos x\right]$$

$-\sin x$

$-\cos x$

$$= e^{-x/2}\left[\frac{x}{8}(\sin x-\cos x)+\frac{1}{8}\,\frac{(2D+1)D}{D^2}\cos x\right]$$

$$= e^{-x/2}\left[\frac{x}{8}(\sin x-\cos x)-\frac{1}{8}(2D^2+D)\cos x\right]$$

$$= e^{-x/2}\left[\frac{x}{8}(\sin x-\cos x)-\frac{1}{8}(-2\cos x-\sin x)\right]$$

$$= e^{-x/2}\left[\frac{x}{8}(\sin x-\cos x)+\frac{1}{8}(2\cos x+\sin x)\right]$$

4) Solve $(D^2 - 6D + 13)y = 28e^{2x} \sin 2x$.

Sol: $y = C.F + P.I$

The A.E is $m^2 - 6m + 13 = 0$

$$\Rightarrow m = \frac{6 \pm \sqrt{36 - 4 \times 1 \times 13}}{2 \cdot 1}$$

$$= \frac{6 \pm \sqrt{-16}}{2} = \frac{6 \pm 4i}{2} = 3 \pm 2i$$

$$C.F = e^{3x}(c_1 \cos 2x + c_2 \sin 2x)$$

$$P.I = \frac{1}{D^2 - 6D + 13} 28 e^{2x} \sin 2x = 28 e^{2x} \frac{1}{(D+2)^2 - 6(D+2) + 13} \sin 2x$$

$$= 28e^{2x} \frac{1}{D^2 + 4 + 4D - 6D - 12 + 13} \sin 2x = 28e^{2x} \frac{1}{D^2 - 2D + 5} \sin 2x$$

$$= 28e^{2x}\frac{1}{D^2-2D+5}\sin 2x = 28e^{2x}\frac{1}{-4-2D+5}\sin 2x$$

$$= 28e^{2x}\frac{1}{1-2D}\sin 2x = 28e^{2x}\frac{(1+2D)}{1-4D^2}\sin 2x$$

$$= 28e^{2x}\frac{(1+2D)}{1+4}\sin 2x = \frac{28}{5}e^{2x}(\sin 2x + 4\cos 2x)$$

5) $(D^2+25)y = \underbrace{9x^3+4x^2}$.

$D(9x^3+4x^2)$
$= 27x^2+8x$

$54x+8$

$y = C.F + P.I$

The A.E is $m^2+25=0 \Rightarrow m = \pm 5i$

$C.F = c_1 \cos 5x + c_2 \sin 5x$ ✓

$P.I = \frac{1}{D^2+25} 9x^3+4x^2 = \frac{1}{25\left(1+\frac{D^2}{25}\right)} 9x^3+4x^2$

$= \frac{1}{25}\left(1+\frac{D^2}{25}\right)^{-1}(9x^3+4x^2) = \frac{1}{25}\left(1-\frac{D^2}{25}+\frac{D^4}{625}\cdots\right)(9x^3+4x^2)$

$= \frac{1}{25}\left(1-\frac{D^2}{25}\right)(9x^3+4x^2) = \frac{1}{25}\left(9x^3+4x^2-\frac{1}{25}(54x+8)\right)$

$= \frac{1}{25}\left(9x^3+4x^2-\frac{54}{25}x-8/25\right)$ ✓

METHOD OF VARIATION OF PARAMETERS

Sandeep Kumar Singh

Consider the second order non-homogeneous linear equation

$$a_0(x)y'' + a_1(x)y' + a_2(x)y = r(x), a_0(x) \neq 0$$

$y = C.F + P.I$

To find the particular integral we use the formula

$$P.I = -y_1 \underbrace{\int \frac{y_2 X}{W} dx}_{A(x)} + y_2 \underbrace{\int \frac{y_1 X}{W} dx}_{B(x)}$$

$C.F = c_1 y_1 + c_2 y_2$

where y_1 and y_2 are two linearly independent solutions of

$a_0(x)y'' + a_1(x)y' + a_2(x)y=0$, $X = \frac{r(x)}{a_0(x)}$ and W is the Wronskian of y_1 and y_2.

