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INF FAKULTÄT FÜR INFORMATIK

Introduction to Simulation

Comparing Systems

Comparing Designs

We often want to compare different system designs

This is one of the most important applications of simulation

Examples:

- Different scheduling policies (queueing strategies)
- Different system structures
- Different times or probabilities
- Validation

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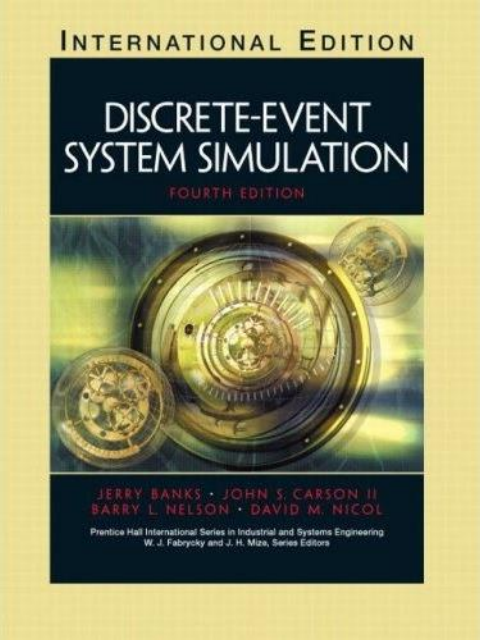
Background Reading

Relevant sections of the book:

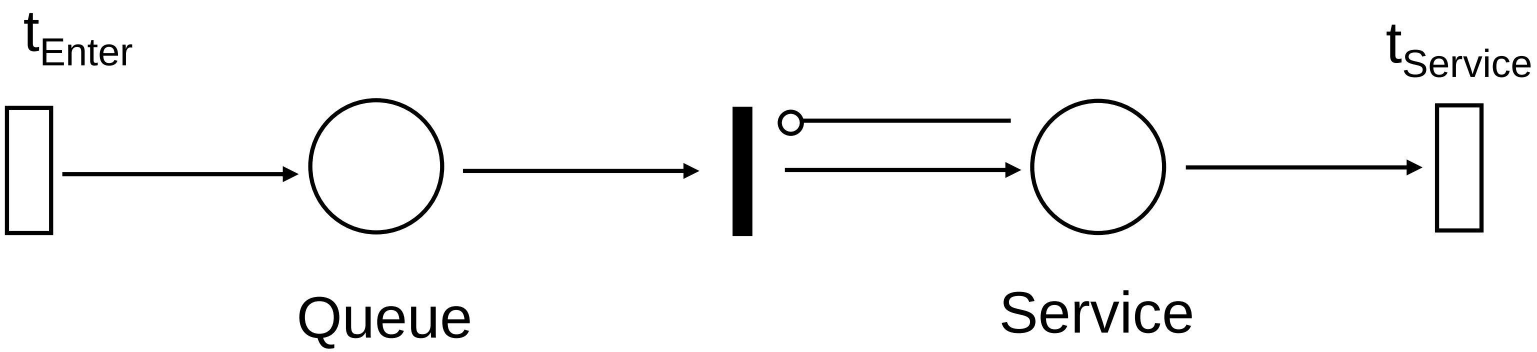
- 12.1



Example: Simple Queue

Arrivals are ~ exp(11)

