First- and Second-Order Recurrence Relations

Recurrence Relation: A formula that specifies how each term in a sequence is produced from earlier terms.

Examples:

(a) One of the most famous sequences of numbers is: 0, 1, 1, 2, 3, 5, 8, 13, 21, ...

These numbers are the Fibonacci numbers and we can specify the sequence in a recurrence relation by:

$$F_n = F_{n-1} + F_{n-2} \qquad F_0 = 0, F_1 = 1$$

NOTE: The initial term in a recurrence relation has a zero subscript. ie: a_0.

(b) Calculate the first 5 terms of the recurrence relation (this means term a_0 through a_4):

$$a_n = 4a_{n-1} - 3a_{n-2} \qquad a_0 = 1, a_1 = 2$$

Solution:
$a_0 = 1$
$a_1 = 2$
$a_2 = 4(2) - 3(1) = 5$
$a_3 = 4(5) - 3(2) = 14$
$a_4 = 3(14) - 3(5) = 27$

(c) What is the 79th term in the sequence in example (b)? To answer this, we could: (1) calculate 79 terms recursively OR (2) Find an "explicit formula" or "closed form". In this case, the closed form will turn out to be:

$$a_n = \frac{1}{2} + \frac{1}{2}(3^n)$$

Using this closed form, we may find the 79th term by plugging in 79 for n:

$$a_{79} = \frac{1}{2} + \frac{1}{2}(3^{79}) \approx 2.46 \times 10^{37}$$

How do we find this closed form? There are a few ways. We're going to look at a few cases now!

First Order Recurrence Relations

First Order Recurrence Relation: A recurrence relation where a_n can be expressed in terms of just the previous element in the sequence a_{n-1}.

Example 1: Find the closed form of $a_n = 3a_{n-1}$ where $a_0 = 4$

Solution: Let's look at the first few terms in the sequence:
$a_0 = 4$
$a_1 = 3(a_0) = 3(4)$
$a_2 = 3(a_1) = 3(3(4))$
$a_3 = 3(a_2) = 3(3(3(4)))$

In this case, it's fairly easy to write down the explicit formula for a_n. We have $a_n = (3^n)(4)$.

Example 2: Suppose a sequence is defined by: $a_n = 2a_{n-1} + 4$ where $a_0 = 1$. Find an explicit formula for a_n.

The first 6 terms in this sequence are: $1, 6, 16, 36, 76, 156, ...$ And, it's not immediately obvious what the explicit formula should be. So, let's instead write them out without calculating exact numbers:

$a_0 = 1$
$a_1 = 2a_0 + 4$
$a_2 = 2(2a_0 + 4) + 4 = 4a_0 + 12$
$a_3 = 2(4a_0 + 12) + 4 = 8a_0 + 28$
$a_4 = 2(8a_0 + 28) + 4 = 16a_0 + 60$

Now, it appears that we may write the recurrence as $a_n = 2^n(a_0) +$ "something", but how do we get the added term? Let's consider the added term in a_4, which is 60.

Where did 60 come from? During our multiplying, we did the following in reverse:

$60 = 2(28) + 4$
$= 2(2(12) + 4)) + 4$
$= 2(2(2(4) + 4) + 4) + 4$

Let's now re-write it as:

$= 2(2^2(4) + 2(4) + 4) + 4$
$= 2^3(4) + 2^2(4) + 2(4) + 4$
$= (2^3 + 2^2 + 2^1 + 2^0)(4)$
$= (2^4 - 1)(4)$

(The last equality is true by Proposition 21.4)

Let's check this. We get that the added term on a_4 is $(2^4 - 1)(4) = 60$. This is correct, and so we can make a conjecture for an explicit formula:

$a_n = 2^n(a_0) +$ "something"
$= 2^n(a_0) + (2^n - 1)(4)$
$= 2^n(a_0) + 2^n(4) - 4$
$= 2^n(a_0 + 4) - 4$
$= (a_0 + 4)2^n - 4.$

So, here, since $a_0 = 1$, we have: $a_n = (5)(2^n) - 4$

Now, we don't want to have to go through this whole process every time we want to find an explicit formula. And, in fact, we don't have to. The following are general forms of explict formulas for First-Order Recurrence relations (which you can see in more detail on page 173-174 of Scheinerman):

General Theorems for First-Order Recurrence Relations:

Proposition 22.1: All solutions to the recurrence relation $a_n = sa_{n-1} + t$ where $s \neq 1$ have the form:

$$a_n = c_1 s^n + c_2$$

where c_1 and c_2 are specific numbers (which we can find).

Proposition 22.3: The solution to the recurrence relation $a_n = a_{n-1} + t$ is:

$$a_n = a_0 + nt$$

Examples:

For the following, give an explicit formula for a_n. Then, calculate a_9.

(a) $a_n = \frac{2}{3}a_{n-1}$ $\qquad$ $a_0 = 4$.

Solution: We have that $a_n = c_1 \left(\frac{2}{3}\right)^n + c_2$. Now, we know that $a_0 = 4$ and $a_1 = \frac{2}{3}(4) = \frac{8}{3}$.

