

Q.1. Example of PMF with
* {Reference for the ques. taken by probability course.com}.
(as) finite range.

Let a fair coin is tossed twice & let X be the no. of heads observed.

∴ here, sample space, $S = \{HH, HT, TH, TT\}$.
no. of heads = $\begin{matrix} \downarrow & \downarrow & \downarrow & \downarrow \\ 2 & 1 & 1 & 0 \end{matrix}$.

∴ range of X , $R_X = \{0, 1, 2\}$
→ this is finite set

* Now, the PMF for a finite set is defined as:-

$$P_X(k) = P(X=k) \text{ for } k=0,1,2 \quad \text{--- (1)}$$

$$\bullet P_X(k=0) = P(X=0) = P(TT) = 1/4.$$

$$\bullet P_X(1) = P(X=1) = P(\{HT, TH\}) = \frac{1}{4} \times 2 = 1/2$$

$$\bullet P_X(2) = P(X=2) = P(HH) = 1/4.$$

* The conditions of PMF are :-

$$\bullet \forall i, P(\omega_i) \geq 0.$$

$$\bullet \sum_{i=1}^n P(\omega_i) = \sum_{i=1}^n P_i = 1.$$

This condition is satisfied, $\therefore \forall P_X(k) > 0$. (from above)

$$\sum_{i=1}^n P(\omega_i) = P(TT) + P(HT, TH) + P(HH) = \frac{1}{4} + \frac{1}{2} + \frac{1}{4} = 1.$$

∴ this condition is also satisfied.

(b) infinite range.

Let there be an unfair coin, with $P(H) = p$, $0 < p < 1$, where the coin is tossed repeatedly until ~~one~~ heads is observed for the 1st time.
Let Y be total coin tosses.

* Here, $R_Y = \mathbb{N} = \{1, 2, 3, \dots\}$.
 \downarrow
 set of natural numbers } is an infinite range.

$$\therefore P_Y(k) = P(Y=k), \quad k = 1, 2, 3, 4, \dots$$

$$\hookrightarrow P_Y(1) = P(Y=1) = P(H) = p.$$

$$P_Y(2) = P(Y=2) = P(TH) = (1-p)p.$$

$$P_Y(3) = P(Y=3) = P(TTH) = (1-p)(1-p)p.$$

$$P_Y(k) = P(Y=k) = P(\underbrace{TTT \dots T}_{(k-1)} TH) = (1-p)^{k-1} p.$$

\therefore PMF of Y can be written as:-

$$P_Y(y) = \begin{cases} (1-p)^{y-1} p, & y = 1, 2, 3, \dots \\ 0, & \text{otherwise} \end{cases}$$

\hookrightarrow Here, also, $P(u_i) \geq 0, \forall i$

$$\begin{aligned} \sum_{i=1}^{\infty} P(u_i) &= p + (1-p)p + (1-p)^2 p + \dots + (1-p)^{k-1} p \\ &= p \left[1 + \underbrace{(1-p)} + \underbrace{(1-p)^2} + \dots + (1-p)^{k-1} \right] \end{aligned}$$

\hookrightarrow all these terms are < 1 .

So, let's consider when

$k \rightarrow \infty$.

\hookrightarrow if it forms infinite GP.

$$\therefore p < 1 \\ (1-p) < 1.$$

$$= p \left(\frac{a}{1-r} \right) = p \left(\frac{1}{(1)-(1-p)} \right) = p \left(\frac{1}{p} \right) = 1$$

\therefore Both conditions satisfied. $\boxed{\sum_{i=1}^{\infty} P(u_i) = 1}$

8.2. Equation 10: $\sigma^2 = \frac{(b-a)^2}{12}$

↳ for a uniform density func.,

$$U(a,b) = \begin{cases} \frac{1}{(b-a)}, & a \leq x \leq b \\ 0, & \text{otherwise} \end{cases}$$

* mean.