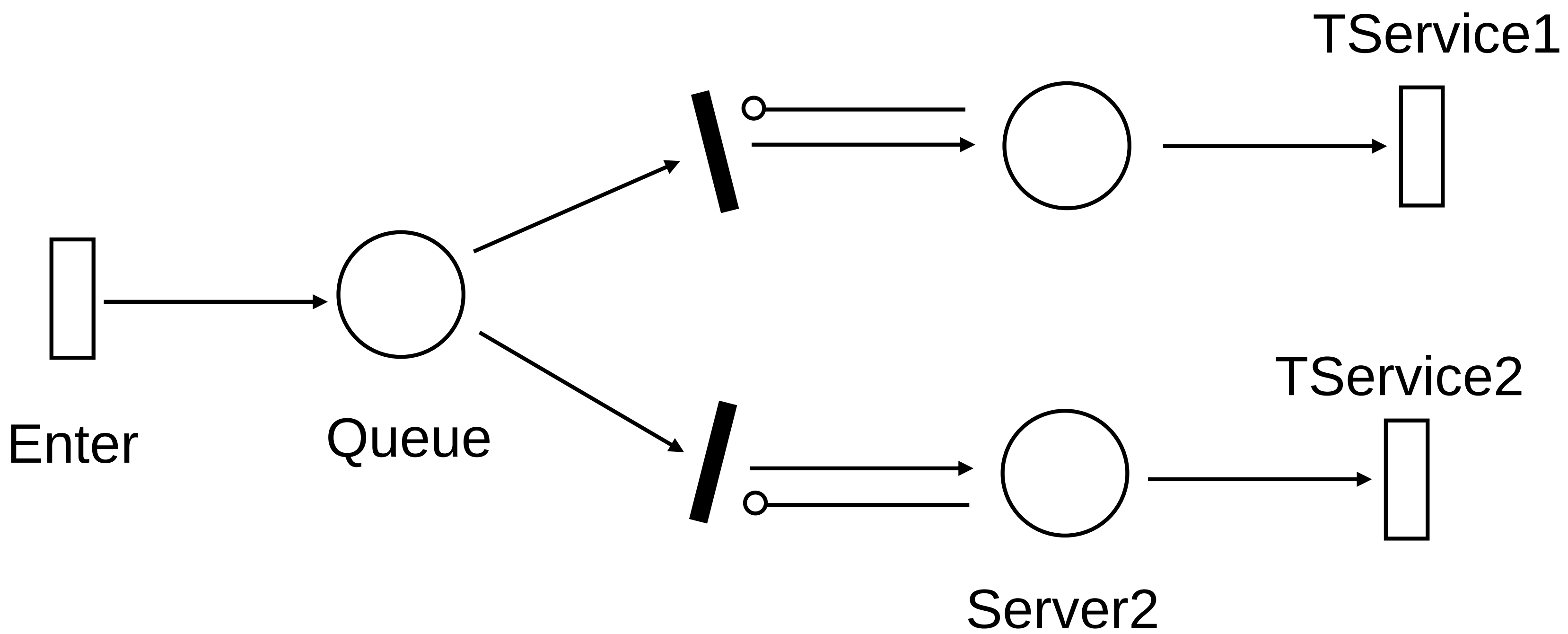
Solution S_1: One server with service time ~ exp(10)



Example: Simple Queue

Solution S_2: Two servers with service time ~ exp(20)

Which system is better?



Example: Simple Queue

Observe the average queue length at T=10000

Simulate both systems and compare results:

Replication	1	2	3	4	5	6	7	8	9	10
Shorter Q.	S_2	S_1	S_2	S_2	S_2	S_2	S_1	S_2	S_1	S_2

Which system is better?

