SMAI - ASSIGNMENT-1 by 2020102004
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Q.1. Example of PMF with
* { Reference for the Ques. taken by probabilitycourse.com] .
(a) finite range.

Let a fair coin is tossed twice & let X be the no. of heads observed.

↳ Here, sample space, $S = \{\underbrace{HH}_{2}, \underbrace{HT}_{1}, \underbrace{TH}_{1}, \underbrace{TT}_{0}\}$.

no. of heads = 2 1 1 0.

$\therefore$ range of X, $\boxed{R_X = \{0, 1, 2\}}$

↳ this is finite set

* Now, the PMF for a finite set is defined as :-

$$\boxed{P_X(k) = P(X = k) \text{ for } k = 0, 1, 2} \quad \text{—} ①.$$

- $P_X(k=0) = P(X=0) = P(TT) = 1/4$.
- $P_X(1) = P(X=1) = P(\{HT, TH\}) = \frac{1}{4} \times 2 = 1/2$
- $P_X(2) = P(X=2) = P(HH) = 1/4$.

* The conditions of PMF are :-
  - $\forall i,\ P(v_i) \geq 0$.
  - $\sum_{i=1}^{n} P(v_i) = \sum_{i=1}^{n} P_i = 1$.

This condition is satisfied, $\because$ $\forall\ P_X(k) > 0$. (from above)

$$\sum_{i=1}^{n} P(v_i) = \underbrace{\frac{1}{4} + \frac{1}{2} + \frac{1}{4}}_{P(TT) + P(\{HT, TH\}) + P(HH)} = 1.$$

$\therefore$ this condition is also satisfied.

(b) infinite range.

Let there be an unfair coin, with $P(H) = p$, $0 < p < 1$, where the coin is tossed repeatedly until ~~one~~ heads is observed for the 1st time.

Let Y be total coin tosses.

* Here, $R_Y = \underbrace{\mathbb{N}}_{\text{set of natural numbers}} = \{1, 2, 3, \ldots\}$. } ~~so~~ an infinite range.

$\therefore P_Y(k) = P(Y = k)$, $k = \cancel{0}, 1, 2, 3, 4, \ldots$

$$P_Y(1) = P(Y=1) = P(H) = p.$$
$$P_Y(2) = P(Y=2) = P(TH) = (1-p)p.$$
$$P_Y(3) = P(Y=3) = P(TTH) = (1-p)(1-p)p.$$
$$\vdots$$
$$P_Y(k) = P(Y=k) = P(\underbrace{TTT\ldots TT}_{(k-1)}H) = (1-p)^{k-1}p.$$

$\therefore$ PMF of Y can be written as:-

$$P_Y(y) = \begin{cases} (1-p)^{y-1}p, & y = 1, 2, 3, \ldots \\ 0, & \text{otherwise.} \end{cases}$$

Here, also, $P(x_i) \geq 0$, $\forall i$

$$\sum_{i=1}^{n} P(x_i) = p + (1-p)p + (1-p)^2 p + \ldots + (1-p)^{k-1}p.$$
$$= p\left[1 + \underbrace{(1-p)}_{} + \underbrace{(1-p)^2}_{} + \ldots + (1-p)^{k-1}\right]$$

all these terms are < 1. $\therefore p < 1$, $(1-p) < 1$.

So, let's consider when $k \to \infty$. → it forms infinite GP.

$$= p\left(\frac{a}{1-r}\right) = p\left(\frac{1}{(1)-(1-p)}\right) = p\left(\frac{1}{p}\right) = 1$$

$\therefore$ Both conditions satisfied. $\therefore$ $\boxed{\sum_{i=1}^{n} P(x_i) = 1}$

Q.2. Equation 10: $\sigma^2 = \frac{(b-a)^2}{12}$

↳ For a uniform density func.,

$$U(a,b) = \begin{cases} \frac{1}{(b-a)}, & a \le x \le b \\ 0, & \text{otherwise} \end{cases}$$

* mean.

$$\mu = \int_{-\infty}^{\infty} x\, p(x)\, dx$$

$$= \int_{-\infty}^{\infty} x \left(\frac{1}{b-a}\right) dx$$

$$= \int_{-\infty}^{a} x(0)\, dx + \int_{a}^{b} \frac{x}{b-a}\, dx + \int_{b}^{\infty} (0)\, dx$$

$$= \int_{a}^{b} \frac{x\, dx}{(b-a)} = \frac{1}{(b-a)} \frac{x^2}{2}\Big|_a^b$$

$$= \frac{1}{(b-a)} \frac{1}{2} (b^2 - a^2)$$

$$\boxed{\mu = \frac{b+a}{2}}$$

* from eqn 5, $\sigma^2 = E[x^2] - (E[x])^2$

and, $E[x] \equiv \mu = \int_{-\infty}^{\infty} x\, p(x)\, dx$.

