1

Matrix identities used are at the back (Page 4)

$y_{ij} | \theta_{ij} \sim \text{Poisson}\,(e^{\theta_{ij}})$

$\underline{\theta}_i \mid \underline{u}_i \sim N(\mu + \Lambda u_i, \Psi)$

$\underline{u}_i \sim N(\underline{0}, I_q)$

$$f(y_i, u_i, \underline{\theta}_i) \propto \prod_{j=1}^{d} e^{\theta_{ij} y_{ij}} e^{-e^{\theta_{ij}}} \exp\left\{-\frac{1}{2}(\underline{\theta}_i - \mu - \Lambda u_i)' \Psi^{-1}(\underline{\theta}_i - \mu - \Lambda u_i)\right\} \exp\left\{-\frac{1}{2}(\underline{u}_i' u_i)\right\}$$

$$f(\underline{\theta}_i | y_i, \underline{u}_i) \propto \prod_{j=1}^{d} e^{\theta_{ij} y_{ij}} e^{-e^{\theta_{ij}}} \cdot \exp\left\{-\frac{1}{2}(\underline{\theta}_i' \Psi^{-1} \underline{\theta}_i - 2\,\underline{\theta}_i' \Psi^{-1}(\mu + \Lambda \underline{u}_i)\right\}$$

$$f(\underline{u}_i | \underline{\theta}_i, y_i) \propto \exp\left\{-\frac{1}{2}\left(-2\,\underline{u}_i' \Lambda' \Psi^{-1}(\underline{\theta}_i - \mu) + \underline{u}_i' \Lambda' \Psi^{-1} \Lambda \underline{u}_i + \underline{u}_i' \underline{u}_i\right)\right\}$$

$$= \exp\left\{-\frac{1}{2}\left(-2\,\underline{u}_i' \Lambda' \Psi^{-1}(\underline{\theta}_i - \mu) + \underline{u}_i' (\Lambda' \Psi^{-1} \Lambda + I)\,\underline{u}_i\right)\right\}$$

..... functional form of multivariate normal distribution.

$\therefore \text{var}(\underline{u}_i | \underline{\theta}_i) = (\Lambda' \Psi^{-1} \Lambda + I)^{-1}$

$= (I + \Lambda' \Psi^{-1} \Lambda)^{-1}$

Using identity # 1, we get

(2)

$$= I^{-1} - I^{-1}\Lambda'(\Psi + \Lambda I^{-1}\Lambda')^{-1}\Lambda$$

$$= I - \Lambda'(\Psi + \Lambda\Lambda')^{-1}\Lambda$$

$$= I - \Lambda'\Sigma^{-1}\Lambda$$

Say $\Lambda'\Sigma^{-1} = \beta$

$$= I - \beta\Lambda$$

For expected value of $\underline{v}_i$, we have

$$-2\,\underline{v}_i'\,\Lambda'\,\Psi^{-1}\,(\underline{\theta}_i - \mu)$$

$$= -2\,\underline{v}_i'\,(\mathrm{var}(v_i))^{-1}\,\mathrm{var}(v_i)\,\Lambda'\Psi^{-1}(\underline{\theta}_i - \mu)$$

$$= -2\,\underline{v}_i'\,(\Lambda'\Psi^{-1}\Lambda + I)(\Lambda'\Psi^{-1}\Lambda + I)^{-1}\Lambda'\Psi^{-1}(\underline{\theta}_i - \mu)$$

$$\therefore\ E[\underline{v}_i \mid \theta_i] = \underbrace{(\Lambda'\Psi^{-1}\Lambda + I)^{-1}\Lambda'\Psi^{-1}}_{\text{See simplifying this in the next page}}(\underline{\theta}_i - \mu)$$

$$= \Lambda'\Sigma^{-1}(\theta_i - \mu)$$

$$= \beta(\theta_i - \mu)$$

$$\therefore\ v_i \mid \theta_i \sim N\underbrace{(\beta(\theta_i - \mu),\ I - \beta\Lambda)}_{\text{Same as factor analyzers.}}$$

3

$$(\Lambda' \Psi^{-1}\Lambda + I)^{-1}\Lambda'\Psi^{-1}$$

$$= (I + \underbrace{\Lambda'}_{A} \underbrace{\Psi^{-1}\Lambda}_{B})^{-1} \underbrace{\Lambda'}_{A} \Psi^{-1}$$

Using identity # 2

$$= \Lambda'(I + \Psi^{-1}\Lambda\Lambda')^{-1}\Psi^{-1}$$

Since Ψ is invertible

$$= \Lambda' (\Psi^{-1}\Psi(I + \Psi^{-1}\Lambda\Lambda'))^{-1}\Psi^{-1}$$

$$= \Lambda' (\Psi^{-1}(\Psi + \Lambda\Lambda'))^{-1}\Psi^{-1}$$

$$= \Lambda'(\Psi^{-1}\Sigma)^{-1}\Psi^{-1}$$

Using identity # 3

$$= \Lambda' \Sigma^{-1} \underbrace{\Psi \Psi^{-1}}_{I}$$

$$= \Lambda'\Sigma^{-1}$$

$$= \Lambda'(\Psi + \Lambda\Lambda')^{-1}$$

Matrix identities Used:

1. Woodbury identity

$$(A + UCV)^{-1} = A^{-1} - A^{-1} U (C^{-1} + VA^{-1}U)^{-1} VA^{-1}$$

2. $A + ABA = A(I + BA) = (I + AB)A$

$$\therefore \underbrace{A(I+BA)(I+BA)^{-1}}_{I} = (I + AB)A(I+BA)^{-1}$$

$$A = (I+AB)A(I+BA)^{-1}$$

$$(I+AB)^{-1}A = \underbrace{(I+AB)^{-1}(I+AB)}_{I} A (I+BA)^{-1}$$

$$\boxed{\therefore (I + AB)^{-1} A = A(I + BA)^{-1}}$$

3. $(AB)^{-1} = B^{-1}A^{-1}$

Note: I is an identity matrix

$\therefore I^{-1} = I$

&

$AI = A$