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The General Procedure

Perform R_1 replications on S_1, obtaining $Y_{r1} \quad r=1...R_1$

Perform R_2 replications on S_2, obtaining $Y_{r2} \quad r=1...R_2$

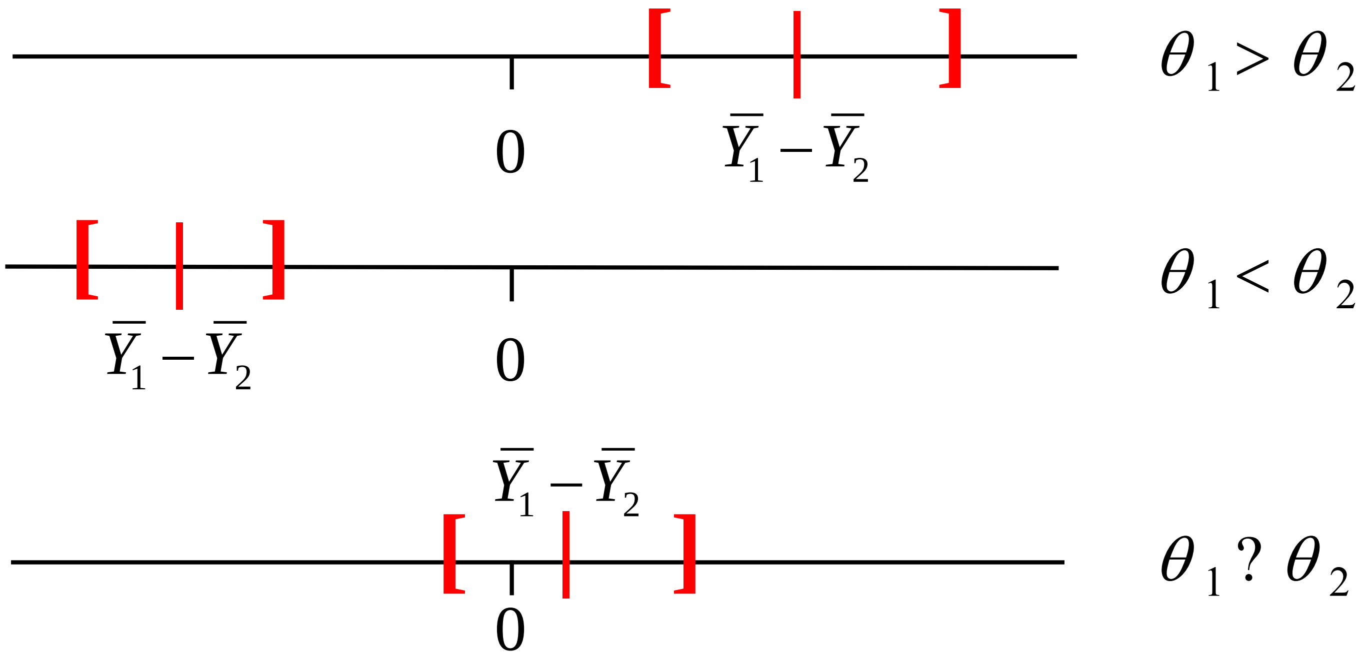
Compute averages $\bar{Y}_1, \bar{Y}_2$

Compute a confidence interval around $\bar{Y}_1 - \bar{Y}_2$:

$$(\bar{Y}_1 - \bar{Y}_2) - \hat{\sigma} \cdot t_{\alpha/2,f} \leq \theta_1 - \theta_2 \leq (\bar{Y}_1 - \bar{Y}_2) + \hat{\sigma} \cdot t_{\alpha/2,f}$$

The General Procedure

There are three possibilities for the confidence interval:



Significance

We consider the difference in system performance ...

Practical significance:

"The actual difference in performance is significant with respect to our goals."

Statistical significance:

"Is our simulation experiment good enough to reveal the difference between the systems, or are we just seeing the result of randomness?"

Computing σ

We need the value of σ for the confidence interval

$$(\bar{Y}_1 - \bar{Y}_2) - \hat{\sigma} \cdot t_{\alpha/2,f} \leq \theta_1 - \theta_2 \leq (\bar{Y}_1 - \bar{Y}_2) + \hat{\sigma} \cdot t_{\alpha/2,f}$$

There are three cases to be considered:

- Independent sampling with equal variances
- Independent sampling with unequal variances
- Correlated sampling

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Indep. Sampling, Equal Var.

We make the following assumptions:

- We will perform independent replications
- The variances in $\bar{Y}_1,\ \bar{Y}_2$ are (approximately) equal

We can pool our estimates as follows:

$$S^2 = \frac{(R_1 - 1)S_1^2 + (R_2 - 1)S_2^2}{R_1 + R_2 - 2}$$

We then obtain σ using: $\sigma = S\sqrt{\frac{1}{R_1} + \frac{1}{R_2}}$

The confidence interval then has $R_1 + R_2 - 2$ d.o.f.

Indep. Sampling, Unequal Var.

What if the variances in $\bar{Y}_1$, $\bar{Y}_2$ are not equal?

Then we must use the following for the standard error:

$$\sigma = \sqrt{\frac{S_1^2}{R_1} + \frac{S_2^2}{R_2}}$$

The formula for the number of d.o.f. is very complicated! (Refer to literature)

Correlated Sampling

Until now, replications were always independent

One technique, however, does this differently

This is the method of *Correlated Sampling*

- It is the most common example of *variance reduction*
- It is used to improve the confidence intervals
- It can be difficult to implement
- It can also be very effective

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Correlated Sampling

Motivation:

- The width of our confidence interval depends on σ
- Is there a way to reduce the size of σ?

σ^2 is an estimator for $\mathrm{var}(\bar{Y}_1 - \bar{Y}_2)$

In the general case, we have

$$\mathrm{var}(\bar{Y}_1 - \bar{Y}_2) = \mathrm{var}(\bar{Y}_1) + \mathrm{var}(\bar{Y}_2) - 2\,\mathrm{cov}(\bar{Y}_1, \bar{Y}_2)$$

Or, more specifically, (with $R_1 = R_2 = R$):

$$\mathrm{var}(\bar{Y}_1 - \bar{Y}_2) = \frac{\sigma_1^2}{R} + \frac{\sigma_2^2}{R} - \frac{2\rho_{12}\sigma_1\sigma_2}{R}$$

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Correlated Sampling

We have $\operatorname{var}(\bar{Y}_1 - \bar{Y}_2) = \dfrac{\sigma_1^2}{R} + \dfrac{\sigma_2^2}{R} - \dfrac{2\rho_{12}\sigma_1\sigma_2}{R}$

(ρ_{12} is the correlation)

Therefore, if we could achieve $\rho_{12} > 0$ ...

