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UNIVERSITÄT
MAGDEBURG

INF

FAKULTÄT FÜR
INFORMATIK

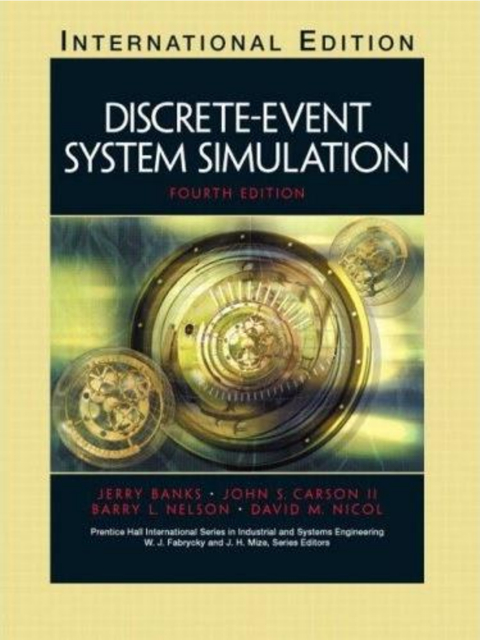
Introduction to Simulation

Output Analysis

Background Reading

Relevant sections of the book:

- 11.1
- 11.2
- 11.3
- 11.4 (parts)
- 11.5 (parts)



A Question

Let X be the random variable

- "Result from throwing a die"

Question: What is the expected value $E(X)$?

Would you throw the die just once and take the result as your answer?

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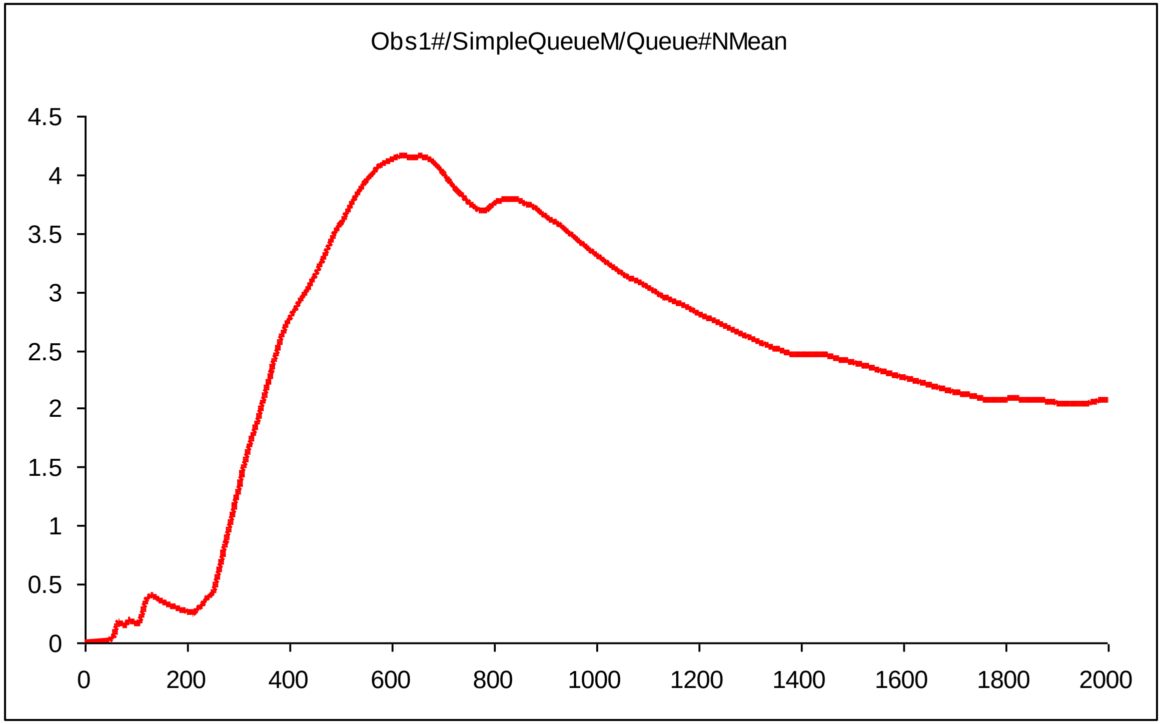
What Are We Doing Wrong?

We have been leaving something out in our simulations

- We have made one observation only
- Example – the average length of the queue in the bank:



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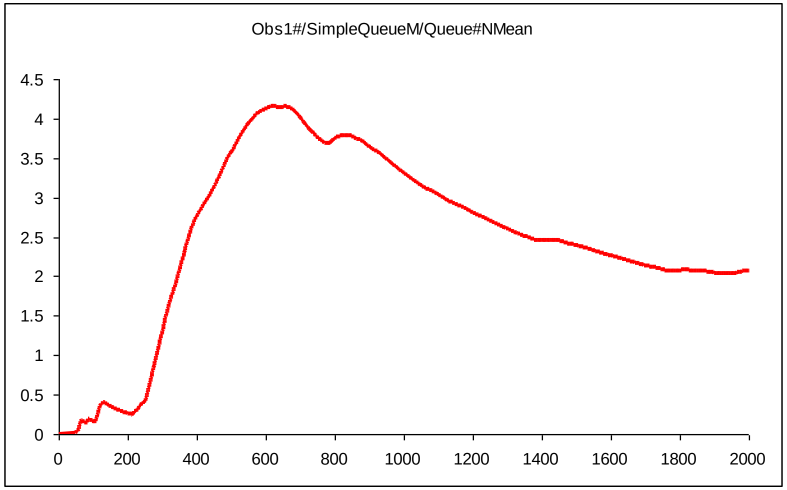
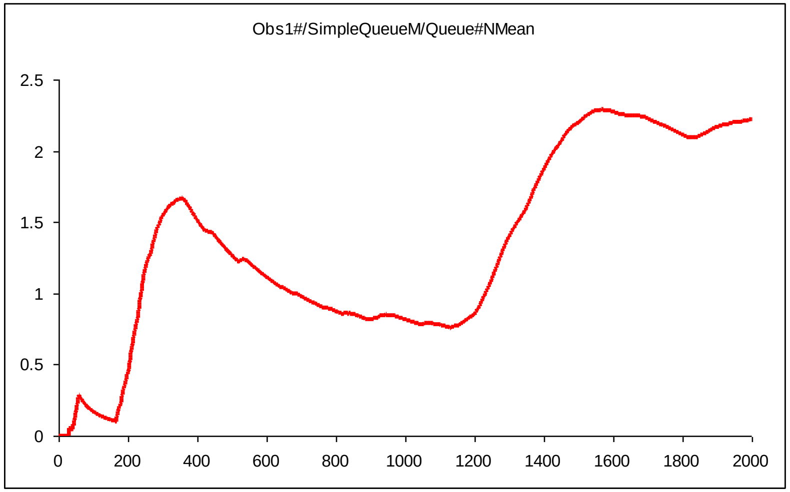
Example

Consider our bank example

- Observe the average queue length $0 < T < 2000$
- Simulate with different sets of random numbers (with the same distribution!)





Example

Consider our bank example

- Observe the queue length at $T = 2000$
- Repeat the simulation using different sets of random numbers

Results:

- 2, 12, 3, 7, 0, 10, 2, 5

Which is the "right" answer?

What Are We Doing Wrong?

Real systems behave randomly

- They contain random variables

Our simulation results are also random

- They depend on random numbers

Running a simulation means taking one sample of a RV

- $\Rightarrow$ We need a more sophisticated approach!