So, $a_0 = c_1 \left(\frac{2}{3}\right)^0 + c_2 = c_1 + c_2 = 4$

And, $a_1 = c_1 \left(\frac{2}{3}\right)^1 + c_2 = \frac{2}{3}c_1 + c_2 = \frac{8}{3}$.

Now, we just need to solve the system of equations:
$c_1 + c_2 = 4$
$\frac{2}{3}c_1 + c_2 = \frac{8}{3}$

When we do, we get that $c_1 = 4$ and $c_2 = 0$. Therefore, the explicit form for this recurrence relation is:

$$a_n = 4\left(\frac{2}{3}\right)^n$$

Thus, we have $a_9 = 4\left(\frac{2}{3}\right)^9 = \frac{2048}{19683}$

(b) $a_n = 3a_{n-1} - 1$ $\qquad$ $a_0 = 10$

We have by the proposition that $a_n = c_1(3)^n + c_2$.

Now, $a_0 = 10$ and $a_1 = 29$, so we have:

$a_0 = c_1(3)^0 + c_2 = c_1 + c_2 = 10$
$a_1 = c_1(3)^1 + c_2 = 3c_1 + c_2 = 29$

So, we need to solve the following equations to find c_1 and c_2:

$c_1 + c_2 = 10$
$3c_1 + c_2 = 29$

Solving, we have: $c_1 = 9.5$ and $c_2 = 0.5$. Therefore, the explicit form for this recurrence relation is:

$$a_n = 9.5\,(3)^n + 0.5$$

And, so, $a_9 = 186989$.

Second Order Recurrence Relations

Second Order Recurrence Relation: A recurrence relation that gives each term of the sequence in terms of the previous two terms.

Example: $a_n = 4a_{n-1} - 3a_{n-2}$

How do we find the explicit form of a second order recurrence relation? The following theorems give us some explicit steps to calculate it (please see pages 175-180 in Scheinerman for more details):

Theorem 22.5: Let s_1, s_2 be numbers and r_1, r_2 be roots of the equation $x^2 - s_1x - s_2 = 0$. If $r_1 \neq r_2$, then every solution to the recurrence

$$a_n = s_1a_{n-1} + s_2a_{n-2}$$

is of the form:

$$a_n = c_1r_1^n + c_2r_2^n$$

Theorem 22.9: Let s_1, s_2 be numbers so that the quadratic equation $x^2 - s_1x - s_2 = 0$ has exactly one root, $r \neq 0$. Then, every solution to the recurrence relation

$$a_n = s_1a_{n-1} + s_2a_{n-2}$$

is of the form

$$a_n = c_1r^n + c_2nr^n$$

<u>Examples</u>

(a) Find the solution to the recurrence relation

$$a_n = 3a_{n-1} + 4a_{n-2} \qquad a_0 = 6, a_1 = 4$$

First, we find the roots of the quadratic equation:

$$x^2 - 3x - 4 = 0$$

(Note that the 3 comes from the a_{n-1} term and the 4 comes from the a_{n-2} term.)

We can factor the quadratic into linear roots to get: $(x-4)(x+1) = 0$. So, the roots of the equation are $r_1 = 4$ and $r_2 = -1$. Since they are distinct roots (ie $r_1 \neq r_2$), we can use Theorem 22.5 and we see that a_n has the closed form:

$$a_n = c_14^n + c_2(-1)^n$$

To find c_1 and c_2, we see that:

$$a_0 = c_1 4^0 + c_2(-1)^0 = c_1 + c_2 = 6$$

$$a_1 = c_1 4^1 + c_2(-1)^1 = 4c_1 - c_2 = 4$$

So, we only need to solve the two equations:
$c_1 + c_2 = 6$
$4c_1 - c_2 = 4$

Solving, we get $c_1 = 2$ and $c_2 = 4$. Therefore, the explicit formula for a_n is:

$$a_n = (2)4^n + (4)(-1)^n$$

(b) Find the solution to the recurrence relation

$$a_n = -2a_{n-1} - a_{n-2} \qquad\qquad a_0 = 5, a_1 = 1$$

First, we find the roots of the quadratic equation:

$$x^2 + 2x + 1 = 0$$

This quadratic factors, and we have: $(x+1)(x+1) = 0$. So, there is only 1 (repeated) root, $r = -1$. Therefore, by Theorem 22.9, we have that the recurrence relation has a closed form:

$$a_n = c_1(-1)^n + c_2 n(-1)^n$$

Now, we have that:
$a_0 = c_1(-1)^0 + c_2(0)(-1)^0 = c_1 = 5$
$a_1 = c_1(-1)^1 + c_2(1)(-1)^1 = -c_1 - c_2 = 1$

Thus, we only need to solve the system of linear equations:
$c_1 = 5$
$-c_1 - c_2 = 1$

So, we have: $c_1 = 5$ and $c_2 = -6$. Thus, the closed form for a_n is

$$a_n = 5(-1)^n - 6n(-1)^n$$