- We could reduce $\operatorname{var}(\bar{Y}_1 - \bar{Y}_2)$ ...
- And thus reduce the width of the C.I.

How can we achieve a positive correlation between $\bar{Y}_1$, $\bar{Y}_2$?

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Correlated Sampling

Idea:

- Use the same random numbers in both models

More precisely:

- Each replication uses different random numbers
- In each replication, use the same RNs for each model

Then,

- Different replications will be independent
- Every pair of results Y_{r1}, Y_{r2} will be positively correlated

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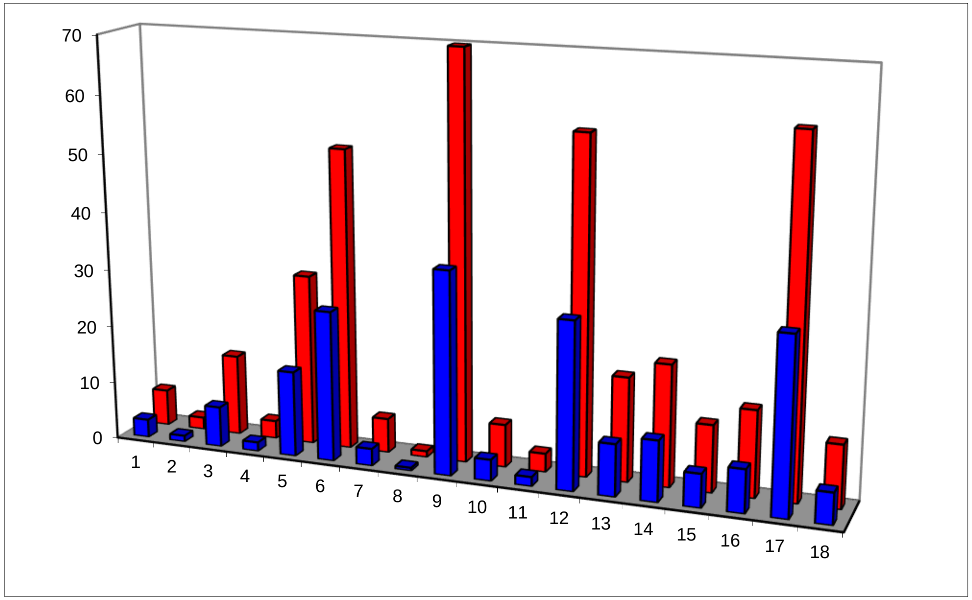
Correlated Sampling

Example: Single-server and two server comparison

- Use same RNs for arrival times and service times
- We get *identical* arrival times for both models
- We get *similar* service times for both models:



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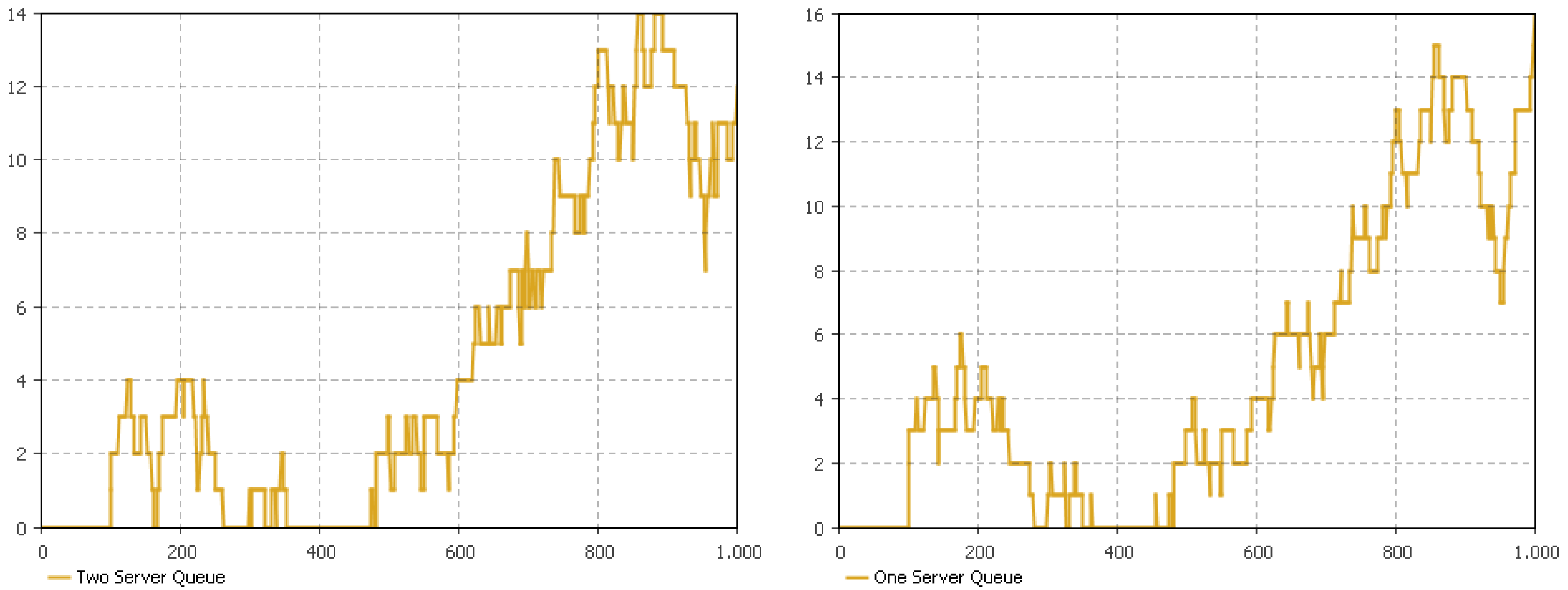
Correlated Sampling

The result is very similar behaviour for queue lengths:



The variance of the difference between these is small

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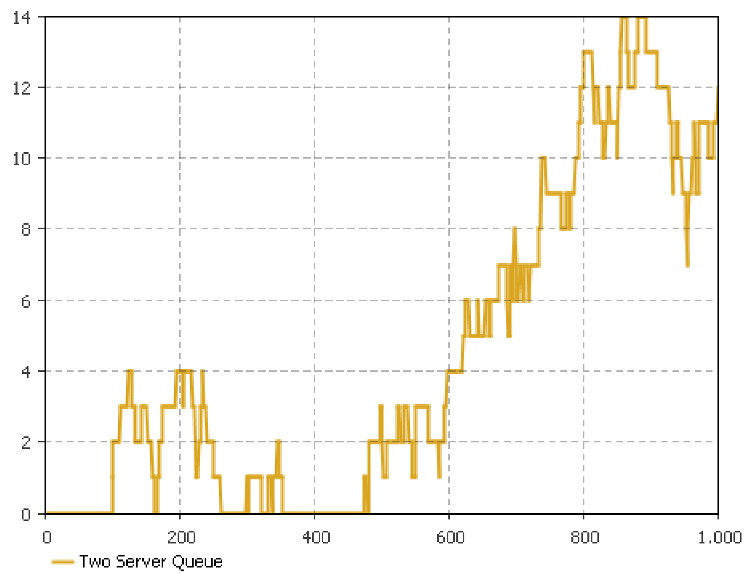
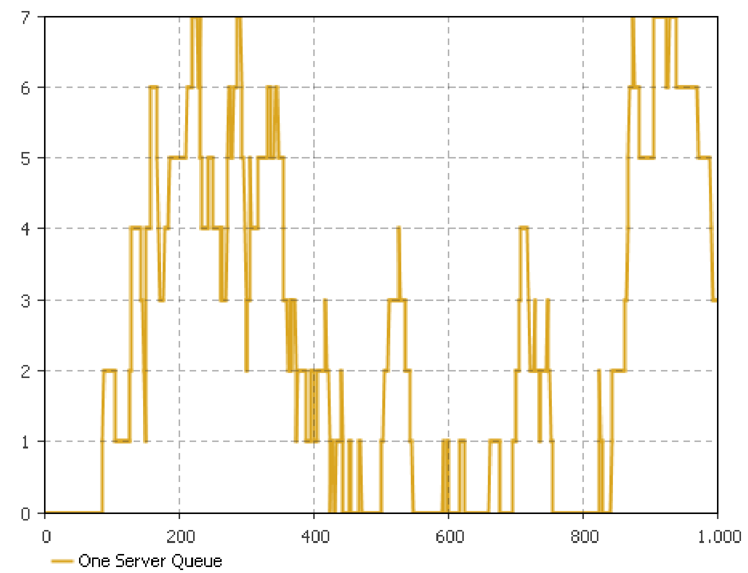
Independent Sampling

Compare with the standard case using independent RN:

- The result is different behaviour for queue lengths:





The variance of the difference between these is large

Correlated Sampling

Algorithm for comparing two systems:

- Compute $D_r = Y_{r1} - Y_{r2}$
- Compute $\overline{D} = \frac{1}{R} \sum D_r$
- Compute $S^2 = \frac{1}{R-1} \sum (D_r - \overline{D})^2$
- Compute $\sigma = \dfrac{S}{\sqrt{R}}$
- Choose α and compute the confidence interval

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Correlated Sampling

If the correlation is positive...