Mean & Sample Variance

Consider a random variable Y

Take a set of observations $Y_i \; i = 1 \ldots n$

The *Sample Mean* is defined as: $\bar{Y} = \frac{1}{n} \sum Y_i$

The *Sample Variance* is defined as: $S^2 = \frac{1}{n-1} \sum (Y_i - \bar{Y})^2$

Then $S^2 \approx \mathrm{var}(Y)$

(The sample variance approximates the actual variance in Y)

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Bias

Given a value θ and an estimator for it $\hat{\theta}$

In general, we may have $E(\hat{\theta}) = \theta + b$
(i.e. the estimator may be biased)

Bias means we have a systematic error

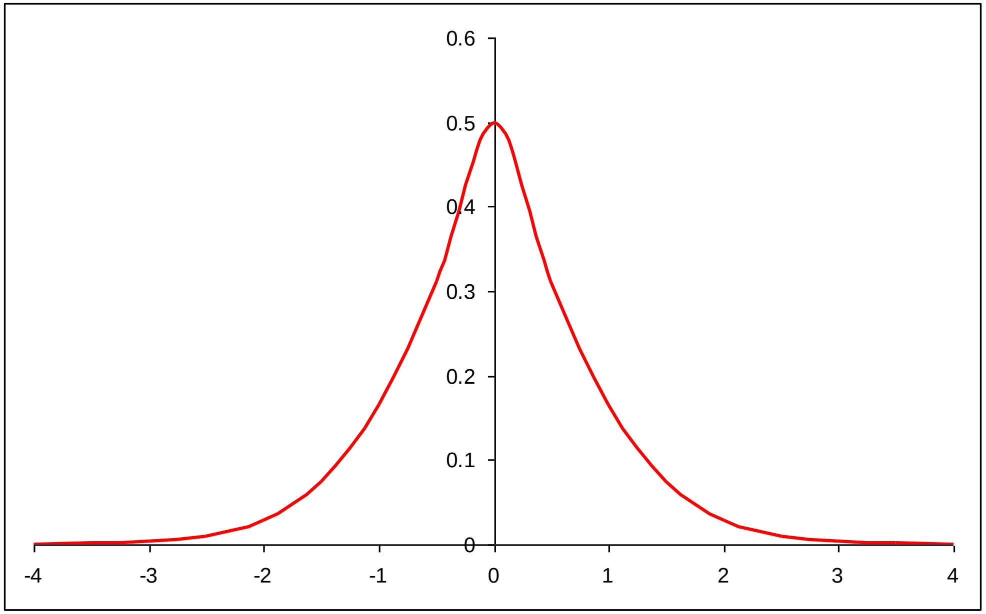
- Our estimate will be too large / too small

Many statistical methods assume (require) that $b = 0$

Student's *t*–Distribution

Introduced by W. Gosset (a.k.a. "Student")

- Has one parameter *f* ("degrees of freedom")
- Used for hypothesis testing
- Tables of values are available



W. Gosset
1876-1937

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Student's *t*–Distribution

Given:

- A measure of the real system θ
- An estimator $\hat{\theta}$ for θ
- An estimator $\hat{\sigma}(\hat{\theta})$ for $\sigma(\theta)$

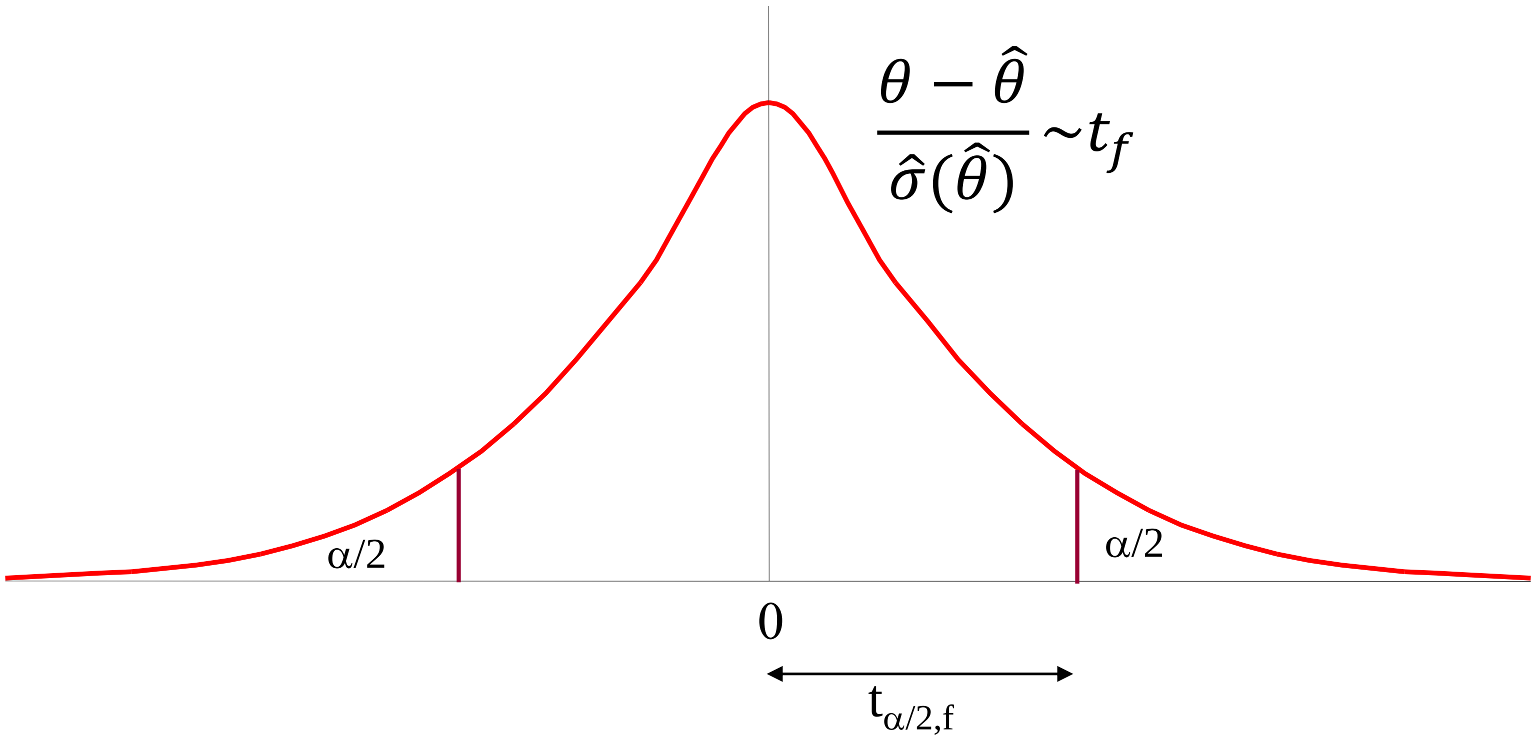
If $\theta = E(\hat{\theta})$ then the value

$$t = \frac{\theta - \hat{\theta}}{\hat{\sigma}(\hat{\theta})}$$

is *t*–distributed with n – 1 d.o.f.

Confidence Intervals

What are we doing?



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Confidence Intervals

Choose a level of significance α

Rearrange the expression

$$0 - t_{\alpha/2,f} \leq \frac{\theta - \hat{\theta}}{\hat{\sigma}(\hat{\theta})} \leq 0 + t_{\alpha/2,f}$$

to obtain the confidence interval

$$\hat{\theta} - \hat{\sigma}(\hat{\theta}) \cdot t_{\alpha/2,f} \leq \theta \leq \hat{\theta} + \hat{\sigma}(\hat{\theta}) \cdot t_{\alpha/2,f}$$

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Confidence Intervals

What does the confidence interval mean?

$$\hat{\theta} - \hat{\sigma}(\hat{\theta}) \cdot t_{\alpha/2,f} \leq \theta \leq \hat{\theta} + \hat{\sigma}(\hat{\theta}) \cdot t_{\alpha/2,f}$$

The value of θ lies in the c.i. with certainty $1-\alpha$

This is the preferred way to present the results of a simulation

Reminder:

- θ is the (theoretical) output of the simulation model
- $\hat{\theta}$ is the result of a (finite) simulation experiment

Confidence Intervals

How to obtain $\hat{\sigma}(\hat{\theta})$?

Answer: Use the approximation $$\frac{S}{\sqrt{n}} \approx \hat{\sigma}(\hat{\theta})$$

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Confidence Intervals

Improving accuracy:

We have $$\frac{S}{\sqrt{n}} \approx \hat{\sigma}(\hat{\theta})$$

If we wish to halve the width of the the c.i. ...