$$\mu = \int_{-\infty}^{\infty} x p(x) dx$$

$$= \int_{-\infty}^{\infty} x \left(\frac{1}{b-a} \right) dx$$

$$= \int_{-\infty}^a x(0) dx + \int_a^b \frac{x}{b-a} dx + \int_b^{\infty} (0) dx$$

$$= \int_a^b \frac{x dx}{(b-a)} = \frac{1}{(b-a)} \left. \frac{x^2}{2} \right|_a^b$$

$$= \frac{1}{(b-a)} \frac{1}{2} (b^2 - a^2)$$

$$\boxed{\mu = \frac{b+a}{2}}$$

* from eqn 5, $\sigma^2 = E[x^2] - (E[x])^2$

• and, $E[x] \equiv \mu = \int_{-\infty}^{\infty} x p(x) dx$

$$= \frac{b+a}{2}$$

$$E[x^2] = \int_{-\infty}^{\infty} x^2 p(x) dx$$

→ replace x with x^2 here.

$$= \int_{-\infty}^a (0) dx + \int_a^b x^2 \left(\frac{1}{b-a} \right) dx + \int_b^{\infty} (0) dx$$

$$= \frac{1}{(b-a)} \left(\frac{1}{3} \right) (b^3 - a^3) = \frac{1}{3(b-a)} (b-a)(a^2 + ab + b^2)$$

$$\therefore E[x^2] = \frac{a^2 + b^2 + ab}{3}$$

$$\begin{aligned} \text{Now, } \sigma^2 &= E[x^2] - (E[x])^2 \\ &= \frac{a^2 + b^2 + ab}{3} - \left(\frac{b+a}{2}\right)^2 \\ &= \frac{a^2 + b^2 + ab}{3} - \frac{a^2 + b^2 + 2ab}{4} \\ &= \frac{4a^2 + 4b^2 + 4ab - 3a^2 - 3b^2 - 6ab}{12} \\ &= \frac{a^2 + b^2 - 2ab}{12} = \frac{(a-b)^2}{12} \end{aligned}$$

$$\therefore \boxed{\sigma^2 = \frac{(b-a)^2}{12}}$$

hence, proved!

8.3. let's take 1 example of a normal distribution function & other of uniform distribution.

* for uniform distribution,

$$U(a, b) = \begin{cases} \frac{1}{b-a}, & a \leq x \leq b. \\ 0, & \text{otherwise.} \end{cases}$$

$$\mu = \frac{a+b}{2}, \quad \sigma^2 = \frac{(b-a)^2}{12}$$

let $a = 5$, $b = 3a$, $\mu = 2a$
 $\mu = 10$

$$\sigma^2 = \frac{(3a-a)^2}{12} = \frac{4a^2}{3} = 2a^2$$

$$\sigma^2 = 20$$

* for normal distribution,

$$N(\mu, \sigma) = \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{(x-\mu)^2}{2\sigma^2}}$$

$$\mu = 10, \quad \sigma^2 = 50$$

For both of above, with same mean and variance, we get different plots for the distribution.

Q.4. To Prove : Alternate expression for variance given in eqⁿ 5 holds for discrete random variables.

* Expressions for variance :-

$$\begin{aligned} & \bullet \sigma^2 = E[(x - \mu)^2] \\ \hookrightarrow & \bullet \sigma^2 = E[x^2] - (E[x])^2. \end{aligned}$$

* Now, expected value for a discrete r.v., x :-

$$\bullet E[x] \equiv \mu = \sum_{x \in X} x P(x) = \sum_{i=1}^n v_i P(v_i).$$

$$\begin{aligned}
 * \quad \sigma^2 &= E[(x-\mu)^2] = \sum_{i=1}^n (x_i - \mu)^2 p(x_i) \\
 &= \sum_n (x_i^2 + \mu^2 - 2\mu x_i) p(x_i) \\
 &= \underbrace{\sum_n x_i^2 p(x_i)}_{E[x^2]} + \mu^2 \underbrace{\left(\sum_n p(x_i)\right)}_{=1} - 2\mu \underbrace{\sum_n x_i p(x_i)}_{E[x]} \\
 &= E[x^2] + \mu^2(1) - 2\mu E[x] \\
 &= E[x^2] + (E[x])^2 - (E[x])^2 + \mu^2 - 2\mu E[x] \\
 &\quad \left\{ \text{Adding \& subtracting } (E[x])^2 \right\} \\
 &= E[x^2] + (E[x] - \mu)^2 - (E[x])^2 \\
 &= E[x^2] - (E[x])^2 + \cancel{(E[x] - \mu)^2}^0
 \end{aligned}$$