- The variance will be smaller than in the independent case
- Our C.I. should be narrower

Plus …

- We get this improvement at no extra cost!

But:

- How to implement common random numbers?

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Common Random Numbers

The main idea:

- Each random number used in model 1 must be used for the same purpose in model 2

i.e. the random numbers must be *synchronised*

Example:

- The same random number must used to generate the *i* th arrival time in model 1 and in model 2

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Common Random Numbers

If the models are very different, this might be difficult

Important principles:

- The systems should be subjected to similar load
- The systems should show similar reactions
- Simulate incomparable subsystems independently

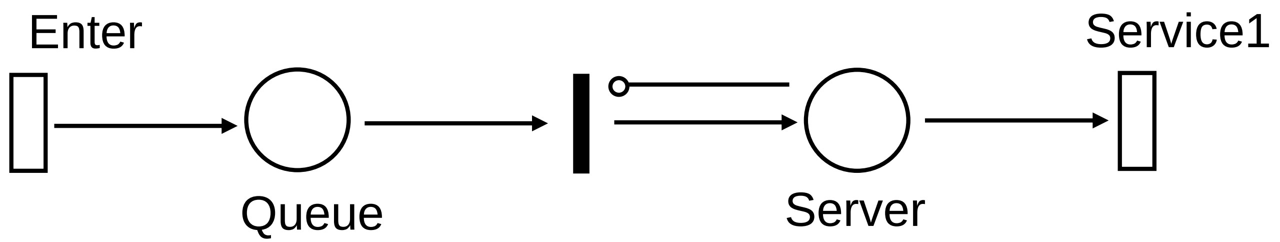
Solution:

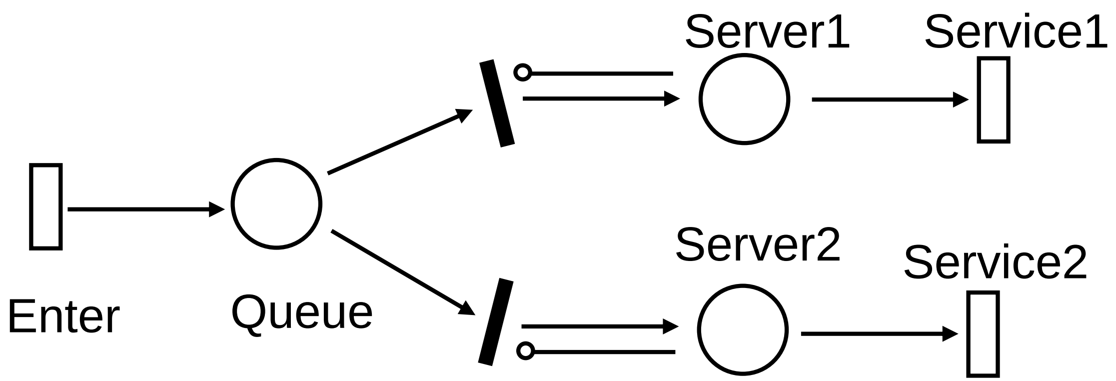
- Use multiple random number streams
- Assign one stream to the same activity in each model

N.B. This does *not* guarantee a reduction in the variance!

Common Random Numbers

Example: Compare one– and two–server models





Common Random Numbers

Results ($\alpha = 0.1, R = 100$):

	Indep.	Arr.	Serv.	Both
ρ_{12}	0.001	0.539	0.37	0.996
S^2	15.317	7.584	9.775	0.057
width of c.i.	1.552	1.092	1.24	0.094

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Learning Goals

Learning questions for the exam:

- What is meant by practical and statistical significance?
- What are synchronised random numbers?
- How does the method of correlated sampling work?
- What is the key idea behind the method of correlated sampling?
- How does the method of correlated sampling achieve a reduced confidence interval at no extra cost?
- What conditions must be achieved in order that the method of correlated sampling can work?