$P.I = A(x) y_1 + B(x) y_2$

$A(x) = -\int \frac{y_2 X}{W} dx$, $B(x) = \int \frac{y_1 X}{W} dx$

Alternate form of writing the solution $y = A(x)y_1 + B(x)y_2$ is the P.I

$$y = C.F + P.I$$

$$= c_1 y_1 + c_2 y_2 - y_1 \int \frac{y_2 X}{W} dx + y_2 \int \frac{y_1 X}{W} dx$$

$$y = \left(-\int \frac{y_2 X}{W} dx + c_1\right) y_1 + \left(\int \frac{y_1 X}{W} dx + c_2\right) y_2$$

General sol $y = A(x)y_1 + B(x)y_2$

$$\checkmark\ A(x) = -\int \frac{y_2 X}{W} dx + c_1$$

$$\checkmark\ B(x) = \int \frac{y_1 X}{W} dx + c_2$$

$A(x) = -\int \frac{y_2 X}{W} dx$

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1) Find the general solution of $y'' + 3y' + 2y = 2e^x$ using the method of variation of parameters.

Sol: $\to y = C.F + P.I$

The A.E is $m^2+3m+2=0 \Rightarrow m^2+2m+m+2=0$

$\Rightarrow m(m+2)+1(m+2)=0 \Rightarrow (m+2)(m+1)=0 \Rightarrow m=-2,-1$

$C.F = c_1 e^{-2x} + c_2 e^{-x}$ ✓

$$P.I = -y_1 \int \frac{y_2 X}{W} dx + y_2 \int \frac{y_1 X}{W} dx$$

$$= -e^{-2x} \int \frac{e^{-x} \times 2e^{x}}{e^{-3x}} dx + e^{-x} \int \frac{e^{-2x} \times 2e^{x}}{e^{-3x}} dx$$

$$= -2e^{-2x} \int e^{3x} dx + 2e^{-x} \int e^{2x} dx$$

$$= -2e^{-2x} \cdot \frac{e^{3x}}{3} + \cancel{2} e^{-x} \frac{e^{2x}}{\cancel{2}} = -\frac{2}{3} e^{x} + e^{x} = \frac{1}{3} e^{x}$$

$y = c_1 e^{-2x} + c_2 e^{-x} + \frac{1}{3} e^{x}$ ✓

$y_1 = e^{-2x}, \; y_2 = e^{-x}$

$X = \frac{2e^x}{1} = 2e^x$

$$W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = \begin{vmatrix} e^{-2x} & e^{-x} \\ -2e^{-2x} & -e^{-x} \end{vmatrix}$$

$= -e^{-3x} + 2e^{-3x} = e^{-3x}$

2)Find the general solution of $y'' + 16y = 32\sec 2x$ using the method of variation of parameters.

$\sin 2\theta = 2\sin\theta\cos\theta$

$\cos 2\theta = 2\cos^2\theta - 1$

Sol: $y = C.F + P.I$

The A.E is $m^2+16=0 \Rightarrow m = \pm 4i$

$C.F = c_1\cos 4x + c_2 \sin 4x$, $y_1 = \cos 4x$, $y_2 = \sin 4x$

$X = 32\sec 2x$

$P.I = -y_1\int \frac{y_2 X}{W}dx + y_2\int\frac{y_1 X}{W}dx$

$W = \begin{vmatrix} \cos 4x & \sin 4x \\ -4\sin 4x & 4\cos 4x \end{vmatrix} = 4\cos^2 4x + 4\sin^2 4x = 4 \neq 0$

$= -\cos 4x\int\frac{\sin 4x \times 32\sec 2x}{4}dx + \sin 4x\int\frac{\cos 4x \times 32\sec 2x}{4}dx$

$= -8\cos 4x\int\frac{\sin 4x}{\cos 2x}dx + 8\sin 4x\int\frac{\cos 4x}{\cos 2x}dx$

$= -8\cos 4x\int\frac{2\sin 2x\cancel{\cos 2x}}{\cancel{\cos 2x}}dx + 8\sin 4x\int\frac{2\cos^2 2x - 1}{\cos 2x}dx$

$= -16\cos 4x\left(-\frac{\cos 2x}{2}\right) + 8\sin 4x\left[2\int\cos 2x\,dx - \int\sec 2x\,dx\right]$

$$= -16\cos 4x\left(\frac{-\cos 2x}{2}\right) + 8\sin 4x\left[2\int \cos 2x\,dx - \int \sec 2x\,dx\right]$$

$$= 8\cos 4x\cos 2x + 8\sin 4x\left[\cancel{2}\,\frac{\sin 2x}{\cancel{2}} - \frac{1}{2}\log|\sec 2x+\tan 2x|\right]$$

$$= 8\cos 4x\cos 2x + 8\sin 4x\sin 2x - 4\sin 4x\log|\sec 2x + \tan 2x|$$

$$y = c_1\cos 4x + c_2\sin 4x + 8\cos 4x\cos 2x + 8\sin 4x\cdot\sin 2x - 4\sin 4x\log|\sec 2x+\tan 2x|$$

3) Find the general solution of $x^2y'' + xy' - y = x^3$, *where* $y_1 = x$, $y_2 = \frac{1}{x}$ using the method of variation of parameters.