Now, we know, $E[x] = \mu$ (mean)

$$\therefore \boxed{\sigma^2 = E[(x-\mu)^2] = E[x^2] - (E[x])^2}$$

Hence, proved.

Q.5. Prove: mean & variance of a normal density, $N(\mu, \sigma^2)$ are its parameters μ & σ^2 .

* Normal density function is given as:-

$$N(\mu, \sigma^2) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2}$$

$$\bullet \text{ mean } = E(x) = \int_{-\infty}^{\infty} x(N(\mu, \sigma^2)) dx$$

$$= \int_{-\infty}^{\infty} x \left(\frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2} \right) dx.$$

$$\boxed{\text{let } \frac{x-\mu}{\sigma} = a}$$

$$dx = da \cdot \sigma$$

$$x = a\sigma + \mu$$

$$= \frac{1}{\sigma\sqrt{2\pi}} \int_{-\infty}^{\infty} x e^{-\frac{a^2}{2}} dx$$

$$= \frac{1}{\sigma\sqrt{2\pi}} \int_{-\infty}^{\infty} (a\sigma + \mu) e^{-\frac{a^2}{2}} da$$

$$= \frac{1}{\sigma\sqrt{2\pi}} \left[\underbrace{\sigma \int_{-\infty}^{\infty} a e^{-\frac{a^2}{2}} da}_{I_1} + \underbrace{\mu \int_{-\infty}^{\infty} e^{-\frac{a^2}{2}} da}_{I_2} \right]$$

• on solving further, we get $\boxed{I_1 = 0}$.

• for I_2 , we'll use Gaussian integral.

$$\hookrightarrow \int_{-\infty}^{\infty} e^{-x^2} dx = \sqrt{\pi}$$

$$\left\{ \begin{array}{l} \text{let } \frac{a^2}{2} = t^2 \\ \frac{2a da}{2} = 2t dt \\ a da = 2t dt \\ da = \frac{2t dt}{\sqrt{2t}} \\ = \sqrt{2} dt \end{array} \right.$$

$$\Rightarrow \int e^{-\frac{a^2}{2}} da = \int_{-\infty}^{\infty} \frac{1}{\sqrt{2}} e^{-t^2} dt$$

$$= \frac{1}{\sqrt{2}} \int_{-\infty}^{\infty} e^{-t^2} dt = \frac{1}{\sqrt{2}} \sqrt{\pi} = \frac{\sqrt{\pi}}{\sqrt{2}}$$

$$\therefore I_2 = \mu \frac{\sqrt{\pi}}{\sqrt{2}}$$

$$\text{mean} = \frac{1}{\sigma\sqrt{2\pi}} (0 + \mu \sqrt{\pi})$$

$$\boxed{\text{mean} = \mu}$$

$$* \text{Var}(x) = E[(x-\mu)^2]$$

$$= \int_{-\infty}^{\infty} (x-\mu)^2 p(x) dx$$

$$= \int_{-\infty}^{\infty} (x-\mu)^2 \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2} dx$$

$$= \int_{-\infty}^{\infty} \frac{(a\sigma)^2}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}a^2} da$$