- … we must use $4n$ samples!

This can imply:

- High accuracy can be expensive in simulation!

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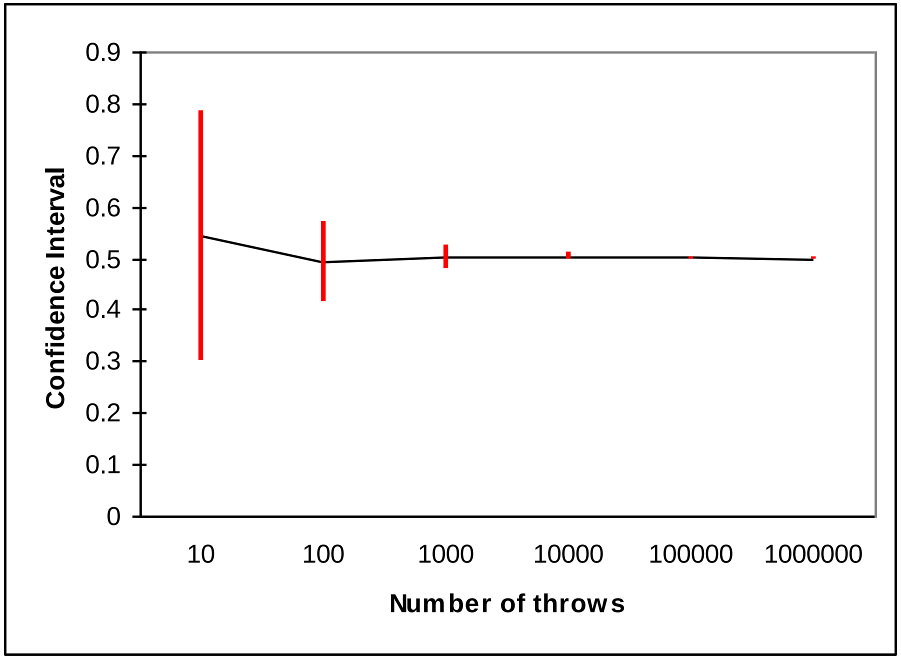
Confidence Intervals

Example: Throwing a coin (interpret the result as 0 or 1)

- θ is the (true, theoretical) expected value
- $\hat{\theta}$ is the result of a (finite) experiment

# Throws	Lower	Mean	Upper
10	0.302	0.545	0.788
100	0.416	0.493	0.570
1000	0.479	0.503	0.527
10000	0.496	0.503	0.511
100000	0.499	0.501	0.503
1000000	0.499	0.500	0.501



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Confidence Intervals

A confidence interval is only accurate if:

- $\hat{\theta}$ is an unbiased estimator of θ
- $\hat{\sigma}^2(\hat{\theta})$ is an unbiased estimator of $\sigma^2(\hat{\theta})$

If the observations are not independent...

- ... then the estimator will be biased
- … the confidence interval will be shifted

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Example

Consider a simple queue

- One arrival stream
- One server

Intervals:

- Arrivals: Exponential distribution, mean = 13s
- Service: Normal distribution, $\mu = 10$s, $\sigma = 2$s

How does the queue length behave over time?

Simulation experiment:

- 100 samples (replications), $\alpha = 0.05$

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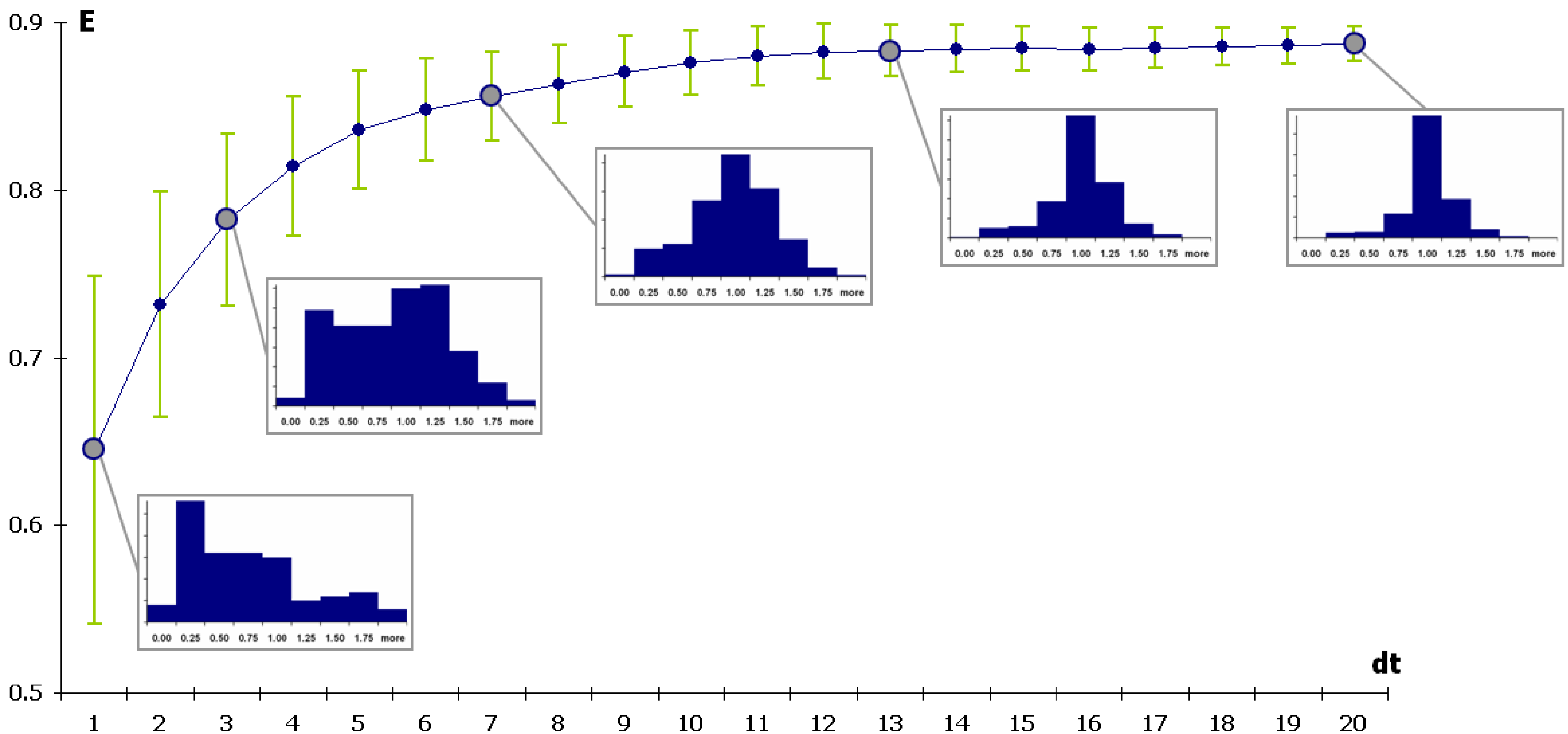
Example