Sol: $y = C.F + P.I$, $C.F = c_1y_1 + c_2y_2 = c_1x + c_2\cdot\frac{1}{x}$

$W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = \begin{vmatrix} x & 1/x \\ 1 & -1/x^2 \end{vmatrix} = -\frac{1}{x} - \frac{1}{x} = -2/x.$

$X = \frac{x^3}{x^2} = x$

$P.I = -y_1\int \frac{y_2 X}{W}dx + y_2\int\frac{y_1 X}{W}dx = -x\int\frac{\cancel{\frac{1}{x}}\times\cancel{x}}{\frac{-2}{x}}dx + \frac{1}{x}\int\frac{x\times x}{\frac{-2}{x}}dx$

$= x\int\frac{x}{2}dx - \frac{1}{2x}\int x^3 dx = \frac{x}{2}\frac{x^2}{2} - \frac{1}{2x}\frac{x^4}{4} = \frac{x^3}{4} - \frac{x^3}{8} = \frac{x^3}{8}$

$y = c_1x + c_2\cdot\frac{1}{x} + \frac{1}{8}x^3$

This method is applicable for both constant coefficients and variable coefficients problems. The method can also be easily extended to equation of any order.

For example consider the third order equation

$a_0(x)y''' + a_1(x)y'' + a_2(x)y' + a_3(x)y = r(x), a_0(x) \neq 0$

The complementary function is $y = Ay_1(x) + By_2(x) + Cy_3(x)$

Where y_1, y_2, y_3 are linearly independent solutions of corresponding homogeneous equation and A, B, C are arbitrary constants. We assume the solution as $y = A(x)y_1(x) + B(x)y_2(x) + C(x)y_3(x)$.

For determining $\mathrm{A}(x), B(x), C(x)$ we have the following equations

$A'(x)y_1 + B'(x)y_2 + C'(x)y_3 = 0$

$A'(x)y_1' + B'(x)y_2' + C'(x)y_3' = 0$

$A'(x)y_1'' + B'(x)y_2'' + C'(x)y_3'' = \frac{r(x)}{a_0(x)} = g(x)$. We solve this system of equations by Cramer's rule.

Q)Find the general solution of $y''' - 6y'' + 11y' - 6y = e^{-x}$ using the method of variation of parameters.

Sol: The A.E is $m^3 - 6m^2 + 11m - 6 = 0 \Rightarrow m = 1, 2, 3$

$C.F = c_1 e^{x} + c_2 e^{2x} + c_3 e^{3x}$, $y_1 = e^{x}$, $y_2 = e^{2x}$, $y_3 = e^{3x}$

The solution is $y = A(x)e^{x} + B(x)e^{2x} + C(x)e^{3x}$

The eqn to determine $A(x), B(x), C(x)$ are as follows

A', B', C'

$$\left.\begin{aligned} A'e^{x} + B'e^{2x} + C'e^{3x} &= 0 \\ A'e^{x} + 2B'e^{2x} + 3C'e^{3x} &= 0 \\ A'e^{x} + 4B'e^{2x} + 9C'e^{3x} &= e^{-x} \end{aligned}\right\} \quad \begin{aligned} &\Delta = \\ &x\Delta = \end{aligned}$$

$$W = \begin{vmatrix} e^{x} & e^{2x} & e^{3x} \\ e^{x} & 2e^{2x} & 3e^{3x} \\ e^{x} & 4e^{2x} & 9e^{3x} \end{vmatrix} = e^{6x} \begin{vmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 1 & 4 & 9 \end{vmatrix}$$

$$= e^{6x}[1 \cdot 6 - 6 + 2] = \underline{2e^{6x}}$$

$$\underbrace{WA'} = \begin{vmatrix} 0 & e^{2x} & e^{3x} \\ 0 & 2e^{2x} & 3e^{3x} \\ e^{-x} & 4e^{2x} & 9e^{3x} \end{vmatrix} = e^{-x}\left[\underbrace{3e^{5x} - 2e^{5x}}\right] = e^{4x}$$