$\left\{ \begin{array}{l} \text{let } \frac{x-\mu}{\sigma} = a \\ dx = da \\ \Rightarrow x = a\sigma + \mu \end{array} \right.$

$$= \frac{\sigma}{\sqrt{2\pi}} \int_{-\infty}^{\infty} a^2 e^{-\frac{1}{2}a^2} da$$

an even function
 \therefore will be similar to $2 \int_0^{\infty} (\text{func.} \cdot da)$

$$= \frac{\sqrt{2}\sigma}{\sqrt{\pi}} \int_0^{\infty} a^2 e^{-\frac{1}{2}a^2} da$$

$$= \sqrt{\frac{2}{\pi}} \sigma \int_0^{\infty} (2t) e^{-t} \frac{dt}{\sqrt{2t}}$$

$$= 2\sqrt{\frac{1}{\pi}} \sigma \int_0^{\infty} \sqrt{t} e^{-t} dt$$

$$\left\{ \begin{array}{l} \text{let } t = \frac{a^2}{2} \Rightarrow a = \sqrt{2t} \\ \Rightarrow dt = \frac{2a}{2} da \\ da = \frac{dt}{\sqrt{2t}} \\ a^2 = 2t \end{array} \right.$$

• using $\int_0^{\infty} t^b e^{-at} dt = \frac{\Gamma(b+1)}{a^{b+1}}$ (gamma func.)

