EXERCISE 1a

$f(x) = x^2 - 2 = 0$ $\qquad \sqrt{2} = 1.4142135623 7$

$x_{n+1} = x_n + h$ such that $f(x_{n+1})$ is closer to 0

$0 \approx f(x_{n+1}) = f(x_n) + h f'(x_n) + O(h^2)$

$f(x_n) + h f'(x_n) = 0 \quad \Rightarrow h = -\frac{f(x_n)}{f'(x_n)} \quad \Rightarrow x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$

- If we start with $x_0 = 1$, then we have

$f(x_0) = -1 \quad f'(x_0) = 2 \Rightarrow h = \frac{1}{2}$

$x_1 = x_0 + \frac{1}{2} = \frac{3}{2}$

$x_2 = x_1 + h = \frac{3}{2} - \frac{f(x_1)}{f'(x_1)} = \frac{3}{2} - \frac{\frac{9}{4} - \frac{8}{4}}{3} = \frac{18}{12} - \frac{1}{12} = \frac{17}{12}$

$x_3 = x_2 + h = \frac{17}{12} - \frac{\frac{289}{144} - \frac{288}{144}}{\frac{17}{6}} = \frac{17}{12} - \frac{1}{17 \cdot 24} = \frac{577}{408}$

$x_4 = x_3 + h = \frac{577}{408} - \frac{\frac{577^2}{408^2} - \frac{2 \cdot 408^2}{408^2}}{\frac{577}{204}} = \frac{577}{408} - \frac{1}{408^2} \cdot \frac{204}{577} = \frac{665858}{470832} = 1.41421568627$

we can see that it is getting closer to the actual value of $\sqrt{2}$

- starting from $x_0 = 2$

$x_1 = 2 - \frac{2}{4} = \frac{3}{2} \qquad x_2 = \frac{3}{2} - \frac{\frac{9}{4} - \frac{8}{4}}{3} = \frac{3}{2} - \frac{1}{12} = \frac{17}{12}$

$x_3 = \frac{17}{12} - \frac{\frac{17^2}{12^2} - \frac{2 \cdot 12^2}{12^2}}{\frac{17}{6}} = \frac{17}{12} - \frac{\cancel{6}}{17} \cdot \frac{1}{\cancel{144}_{24}} = \frac{577}{408}$ --- it's like the previous case

- Even if we start from very different initial values, the solution will converge: if we choose for example

$x_0 = 100$, then $x_1 = 100 - \frac{f(100)}{f'(100)} = 4.999$

$x_2 = x_1 - \frac{4.999^2 - 2}{2 \cdot 4.999} = 2.69954000$

$x_3 = x_2 - \frac{(2.6995400)^2 - 2}{2 \cdot 2.6995400} = 1.72020348$

$x_4 = \dots = 1.44142831 \qquad x_5 = 1.41447047$ so it is converging

EXERCISE 2c

$\frac{d^2\vartheta}{dt^2} = -\sin\theta + I \quad (*)$

by calling $y_1 = \vartheta$ and $y_2 = y_1'$, we have $y_2' = y_1'' = \frac{d^2 y_1}{dt^2} = \frac{d^2}{dt^2}\theta = I - \sin\theta = I - \sin y_1$

so equation $(*)$ is equivalent to the system $\begin{cases} y_1' = y_2 \\ y_2' = I - \sin y_1 \end{cases}$