$$\Rightarrow WA' = e^{4x} \Rightarrow 2e^{6x} \cdot A' = e^{4x} \Rightarrow \underbrace{A'} = \frac{1}{2}e^{-2x}$$

$$\Rightarrow A = \frac{1}{2}\int e^{-2x}\,dx + c_1 \Rightarrow \underbrace{A(x)} = -\frac{1}{4}e^{-2x} + c_1$$

$$WB' = \begin{vmatrix} e^{x} & 0 & e^{3x} \\ e^{x} & 0 & 3e^{3x} \\ e^{x} & e^{-x} & 9e^{3x} \end{vmatrix} = -e^{-x}\left[3e^{4x} - e^{4x}\right] = -2e^{3x}$$

$$\Rightarrow \underbrace{B'} = \frac{-2e^{3x}}{2e^{6x}} \Rightarrow B' = -e^{-3x}$$

$$\Rightarrow B = -\frac{e^{-3x}}{-3} + c_2 \Rightarrow B(x) = \frac{1}{3}e^{-3x} + c_2$$

$$Wc' = \begin{vmatrix} e^{x} & e^{2x} & 0 \\ e^{x} & 2e^{2x} & 0 \\ e^{x} & 4e^{2x} & e^{-x} \end{vmatrix} = e^{-x}\left[2e^{3x} - e^{3x}\right] = e^{-x}\cdot e^{3x}$$

$$= e^{2x}$$

$\frac{d}{dx} c(x)$

$$\Rightarrow c' = \frac{e^{2x}}{2e^{6x}} = \frac{1}{2}e^{-4x} \Rightarrow c' = \frac{1}{2}e^{-4x}$$

$$-\frac{1}{4}+\frac{1}{3}-\frac{1}{8}$$

$$\frac{-6+8-3}{24}$$

$$\Rightarrow c = \frac{1}{2}\frac{e^{-4x}}{-4} + c_3 \Rightarrow c(x) = -\frac{1}{8}e^{-4x} + c_3$$

$$y = A(x)e^{x} + B(x)e^{2x} + C(x)e^{3x}$$

$$= \left(-\frac{1}{4}e^{-2x} + c_1\right)e^{x} + \left(\frac{1}{3}e^{-3x} + c_2\right)e^{2x} + \left(-\frac{1}{8}e^{-4x} + c_3\right)e^{3x}$$

$$\Rightarrow y = c_1e^{x} + c_2e^{2x} + c_3e^{3x} - \frac{1}{4}e^{-x} + \frac{1}{3}e^{-x} - \frac{1}{8}e^{-x}$$

$$y = c_1e^{x} + c_2e^{2x} + c_3e^{3x} - \frac{1}{24}e^{-x}$$

METHOD·OF UNDETERMINED COEFFICIENTS

Sandeep Kumar Singh

Process of selecting the Trial solution for particular integral

R.H.S $r(x)$	Trial solution
e^{mx}	Ce^{mx}
$\sin mx$ or $\cos mx$	$A\cos mx + B\sin mx$
x^m	$c_0x^m + c_1x^{m-1} + \cdots + c_m$
$e^{ax}\cos bx$ or $e^{ax}\sin bx$	$e^{ax}(c_1\cos bx + c_2\sin bx)$

$\sin mx$
$m\cos mx$
$-m^2\sin mx$
$-m^3\cos mx$

e^{mx} $\quad me^{mx}$
m^2e^{mx}

$0e^{mx}$

$y = C.F + P.I$

x^3
$x^3 \quad 3x^2$
$6x,\ 6$

$e^{mx} + \sin mx$
$ce^{mx} + (A\cos mx + B\sin mx)$

Remark:

If any term in the choice of the particular integral is also a solution of the corresponding homogeneous equation, that is, a term in the complementary function, then we multiply this term by x or by x^m (if the term in the complementary function corresponds to a multiple root of multiplicity m). If $r(x)$ is the sum of a number of functions then contribution with respect to each of the terms is included in the choice of the particular integral.