• $\Gamma(1+n) = n\Gamma(n)$; $\Gamma(1/2) = \sqrt{\pi}$

$$\therefore \Gamma(1+1/2) = \frac{1}{2} \Gamma(1/2) = \frac{\sqrt{\pi}}{2}$$

* On solving further:-

$$\therefore \boxed{\text{Var}(x) = \sigma^2} \text{ Ans.}$$

Assignment-1

2020102004

Programming based problems

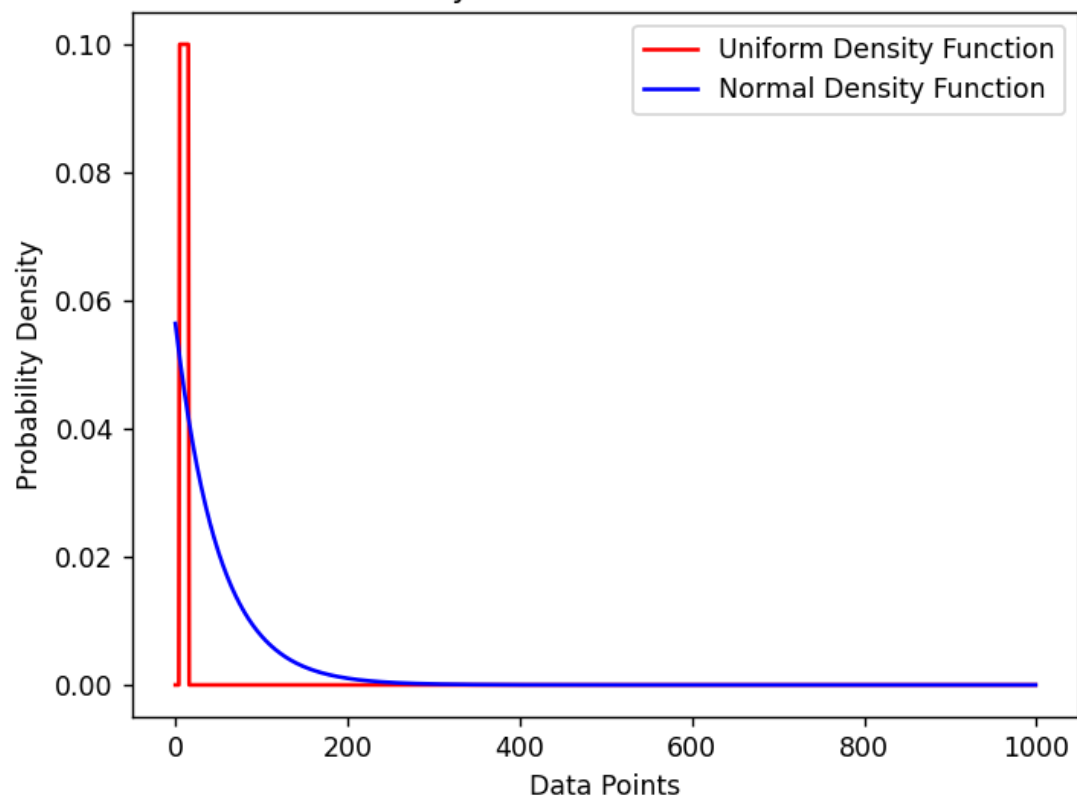
Question-3

Let's take an example of normal distribution and uniform distribution, with mean = 10 and variance = 50

```
1  import numpy as np
2  from matplotlib import pyplot as plt
3
4  mean = 10
5  variance = 50
6  quantile = np.arange(0, 1000)
7
8  ## Uniform distribution
9  a = 5
10 b = 3*a
11 udf = np.zeros(len(quantile))
12 a_idx = quantile.tolist().index(a)
13 b_idx = quantile.tolist().index(b)
14 udf[a_idx:b_idx+1] = (1/(b-a))
15
16 ## Normal distribution
17 ndf = np.exp(-(quantile**2)/(2*variance))*(1/(2*3.1415*variance)**0.5)
18
19 ## Plotting
20 plt.plot(quantile, udf, color = 'red', label = 'Uniform Density Function')
21 plt.plot(quantile, ndf, color = 'blue', label = 'Normal Density Function')
22 plt.xlabel("Data Points")
23 plt.ylabel("Probability Density")
24 plt.title("Uniform and Normal Density functions with Mean = {} and Variance = {}".format(mean, variance))
25 plt.legend()
26 plt.show()
```

PLOT

Uniform and Normal Density functions with Mean = 10 and Variance = 50



Question-6

Code snippets are as follows:

```
1  import math
2  import numpy as np
3  from matplotlib import pyplot as plt
4  from scipy.special import erfinv
5  from scipy.special import erf
6
7  N = 10000
8  data_points = np.random.uniform(0, 1, N)
9
```

```

10 # a. Normal Density
11 sigma1 = 3
12 mu = 0
13 normal_icdf = sigma1*math.sqrt(2)*erfinv(2*data_points -1) + mu
14
15 # b. Rayleigh Density
16 sigma2 = 1.0
17 rayleigh_icdf = np.sqrt((2*( sigma2 )**2)* np.log(1/(1 - data_points)) )
18
19 # c. Exponential Density
20 lambd = 1.5
21 exponential_icdf = -np.log(1 - data_points)/lambd