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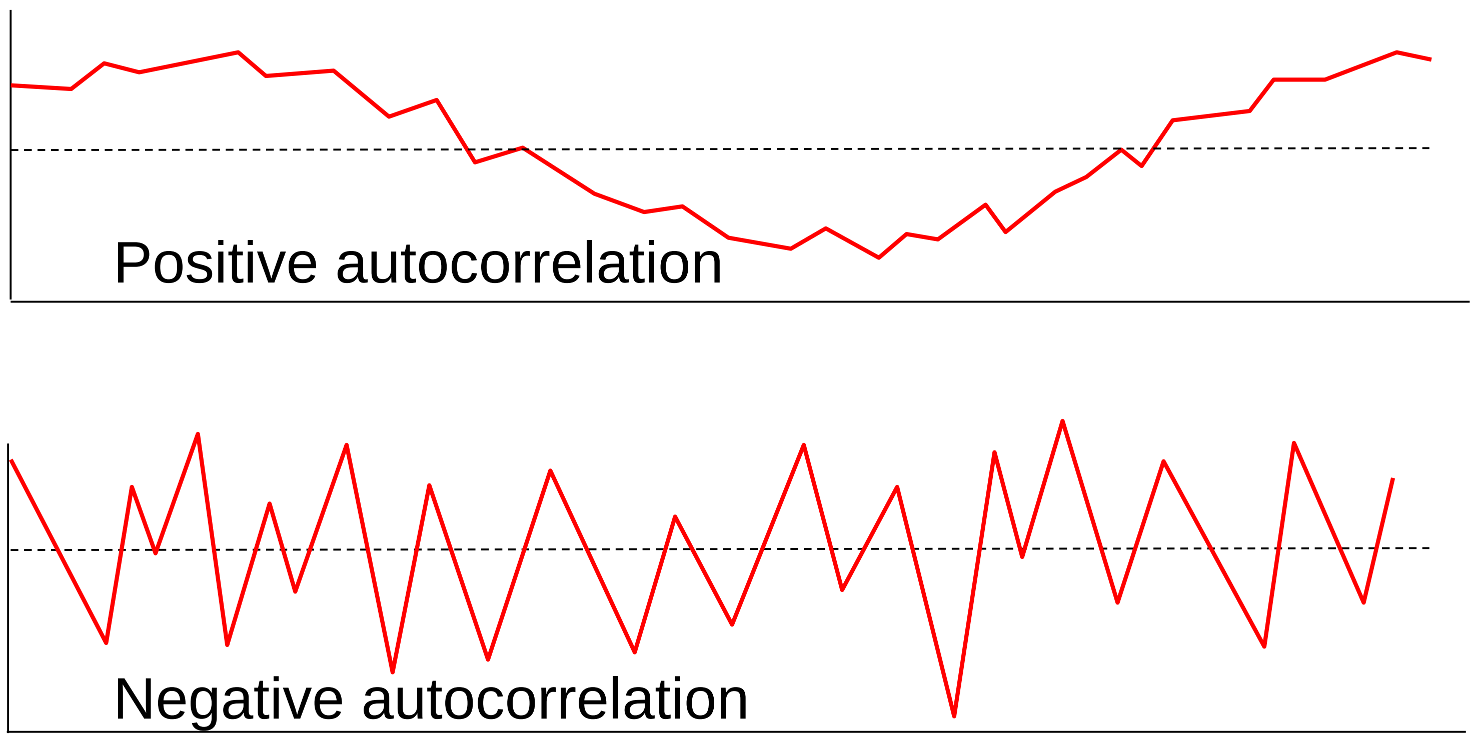
Bias due to Autocorrelation

Consider the Y_i from one simulation run (time series):



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Bias due to Autocorrelation

Positive autocorrelation ...

- leads to under–estimation of S^2
- leads to over–optimistic confidence intervals
- Example: Most queues

Negative autocorrelation ...

- leads to over–estimation of S^2
- leads to under–optimistic confidence intervals
- Example: Some inventory systems

$\Rightarrow$ Don't use time series for confidence intervals!

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Independent Replications

The method of independent replications:

- Run the simulation R times
- Use independent sets of random numbers for each run
- Make the observations $Y_r,\ r = 1 \ldots R$
- Compute $\hat{\theta}$ and S^2 from the Y_r
- Compute a confidence interval from $\hat{\theta}$ and S^2

Independence ensures that $\hat{\theta}$ is unbiased

Terminating Simulations

A terminating simulation is one...

- which runs up to a specified time
- which has known initial conditions
- in which the initial conditions are important

Examples:

- Will the satellite survive for 5 years?
- How full is the bank 2 hours after opening time?

Non–Terminating Simulations

A non–terminating simulation is one...

- which runs for an indefinite period
- in which the steady–state behaviour is of interest
- in which the initial conditions are not important

Examples:

- Any continuously running system
- Computer centre, traffic system, ...

Non–Terminating Simulations

Difficulties with non–terminating simulations:

- Initial bias
- How long to run the simulation?
- Trade off between replication and duration

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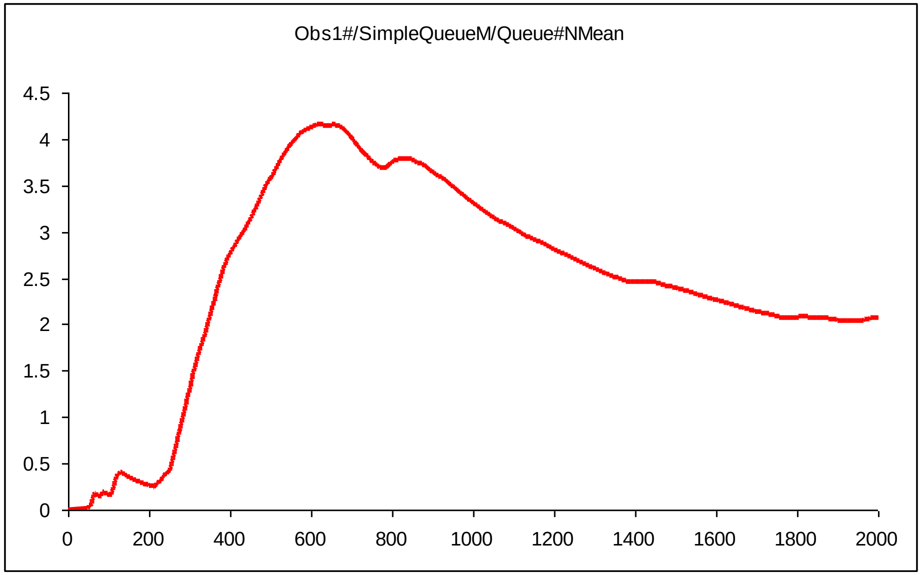
Initial Bias

In a non–terminating simulation

- We are interested in the steady–state behaviour
- The values at the beginning will usually be untypical ("initial bias")
- Example: Queue in bank (starting empty):

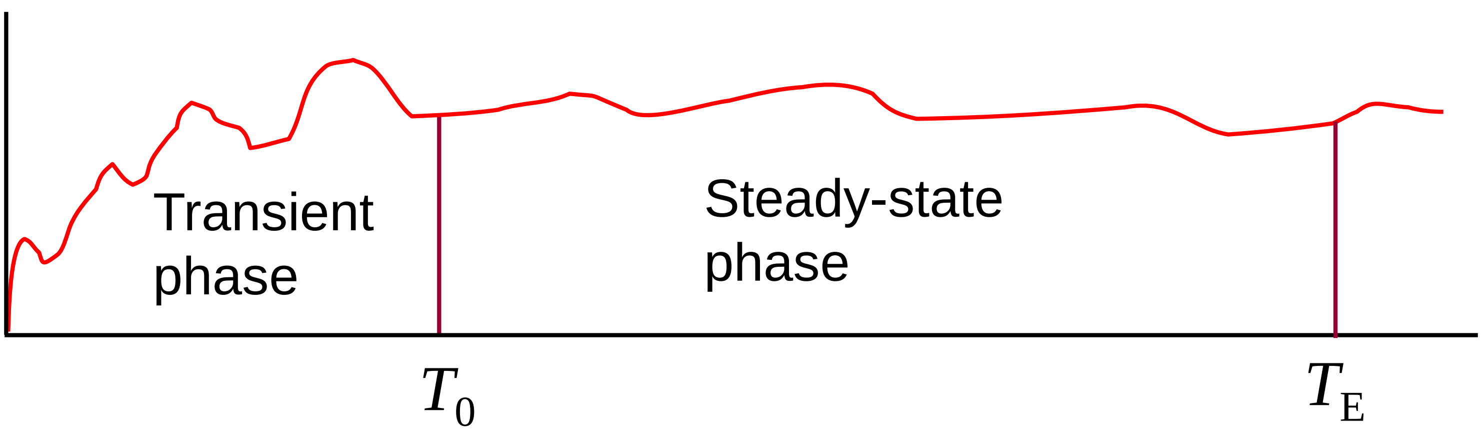


Initial Bias

Solution:

- Delete values from transient phase
- Choose T_0 after transient phase is over
- Observe from T_0 to T_E

How to choose T_0 (and T_E)?