1) Using the method of undetermined coefficients find the general solution of the differential equation $y'' + y = 32\,x^3$

Sol: $y = C\cdot F + P\cdot I$

deg 3

The A·E is $m^2+1=0 \Rightarrow m = \pm i$

$32x^3+5x$

$32x^3+5x+9$

$y_c(x) = C\cdot F = A\cos x + B\sin x$

$y_p(x) = C_0 x^3 + C_1 x^2 + C_2 x + C_3$ $\quad C_0, C_1, C_2, C_3$

$y_p' = 3C_0 x^2 + 2C_1 x + C_2, \quad y_p'' = 6C_0 x + 2C_1$

$(6C_0 x + 2C_1) + (C_0 x^3 + C_1 x^2 + C_2 x + C_3) = 32x^3$

$\Rightarrow \quad C_0 = 32, \quad 6C_0 + C_2 = 0, \quad C_1 = 0, \quad 2C_1 + C_3 = 0$

$C_2 = -6C_0 = -192, \quad C_1 = 0, \quad C_3 = -2C_1 = 0$

$y_p(x) = 32x^3 - 192x$ $\quad y = A\cos x + B\sin x + 32x^3 - 192x$

Using the method of undetermined coefficients find the general solution of the differential equation $y'' - 2y' - 3y = 6e^{-x} - 8e^{x}$.

Sol: $y = C.F + P.I$

The A.E is $m^2 - 2m - 3 = 0 \Rightarrow m^2 - 3m + m - 3 = 0$

$\Rightarrow m(m-3) + 1(m-3) = 0 \Rightarrow (m-3)(m+1) = 0 \Rightarrow m = 3, -1$

$C.F = c_1 e^{3x} + c_2 e^{-x}$

$y_p(x) = Axe^{-x} + Be^{x}$

$y_p' = A(-xe^{-x} + e^{-x}) + Be^{x}$

$y_p'' = -A(-xe^{-x} + e^{-x}) - Ae^{-x} + Be^{x}$

$\cancel{Ax}e^{-x} - 2Ae^{-x} + Be^{x} + 2A\cancel{x}e^{-x} - 2Ae^{-x} - 2Be^{x} - 3A\cancel{x}e^{-x} - 3Be^{x} = 6e^{-x} - 8e^{x}$

$\Rightarrow -4Ae^{-x} - 4Be^{x} = 6e^{-x} - 8e^{x} \Rightarrow -4A = 6, \; -4B = -8$

$A = \frac{6}{-4} = -3/2, \; B = 2$

$y_p(x) = -\frac{3}{2}xe^{-x} + 2e^{x}$ | $y = c_1 e^{3x} + c_2 e^{x} - \frac{3}{2}xe^{-x} + 2e^{x}$

Write the trial solution for the Particular integral of the differential equation $y'' + 9y = \cos 3x$

Sol: $y = C.F + P.I,$

The A.E is $m^2+9=0 \Rightarrow m = \pm 3i$

$y_c(x) = C.F$
$= c_1 \cos 3x + c_2 \sin 3x$

$y_p(x) = x(A\cos 3x + B\sin 3x) = \frac{1}{6}x\sin 3x$

$y_p' = x(-3A\sin 3x + 3B\cos 3x) + (A\cos 3x + B\sin 3x)$

$y_p'' = x(-9A\cos 3x - 9B\sin 3x) + (-3A\sin 3x + 3B\cos 3x)$
$+ (-3A\sin 3x + 3B\cos 3x)$

$-9A\cancel{x}\cos 3x - 9B\cancel{x}\sin 3x - 3A\sin 3x + 3B\cos 3x$
$-3A\sin 3x + 3B\cos 3x + 9A\cancel{x\cos 3x} + 9B\cancel{x}\sin 3x = \cos 3x$

$\Rightarrow -6A = 0, \quad 6B = 1$

$A = 0, \quad B = \frac{1}{6}$

$y = c_1\cos 3x + c_2\sin 3x + \frac{1}{6}x\sin 3x$

Write the trial solution for the Particular integral of the differential equation $y'' + 4y' + 4y = 12e^{-2x}$.

Sol: $y = y_c(x) + y_p(x)$

The A.E is $m^2 + 4m + 4 = 0$

$\Rightarrow (m+2)^2 = 0 \Rightarrow m = -2, -2$

$y_c(x) = (c_1 + c_2 x)e^{-2x} = c_1 e^{-2x} + c_2 x e^{-2x}$

$y_p(x) = A x^2 e^{-2x}$

Write the trial solution for the Particular integral of the differential equation $y'' - 4y' + 13y = 12e^{2x}\sin 3x$.