```

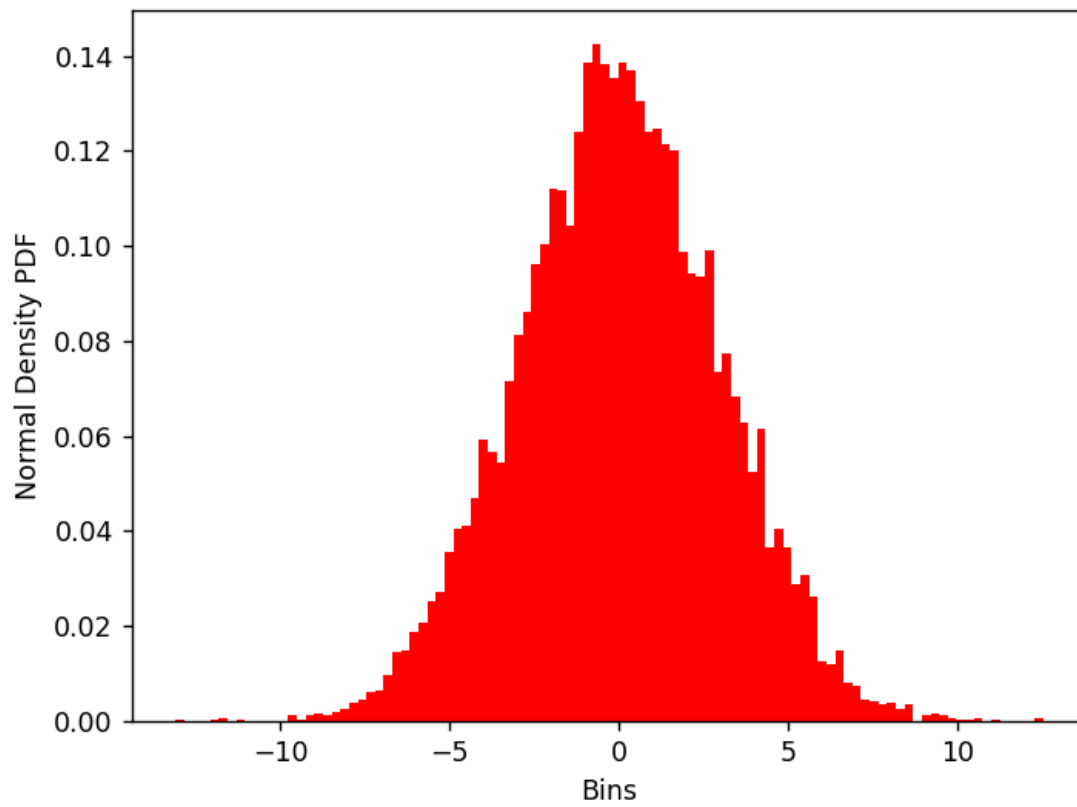
```

23 ## plot
24 plt.hist(normal_icdf, bins = 100, density = True, color = 'red', label = "Normal Density")
25 plt.xlabel("Bins")
26 plt.ylabel("Normal Density PDF")
27 plt.show()
28
29 plt.hist(rayleigh_icdf, bins =100, density = True, color = 'blue', label = "Rayleigh Density")
30 plt.xlabel("Bins")
31 plt.ylabel("Rayleigh Density PDF")
32 plt.show()
33
34 plt.hist(exponential_icdf, bins =100, density = True, color = 'green', label = "Exponential Density")
35 plt.xlabel("Bins")
36 plt.ylabel("Exponential Density PDF")
37 plt.show()

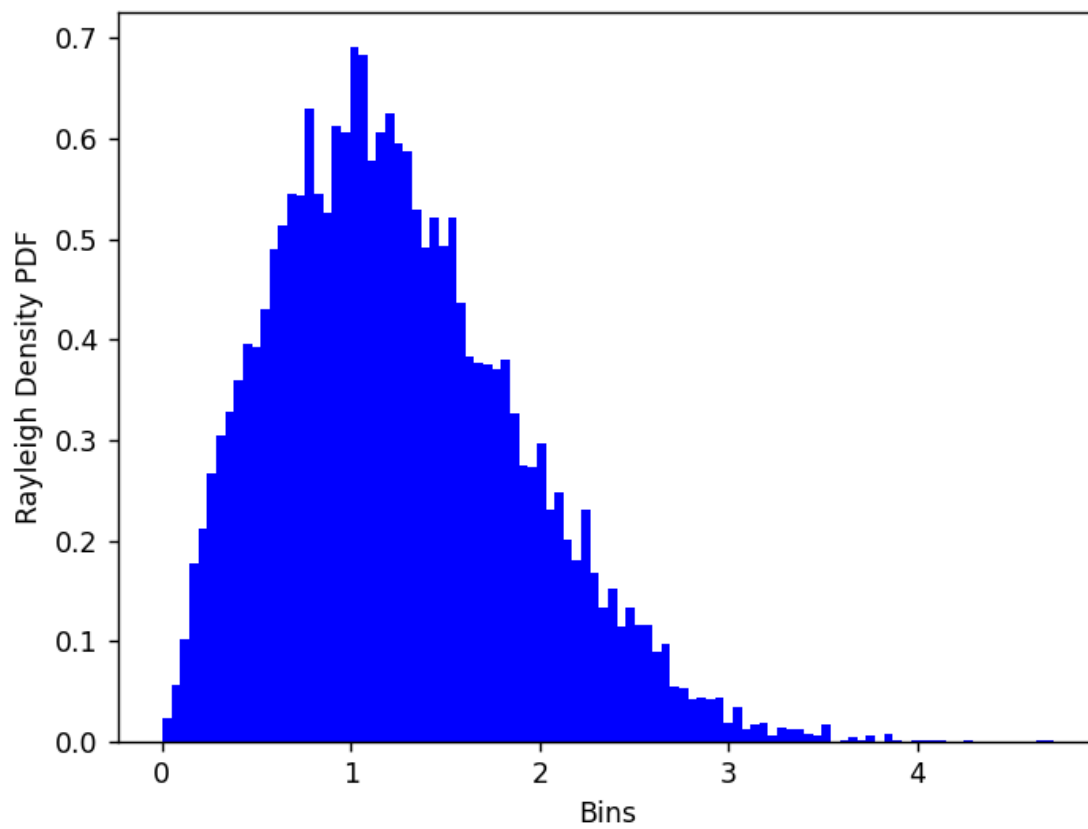
```