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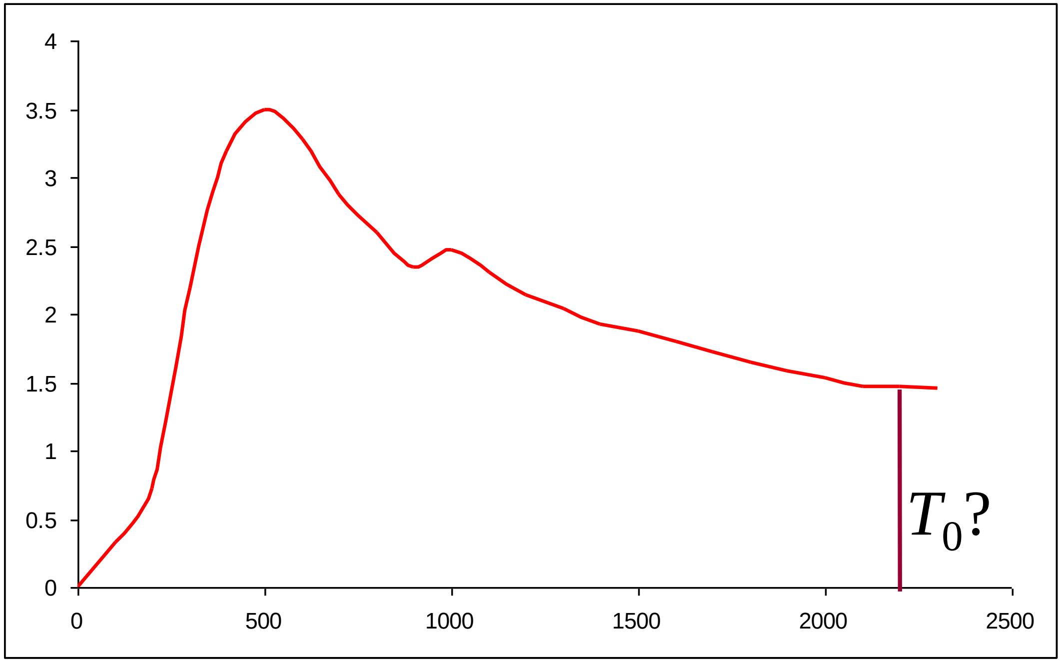
Initial Bias

How long is the transient phase in a non–terminating simulation?



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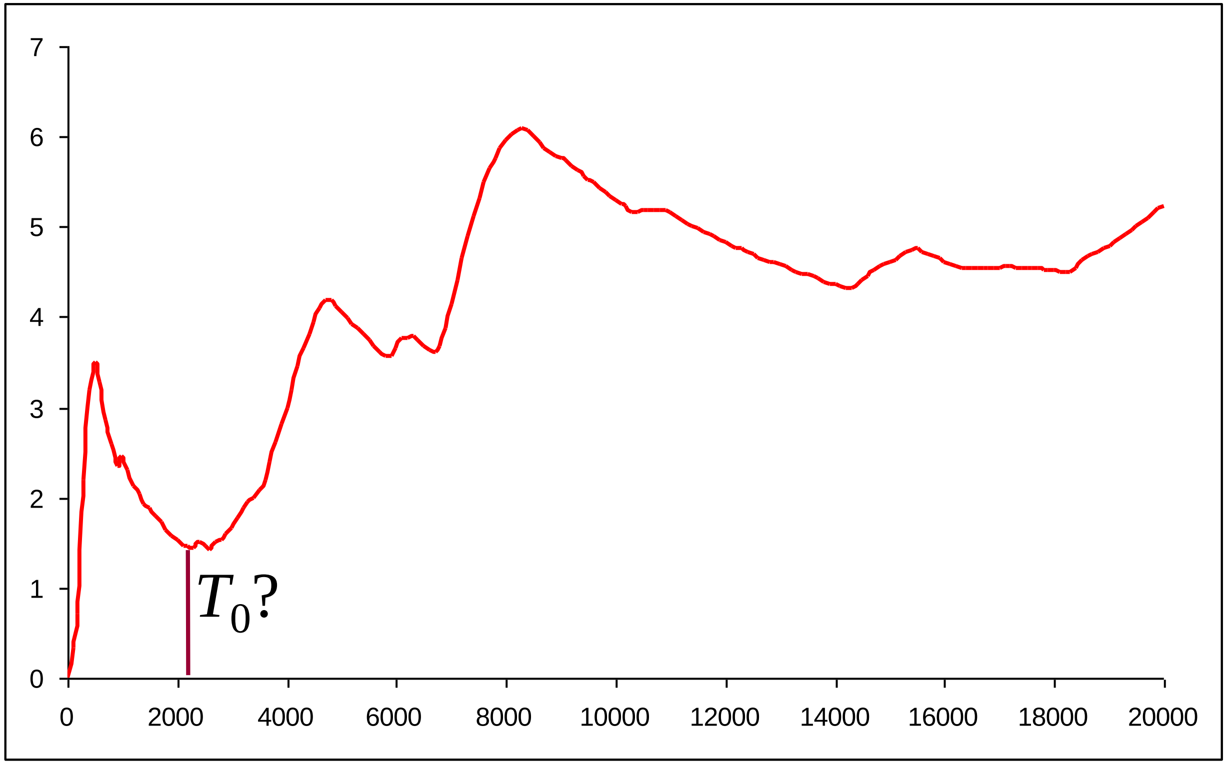
Initial Bias

A difficult question!



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Ensemble Averages

Using just one run to find T_0 is dangerous

- Compute ensemble averages

Perform independent replications to obtain

$$Y_{r,i} \quad i = 1 \ldots n, \; r = 1 \ldots R$$

Compute average values across replications

$$Y_i = \frac{1}{R} \sum_{r=1}^{R} Y_{r,i} \quad i = 1 \ldots n$$

Test the sequence Y_i for the end of the transient phase

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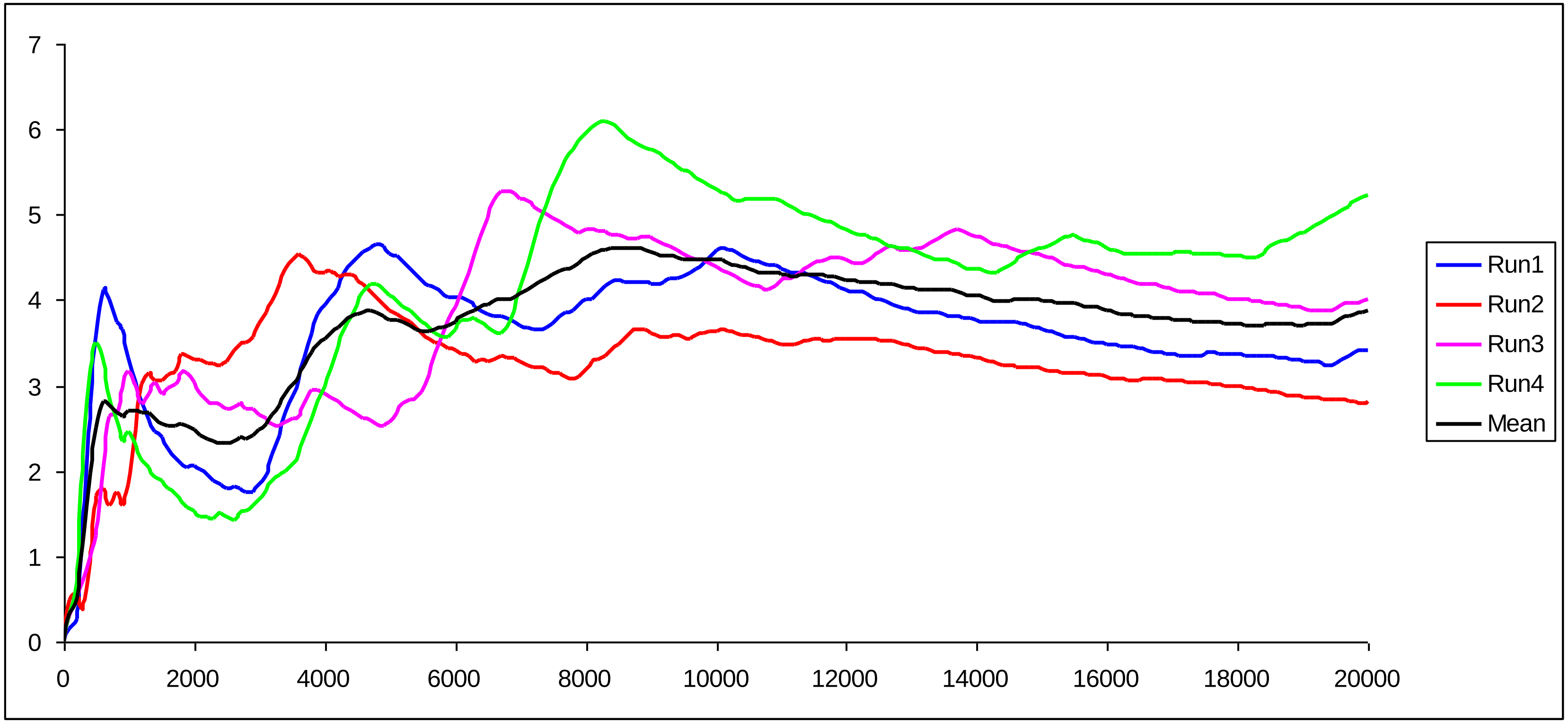
Ensemble Averages