$e^{ax}\sin bx$

Sol:

$$y = y_c(x) + y_p(x)$$

The A.E is $m^2 - 4m + 13 = 0$

$$m = \frac{4 \pm \sqrt{16 - 4\cdot 1\cdot 13}}{2} = \frac{4 \pm 6i}{2}$$

$$= 2 \pm 3i$$

$$y_c(x) = e^{+2x}(C_1 \cos 3x + C_2 \sin 3x)$$

$$y_p(x) = x.e^{2x}(A\cos 3x + B\sin 3x)$$

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Write the trial solution for the Particular integral of the differential equation $y''' - 2y'' - 5y' + 6y = 18e^x$.

Sol:

$y = y_c(x) + y_p(x)$

A·E is $m^3 - 2m^2 - 5m + 6 = 0$

$m = 1, \quad 1 - 2 - 5 + 6 = 0$, $(m-1)$ is a factor

$m = 1, -2, 3$

$y_c(x) = c_1 e^x + c_2 e^{-2x} + c_3 e^{3x}$

$y_p(x) = Axe^x$

Write the trial solution for the Particular integral of the differential equation $y''' - 6y'' + 12y' - 8y = 12e^{2x} + 27e^{-x}$.

Sol:

$$m^3 - 6m^2 + 12m - 8 = 0$$

$$\Rightarrow m^3 - 3\cdot m^2\cdot 2 + 3\cdot m\cdot 2^2 - 2^3 = 0$$

$$\Rightarrow (m-2)^3 = 0 \Rightarrow m = 2, 2, 2$$

$$C\cdot F = (c_1 + c_2 x + c_3 x^2)e^{2x}$$

$$y_p(x) = Ax^3e^{2x} + Be^{-x}$$

Solution of Euler-Cauchy Equation

Sandeep Kumar Singh

Euler-Cauchy equation is the equation which appears in this form
$a_0 x^n y^{(n)} + a_1 x^{n-1} y^{(n-1)} + \cdots + a_{n-1} x y' + a_n y = r(x), x \neq 0,$
where $a_0, a_1, \ldots a_n$ are constants.