$$= \frac{b+a}{2}$$

$$E[x^2] = \int_{-\infty}^{\infty} x^2 p(x)\, dx$$ → replace x with x^2 here.

$$= \int_{-\infty}^{a} (0)\, dx + \int_{a}^{b} x^2 \left(\frac{1}{b-a}\right) dx + \int_{b}^{\infty} (0)\, dx$$

$$= \frac{1}{(b-a)} \left(\frac{1}{3}\right) (b^3 - a^3) = \frac{1}{3\cancel{(b-a)}} \cancel{(b-a)}(a^2 + b^2 + ab)$$

$$\therefore \; E[x^2] = \frac{a^2+b^2+ab}{3}.$$

Now, $\sigma^2 = E[x^2] - (E[x])^2$

$$= \frac{a^2+b^2+ab}{3} - \left(\frac{b+a}{2}\right)^2$$

$$= \frac{a^2+b^2+ab}{3} - \frac{a^2+b^2+2ab}{4}$$

$$= \frac{4a^2+4b^2+4ab-3a^2-3b^2-6ab}{12}$$

$$= \frac{a^2+b^2-2ab}{12} = \frac{(a-b)^2}{12}$$

$$\therefore \; \boxed{\sigma^2 = \frac{(b-a)^2}{12}}$$

Hence, proved!

Q.3. Let's take 1 example of a normal distribution function & other of uniform distribution.

* For uniform distribution, $a \le x \le b$.

• $U(a,b) = \begin{cases} \frac{1}{b-a}, \\ 0, & \text{otherwise.} \end{cases}$

• $\mu = \frac{a+b}{2}$, $\sigma^2 = \frac{(b-a)^2}{2}$.

Let $a = 5$, $b = 3a$, $\mu = 2a$.

$\boxed{\mu = 10}$

$$\sigma^2 = \frac{(3a-a)^2}{2} = \frac{4a^2}{2} = 2a^2$$

$\boxed{\sigma^2 = 50}$

* For normal distribution,

• $N(\mu, \sigma) = \frac{1}{\sqrt{2\pi}\,\sigma} e^{-\frac{(x-\mu)^2}{2\sigma^2}}$.

• $\mu = 10$, $\sigma^2 = 50$.

For both of above, with same mean and variance, we get different plots for the distribution.

Q.4. $\boxed{\text{To Prove :}}$ Alternate expression for variance given in eq^n 5 holds for discrete random variables.

* Expressions for variance :-

- $\sigma^2 = \mathcal{E}[(x-\mu)^2]$
- $\sigma^2 = \mathcal{E}[x^2] - (\mathcal{E}[x])^2$.

* Now, expected value for a discrete r.v., x :-

- $\mathcal{E}[x] \equiv \mu = \sum_{x \in X} x P(x). = \sum_{i=1}^{n} v_i P(v_i)$.

* $\sigma^2 = \mathcal{E}[(x-\mu)^2] = \sum_{i=1}^{n} (x_i-\mu)^2\, p(x_i)$

$$= \sum_{n} (x_i^2 + \mu^2 - 2\mu x_i)\, p(x_i)$$

$$= \underbrace{\sum_n x_i^2\, p(x_i)}_{\shortparallel} + \mu^2 \underbrace{\sum_n p(x_i)}_{=1} - 2\mu \underbrace{\sum x_i\, p(x_i)}$$

$$= \mathcal{E}[x^2] + \mu^2(1) - 2\mu\, \mathcal{E}[x].$$

$$= \mathcal{E}[x^2] + (\mathcal{E}[x])^2 - (\mathcal{E}[x])^2 + \mu^2 - 2\mu\,\mathcal{E}[x]$$

{Adding & subtracting $(\mathcal{E}[x])^2$}

$$= \mathcal{E}[x^2] + (\mathcal{E}[x] - \mu)^2 - (\mathcal{E}[x])^2$$

$$= \mathcal{E}[x^2] - (\mathcal{E}[x])^2 + (\mathcal{E}[x] - \mu)^2 \to 0$$

Now, we know, $\boxed{\mathcal{E}[x] = \mu}$ (mean)

$$\therefore \boxed{\sigma^2 = \mathcal{E}[(x-\mu)^2] = \mathcal{E}[x^2] - (\mathcal{E}[x])^2}.$$

Hence proved.

Q.5. Prove : mean & variance of a normal density, $N(\mu, \sigma^2)$ are its parameters μ & σ^2.