Compute ensemble averages:



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Non–Terminating Simulations

Increasing R will make the c.i. narrower

It will not reduce the initial bias, i.e.

- ...we will get a better c.i. around $\theta + b$!

A computing time tradeoff is necessary between

- ...increasing T_E and increasing R

Solution:

- $R > 25$ is not useful

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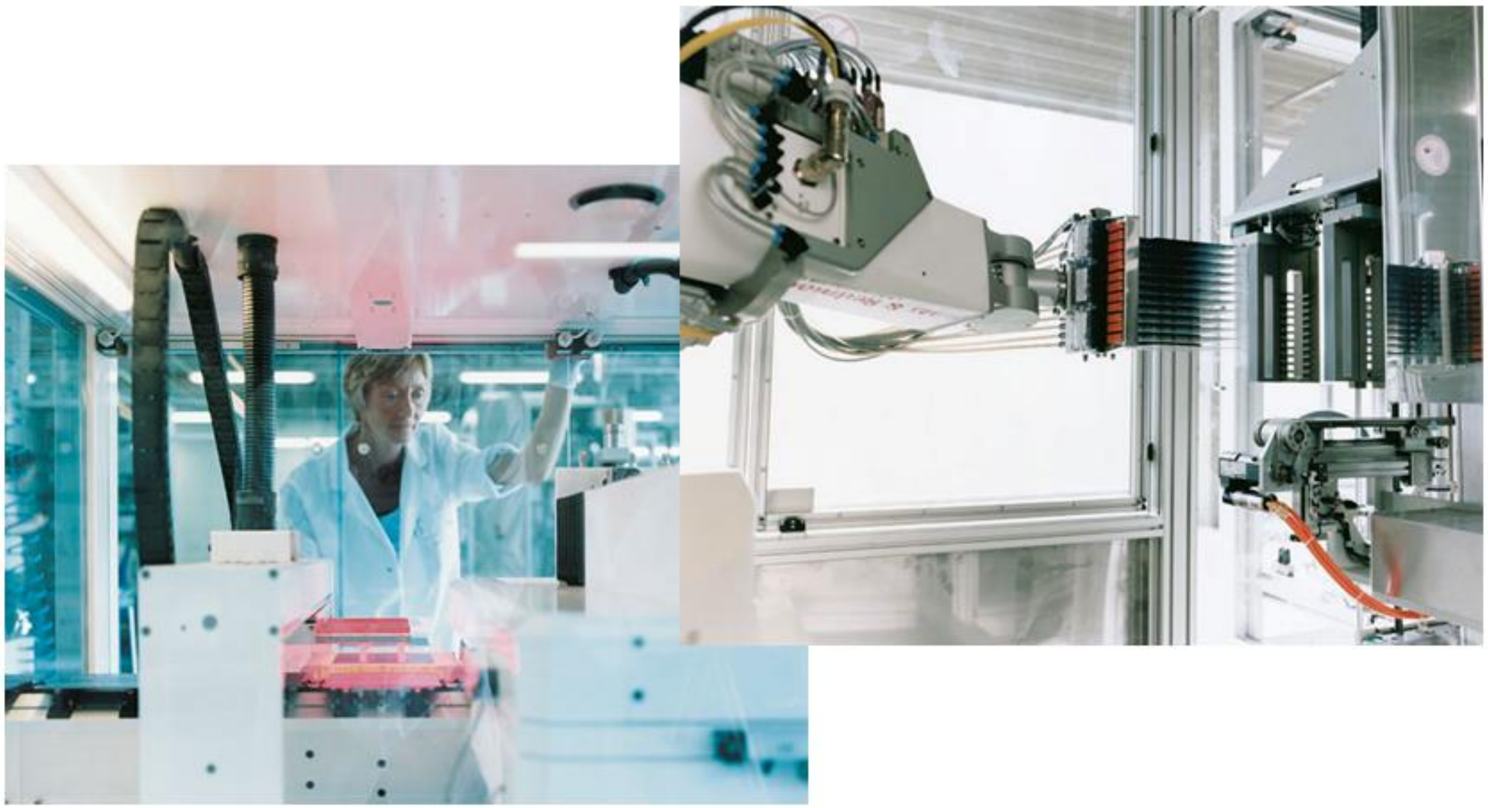
Simulation Project

Utilization of a machine ("Nassbank") at Q-Cells:

- Test dependency of utilization on service time reduction



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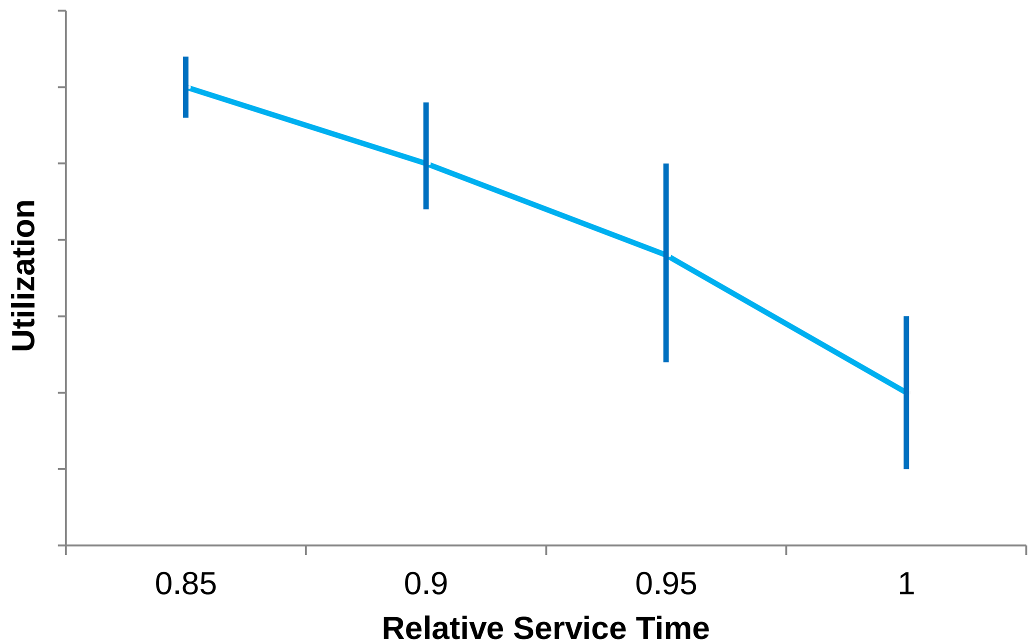
Simulation Project

Simulation experiment:

- 25 replications
- 99% confidence interval



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Simulation Project

Intersection of Gustav-Adolf-Straße and Bundesstraße 1

- Analyse the effect of allowing left turns to the north

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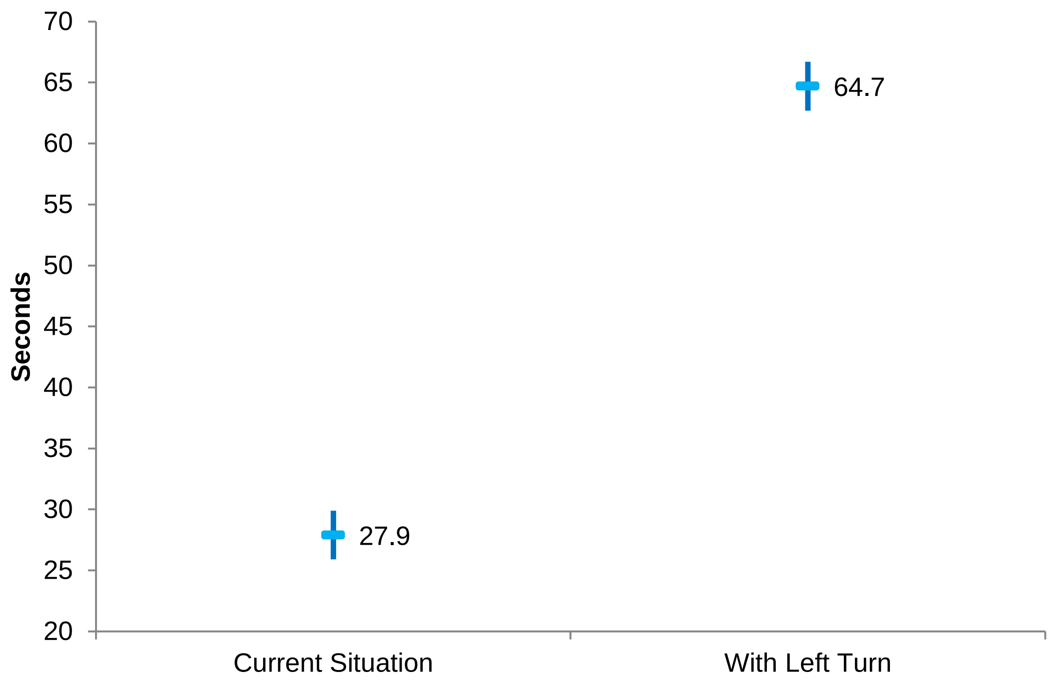
Simulation Project

Simulation experiment for waiting time for westbound traffic:

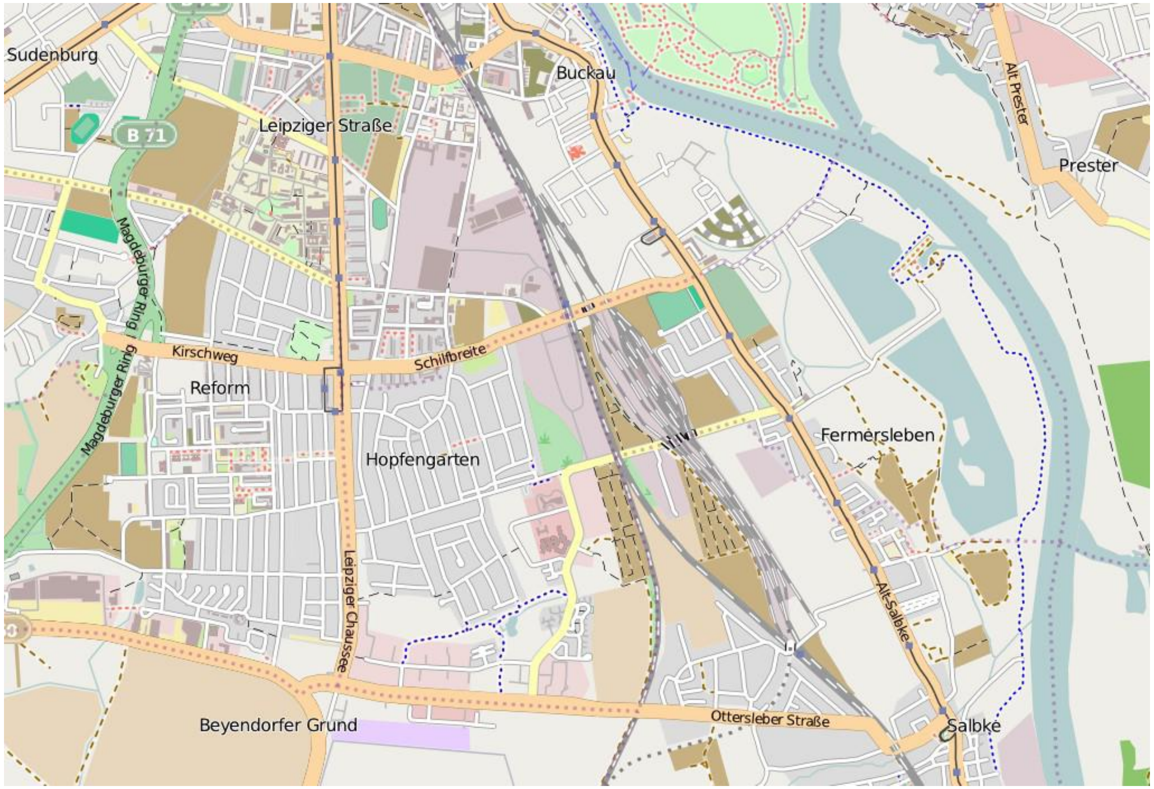
- 50 replications
- 90% confidence interval



Simulation Project

Magdeburg Buckau

- Analyze the effect of a planned street to disburden an existing one of through traffic when developing new industrial areas



Simulation Project

Magdeburg Buckau

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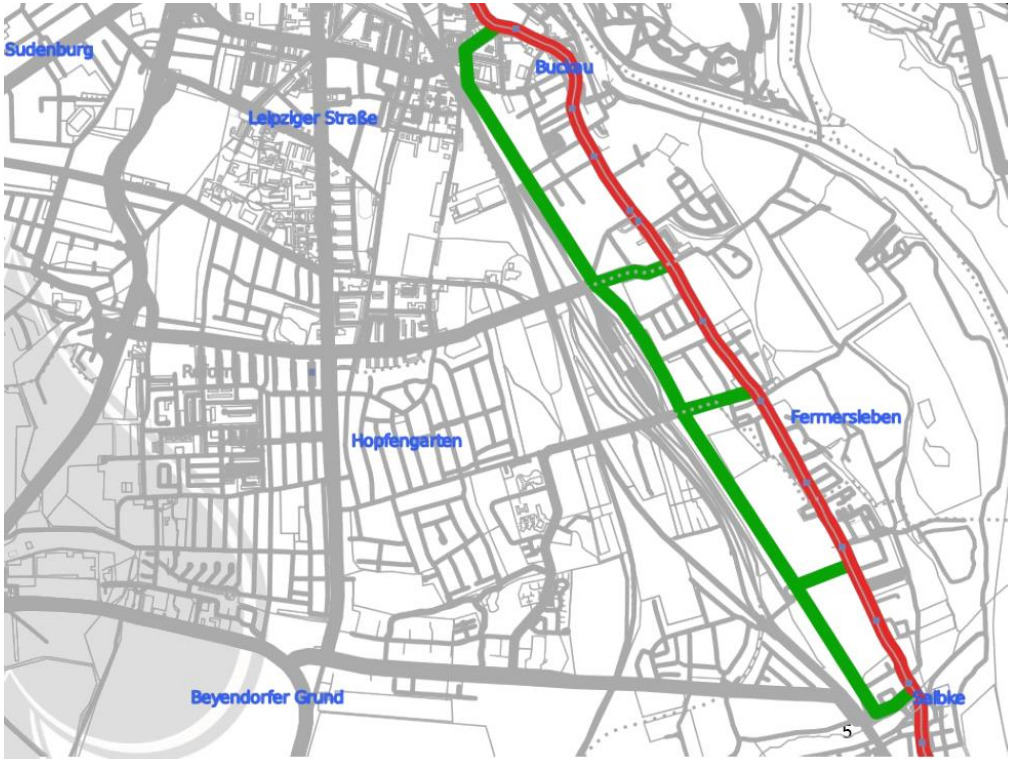