Examples

a) $x^2 y'' + 2xy' - 2y = 0$

b) $2x^2 y'' + xy' - 6y = 0$

c) $2x^2 y'' + 3xy' - 3y = x^3$

d) $x^2 y'' + 5xy' + 3y = \ln x, x > 0.$

1) $x^2y'' + 2xy' - 2y = 0$

$x^2 y'' = D(D-1)y$

$xy' = Dy$

$x^3 y''' = D(D-1)(D-2)y$

$x^4 y^{iv} = D(D-1)(D-2)(D-3)y$

Sol: let $z = \log x$ or $x = e^z$, $D \equiv \frac{d}{dz}$

The above eqn reduces to

$$D(D-1)y + 2Dy - 2y = 0$$

$$\Rightarrow (D^2 - D + 2D - 2)y = 0$$

$$\Rightarrow (D^2 + D - 2)y = 0$$

$y \rightarrow$

$z \rightarrow$

The A.E is $m^2 + m - 2 = 0 \Rightarrow m^2 + 2m - m - 2 = 0$

$\Rightarrow m(m+2) - 1(m+2) = 0 \Rightarrow (m+2)(m-1) = 0$

$\therefore m = -2, 1$

$$y = c_1 e^{-2z} + c_1 e^{z} = c_1 e^{-2\log x} + c_1 e^{\log x}$$

$$y = \frac{c_1}{x^2} + c_2 x$$

$e^{-2\log x} = e^{\log x^{-2}} = x^{-2} = \frac{1}{x^2}$

2) $2x^2y'' + xy' - 6y = 0$

Sol: $z = \log x$ or $x = e^z$, $D \equiv \frac{d}{dz}$

The above eqn reduces to

$2D(D-1)y + Dy - 6y = 0$

$\Rightarrow [2D^2 - 2D + D - 6]y = 0$

$\Rightarrow (2D^2 - D - 6)y = 0$

The A.E is $2m^2 - m - 6 = 0$

$\Rightarrow 2m^2 - 4m + 3m - 6 = 0$

$\Rightarrow 2m(m-2) + 3(m-2) = 0$

$\Rightarrow (m-2)(2m+3) = 0$

$m = 2, -3/2$

$y = c_1 e^{2z} + c_2 e^{-3/2 z}$

$\Rightarrow y = c_1 e^{2\log x} + c_2 e^{-3/2 \log x}$

$\Rightarrow y = c_1 x^2 + \frac{c_2}{x\sqrt{x}}$

3) $2x^2y'' + 3xy' - 3y = x^3$

Sol: $z = \log x$ or $x = e^z$, $D \equiv \frac{d}{dz}$

The above eqn reduces to

$$2D(D-1)y + 3Dy - 3y = e^{3z}$$

$$\Rightarrow [2D^2 - 2D + 3D - 3]y = e^{3z}$$

$$\Rightarrow (2D^2 + D - 3)y = e^{3z}$$

$$y = C.F + P.I$$

The A.E is $2m^2 + m - 3 = 0$

$$\Rightarrow 2m^2 + 3m - 2m - 3 = 0$$

$$\Rightarrow m(2m+3) - 1(2m+3) = 0$$

$$\Rightarrow (2m+3)(m-1) = 0$$

$$m = -3/2, 1$$

$$C.F = c_1 e^{-3/2 z} + c_2 e^{z}$$

$$= \frac{c_1}{x\sqrt{x}} + c_2 x$$

$$P.I = \frac{1}{2D^2 + D - 3} e^{3z}$$

$$= \frac{1}{2\cdot 3^2 + \cancel{3} - \cancel{3}} e^{3z}$$

$$= \frac{1}{18} e^{3z} = \frac{1}{18} x^3$$

$$y = \frac{c_1}{x\sqrt{x}} + c_2 x + \frac{1}{18} x^3$$

4) $x^2y'' + 5xy' + 3y = \ln x, x > 0.$

Sol: Let $z = \log x$ or $x = e^z$, $D \equiv \frac{d}{dz}$

The above eqn reduces to

$$D(D-1)y + 5Dy + 3y = z$$

$$\Rightarrow (D^2 - D + 5D + 3)y = z$$

$$\Rightarrow (D^2 + 4D + 3)y = z$$

$$y = C.F + P.I$$

The A.E is $m^2 + 4m + 3 = 0$

$$\Rightarrow m^2 + 3m + m + 3 = 0$$

$$\Rightarrow m(m+3) + 1(m+3) = 0$$

$$\Rightarrow (m+3)(m+1) = 0$$

$$m = -3, -1$$

$$C.F = c_1 e^{-3\log x} + c_2 e^{-\log x}$$

$$= \frac{c_1}{x^3} + \frac{c_2}{x}$$

$$P.I = \frac{1}{D^2 + 4D + 3} z$$

$$= \frac{1}{3\left(1 + \frac{4}{3}D + \frac{D^2}{3}\right)} z$$

$$= \frac{1}{3}\left(1 + \left(\frac{4}{3}D + \frac{D^2}{3}\right)\right]^{-1} z$$

$$= \frac{1}{3}\left[1 - \frac{4}{3}D - \frac{D^2}{3} + \cdots\right] z$$

$$= \frac{1}{3}\left[z - \frac{4}{3} 1\right] = \frac{z}{3} - \frac{4}{9}$$

$$= \frac{1}{3}\log x - 4/9$$

6) $x^2y'' - 2y = 2x + 6$

Sol: let $z = \log x$, $x = e^z$, $D \equiv \frac{d}{dz}$

The above eqn reduces to

$$D(D-1)y - 2y = 2e^z + 6$$

$$\Rightarrow [D^2 - D - 2]y = 2e^z + 6$$

$$y = C.F + P.I$$

The A.E is $m^2 - m - 2 = 0$

$$\Rightarrow m^2 - 2m + m - 2 = 0$$

$$\Rightarrow m(m-2) + 1(m-2) = 0$$

$$\Rightarrow (m-2)(m+1) = 0$$

$$m = 2, -1$$

$$C.F = c_1 e^{2z} + c_2 e^{-z}$$

$$C.F = c_1 x^2 + \frac{c_2}{x}$$

$$P.I = \frac{1}{D^2 - D - 2}(2e^z + 6)$$

$$= 2\frac{1}{\cancel{1} - \cancel{1} - 2}e^z + \frac{1}{D^2 - D - 2}6e^{0z}$$

$$= -e^z + \frac{1}{0 - 0 - 2}6$$

$$= -e^z - 3$$

$$= -x - 3$$

$$y = c_1 x^2 + \frac{c_2}{x} - x - 3$$