Plots are as follows:

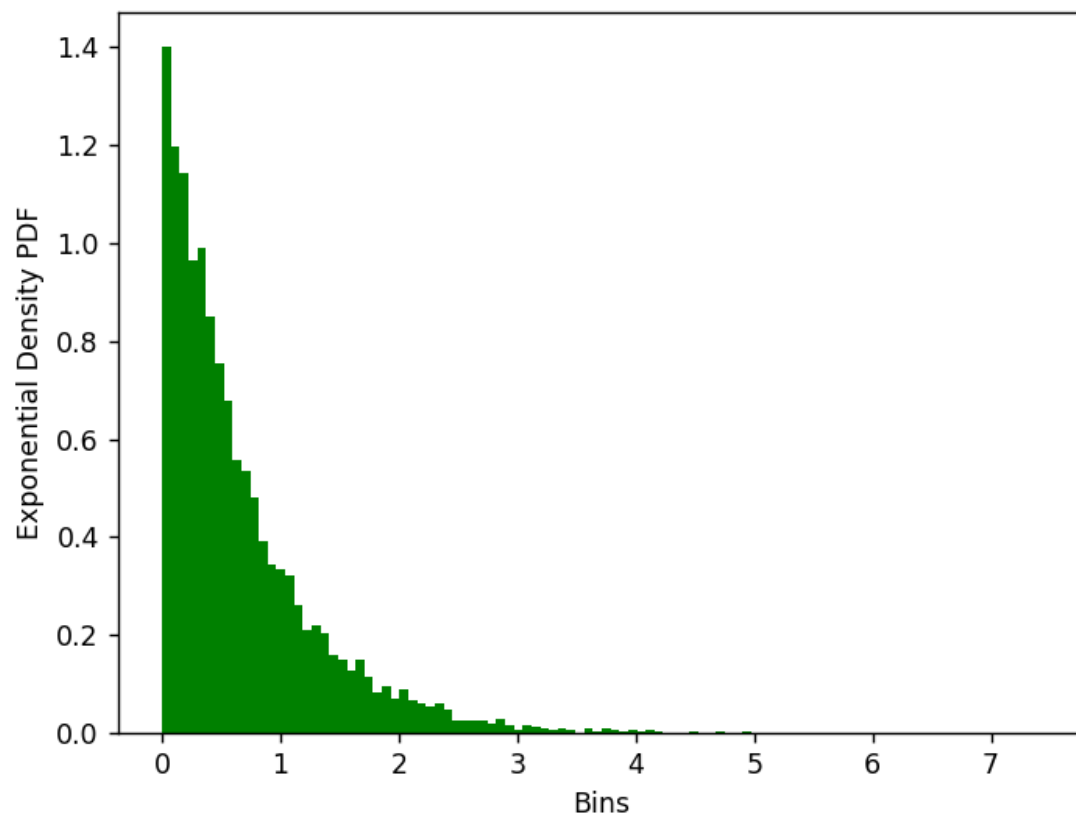
- a. Normal Density, mean = 0, standard deviation = 3



b. Rayleigh Density, standard deviation = 1



c. Exponential Density, Lambda = 1.5



Inference: Inverse CDF can be used in mapping $U[0, 1]$ to any set of random variables X with a PDF.