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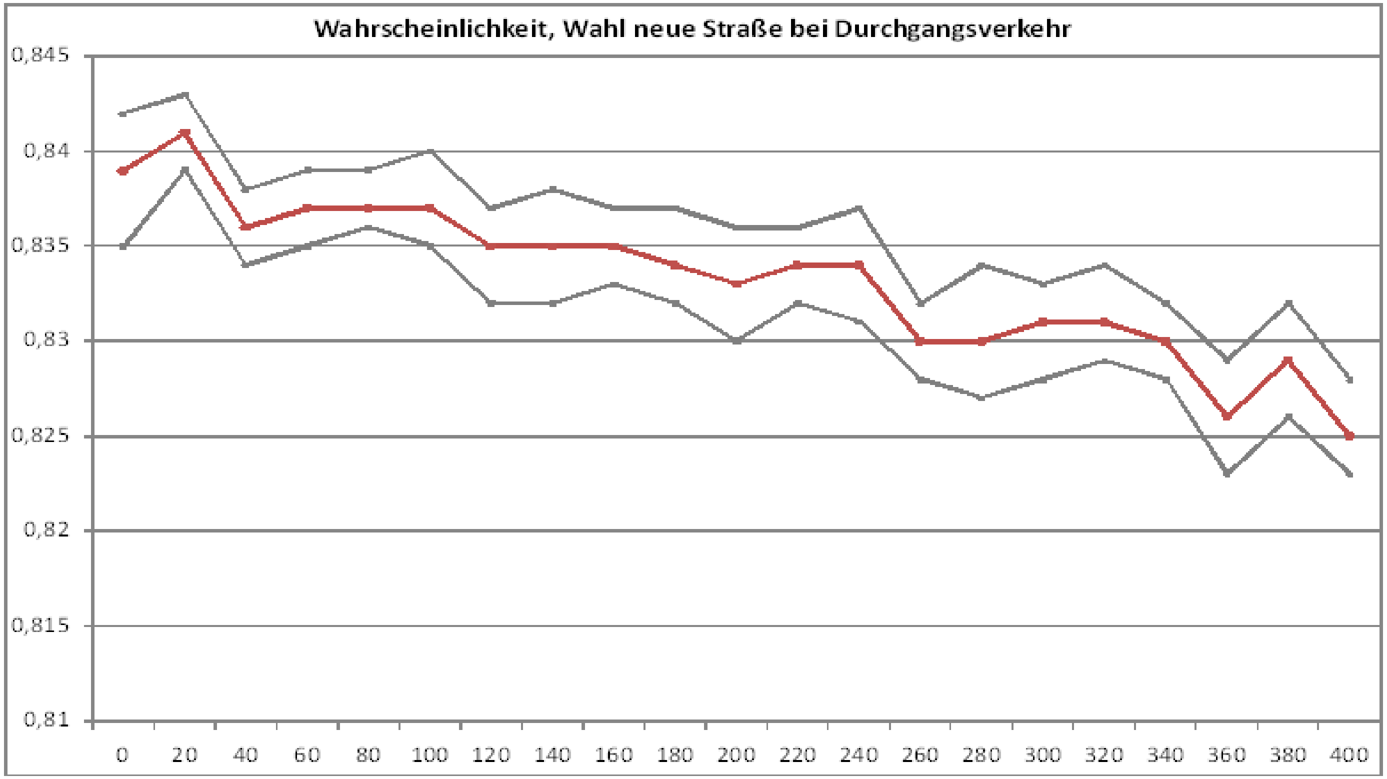
Simulation Project

Probability of choosing the new street when varying the industry density adjacent to the new street:

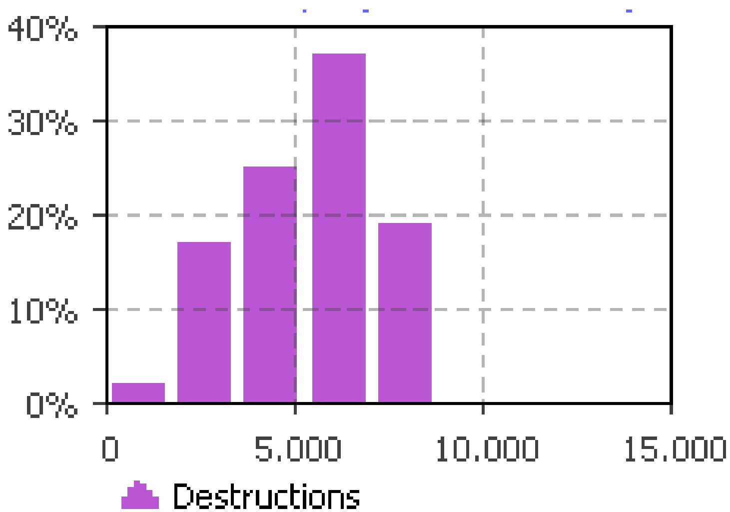
- 10 replications, 90% confidence interval



The Sims – Almost Normal Family Life

You have to compute confidence intervals:

- For how long will the father be unemployed on average?
- How much money will be spent on damaged school property?



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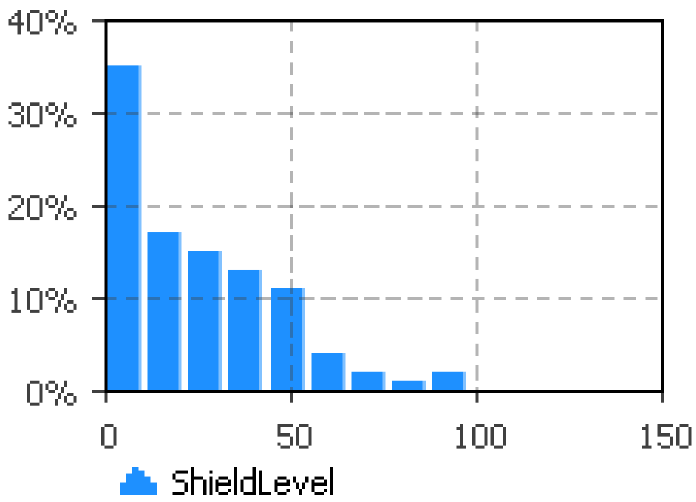
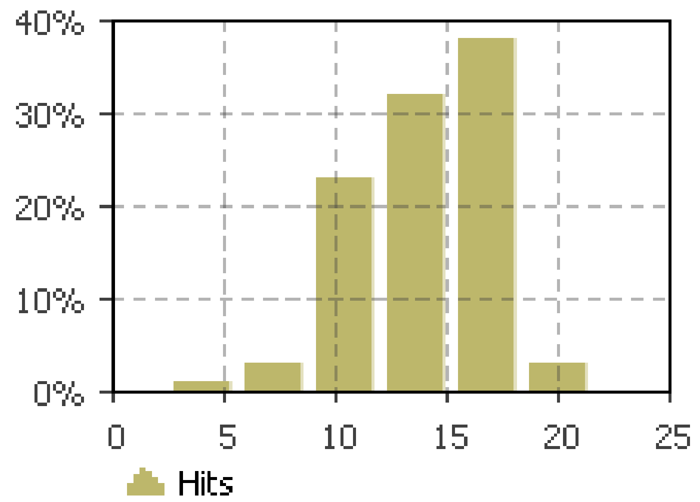
Star Trek – USS Enterprise in Danger

STAR TREK THE NEXT GENERATION

You have to compute confidence intervals:

- What will the shield energy level be after 2 hours?
- How many antimatter particles will hit the shield?





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Learning Goals

Learning questions:

- How is the sample variance of a set of random samples defined?
- What is the method of independent replications?
- How is a confidence interval computed?
- What does a confidence interval signify?
- How would you reduce the width of a confidence interval?
- What are terminating and non–terminating simulations?
- What is initial bias? How can it be avoided?
- What is an ensemble average?