* Normal density function is given as:-

$$N[\mu, \sigma^2] = \frac{1}{\sigma\sqrt{2\pi}}\, e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2}.$$

- mean $= \mathcal{E}(x) = \int_{-\infty}^{\infty} x(N(\mu, \sigma^2))\, dx$

$$= \int_{-\infty}^{\infty} x \left(\frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2} \right) dx.$$

$$\boxed{\text{Let } \frac{x-\mu}{\sigma} = a} \qquad \begin{aligned} dx &= da. \\ x &= a\sigma + \mu. \end{aligned}$$

$$= \frac{1}{\sigma\sqrt{2\pi}} \int_{-\infty}^{\infty} x e^{-\frac{a^2}{2}} dx.$$

$$= \frac{1}{\sigma\sqrt{2\pi}} \int_{-\infty}^{\infty} (a\sigma + \mu) e^{-a^2/2} da.$$

$$= \frac{1}{\sigma\sqrt{2\pi}} \left[\underbrace{\sigma \int_{-\infty}^{\infty} a e^{-a^2/2} da}_{I_1} + \underbrace{\mu \int_{-\infty}^{\infty} e^{-a^2/2} da.}_{I_2} \right].$$

- on solving further, we get $\boxed{I_1 = 0}$.
- for I_2, we'll use Gaussian integral.

$$\hookrightarrow \int_{-\infty}^{\infty} e^{-x^2} dx = \sqrt{\pi}.$$

$$\left\{ \begin{aligned} &\text{let } \frac{a^2}{2} = t^2 \quad \rightarrow a = \sqrt{2t} \\ &\frac{\cancel{2}ada}{\cancel{2}} = \cancel{2t}dt \\ &ada = \cancel{2t}dt \\ &da = \frac{\cancel{2t}dt}{\sqrt{2t}} \\ &= \sqrt{2}dt \end{aligned} \right.$$

$$\Rightarrow \int e^{-a^2/2} da = \int_{-\infty}^{\infty} \sqrt{2} e^{-t^2} dt.$$

$$= \sqrt{2} \int_{-\infty}^{\infty} e^{-t} dt \quad = \sqrt{\pi}$$

$$= \sqrt{2\pi}.$$

$$\therefore I_2 = \mu\sqrt{2\pi}.$$

$$\text{mean} = \frac{1}{\cancel{\sigma}(\cancel{\sqrt{2\pi}})} \left(0 + \mu\cancel{\sqrt{2\pi}}\right).$$

$$\boxed{\text{mean} = \mu}$$

Ans

* $\text{var}(x) = \mathcal{E}[(x-\mu)^2]$

$$= \int_{-\infty}^{\infty} (x-\mu)^2 \, p(x) \, dx .$$

$$= \int_{-\infty}^{\infty} (x-\mu)^2 \frac{1}{\sigma\sqrt{2\pi}} e^{-1/2\left(\frac{x-\mu}{\sigma}\right)^2}$$

Let $\frac{x-\mu}{\sigma} = a$.

$dx = da$

$\Rightarrow x = a\sigma + \mu$.

$$= \int_{-\infty}^{\infty} \frac{(a\sigma)^2}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}a^2}$$

$$= \frac{\sigma}{\sqrt{2\pi}} \underbrace{\int_{-\infty}^{\infty} a^2 e^{-a^2/2} \, da}_{}$$

an even function

$\therefore$ will be similar to $2\int_0^{\infty}(\text{func}.da)$

$$= \frac{\sqrt{2}\,\sigma}{\sqrt{\pi}} \int_0^{\infty} a^2 e^{-a^2/2} da .$$

let $t = \frac{a^2}{2}$. $\Rightarrow a = \sqrt{2t}$

$\Rightarrow$ $dt = \frac{2a\,da}{2}$.

$da = \frac{dt}{\sqrt{2t}}$.

$a^2 = 2t$.

$$= \sqrt{\frac{2}{\pi}}\,\sigma \int_0^{\infty} (2t)\, e^{-t} \frac{dt}{\sqrt{2t}}$$

$$= 2\sqrt{\frac{1}{\pi}}\,\sigma \int_0^{\infty} \sqrt{t}\, e^{-t} dt .$$

- using $\int_0^{\infty} t^b e^{-at} dt = \frac{\Gamma(b+1)}{a^{b+1}}$ (gamma func.).

- $\Gamma(1+n) = n\Gamma(n)$; $\Gamma(1/2) = \sqrt{\pi}$.

  $\therefore \Gamma(1+1/2) = \frac{1}{2}\Gamma(1/2) = \frac{\sqrt{\pi}}{2}$.

* On solving further:-

$\therefore$ $\boxed{\text{var}(x) = \sigma^2}$ Ans.