Mid Sem I Part II **Discrete Structures** **3x5=15 Marks**

1. Let $f : \mathbf{R} \longrightarrow 2^{\mathbf{R}}$ is an arbitrary map from the set of reals to its powerset. Construct a set $S \subseteq \mathbf{R}$ such that S is not in $Image(f)$. What can you conclude about the cardinality of $2^{\mathbf{R}}$ from this construction? $\boxed{3}$

   Soln: Define $S = \{x : x \in \mathbf{R} \text{ and } x \notin f(x)\}$. Now if $S = f(y)$ for any real number y, we get the following contradiction: $y \in S \Leftrightarrow y \notin f(y) \Leftrightarrow y \notin S$. This shows that no f from $\mathbf{R}$ to $2^{\mathbf{R}}$ can be injective. Consequently, the cardinality of $2^{\mathbf{R}}$ cannot be equal to that of $\mathbf{R}$

2. Let R be a relation on a set X. Let $A, B \subseteq X$. Recall that $R(A) = Image(A) = \{y :$ there exists some $a \in A$ satisfying $(a, y) \in R\}$. Here is a proof that $R(A - B) \subseteq R(A) - R(B)$. We split the proof into four parts: (i) Suppose $x \in R(A - B)$. then there exists $a \in (A - B)$ such that $(a, x) \in R$. (ii) This means that $x \in R(A)$ and $x \notin R(B)$. (iii) That is, $x \in R(A) - R(B)$. (iv) Since x has been chosen arbitrarily, we must have $R(A - B) \subseteq R(A) - R(B)$. Is this proof correct? Which among (i),(ii), (iii),(iv) are incorrect. Explain what is the error in reasoning. $\boxed{3}$

   Soln: Step (ii) is error. The existence of an $a \in A - B$ satisfying $x \in R(a)$ does not rule of the existence of another $b \in B$ such that $(x, b) \in R$. In this case we will have both $x \in R(A)$ and $x \in R(B)$ and thus $x \notin (R(A) - R(B))$.

3. Consider the relation NEAR defined on the set of rational numbers $\mathbf{Q}$ as follows: $(x, y) \in$ NEAR if $|x-y| = 1$. Consider the reflexive transitive closure NEAR*. Is NEAR* an equivalence relation? Is it a partial order? Prove/disprove your answers. $\boxed{3}$

   Soln: $NEAR^* = \{(x, y)|x, y \in \mathbf{Q}\}$ such that $|x - y|$ is an integer$\}$. It is easy to see that this relation is reflexive, symmetric and transitive, but not anti-symmetric. Hence $NEAR^*$ is an equivalence relation and not a partial order.

4. You have applied for a mess change from A-mess to C-mess. Here is the notice that appeared in the hostel notice board: *All except those who have not applied for change of mess from A-mess to C-mess or B-mess to C-mess are not permitted to change their mess.* Formulate the notice as an axiom system in (predicate) logic and deduce formally whether or not you are permitted to change your mess. Use variables: A(x): x has applied for change of mess from A-mess to C-mess. B(x): x has applied for change of mess from B-mess to C-mess. C(x): x is permitted to change her/his mess. $\boxed{3}$

   Soln: Given $\{A(me), \forall x(\neg\neg(A(x) \vee B(x)) \Leftrightarrow \neg C(x))\}$ We can simplify this to $\{A(me), \forall x((A(x) \vee B(x)) \Leftrightarrow \neg C(x))\}$. From $\forall x((A(x) \vee B(x)) \Leftrightarrow \neg C(x))$ we deduce $(A(me) \vee B(me)) \Leftrightarrow \neg C(me)$ by instantiating $x = me$. Since $A(me) \Rightarrow (A(me) \vee B(me))$, by hypothetical syllogism, $\neg C(me)$ follows. Thus you are not permitted to change your mess.

5. Suppose A is a set of axioms that is **not** categorical over a variable set V. Is it true that there must always exist a formula f over the variable set V that is independent of A? Prove/Disprove. (Answer on the next page). $\boxed{3}$

   Soln: Since A is not categorical, there exists two distinct truth assignments τ_1, τ_2 that satisfies A. Since $\tau_1 \neq \tau_2$, there must be some variable $v \in V$ such that $\tau_1 \models v$ and $\tau_2 \models \neg v$. Hence $\tau_1 \models A \cup \{v\}$ and $\tau_2 \models A \cup \{\neg v\}$. Thus both $A \cup \{v\}$ and $A \cup \{\neg v\}$ are consistent and hence v is independent of A.