Question-7

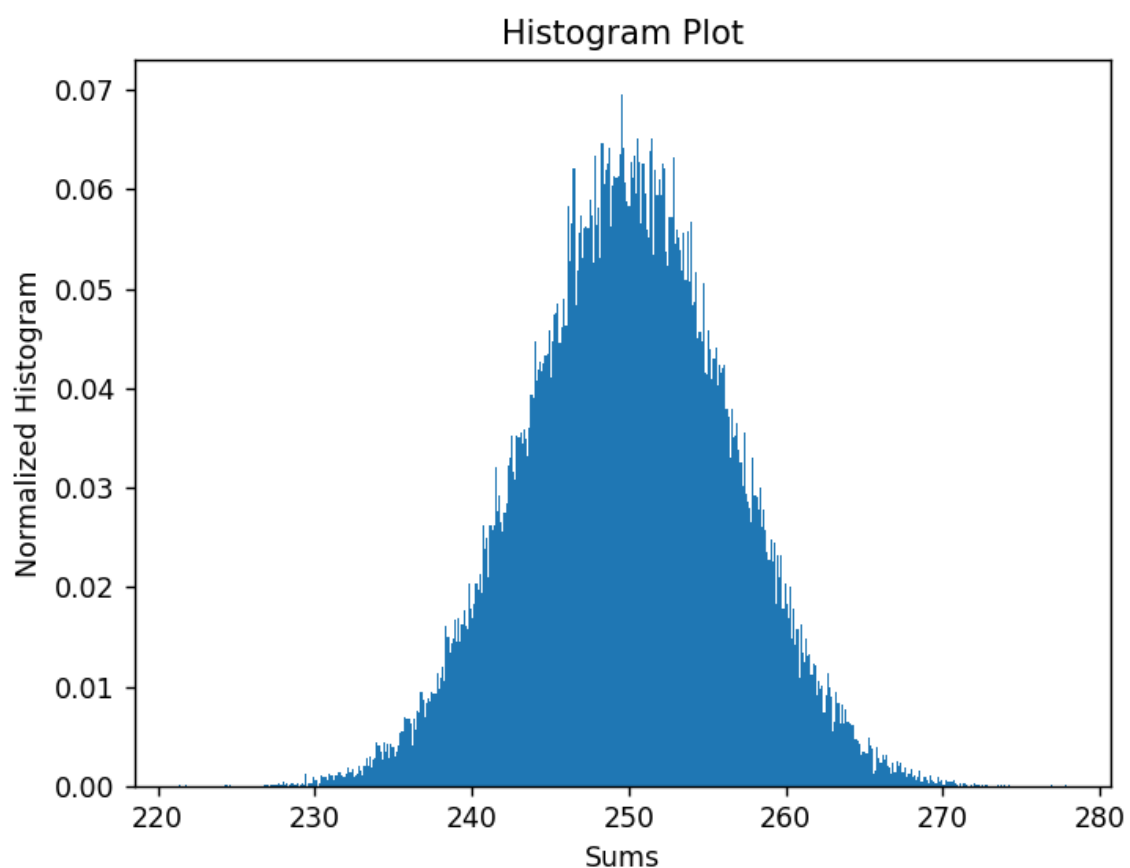
Code snippet

```

1  import numpy as np
2  import matplotlib.pyplot as plt
3  from numpy import random
4  from numpy.random import *
5
6  numGenerate = 500
7  N = 50000
8
9  sum_arr = [np.sum(np.random.uniform(0,1,500)) for i in range(N)]
10
11 plt.hist(sum_arr, bins = 500, density = True)
12 plt.title("Histogram Plot")
13 plt.xlabel("Sums")
14 plt.ylabel("Normalized Histogram")
15 plt.show()

```

PLOT



Inference: The above histogram plot looks similar to the one with a function of normal distribution, also suggesting the mean to lie somewhere in between the middle just like in the normal density function plot.