Math Document Template

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Abstract—This document solves triangle equality using matrices

1 PROBLEM

In an isosceles $\triangle ABC$ with $AB = AC$, the bisectors of $\angle B$ and $\angle C$ intersect each other at O. Join A to O. Show that : a) $OB = OC$ b) AO bisects $\angle A$

2 SOLUTION
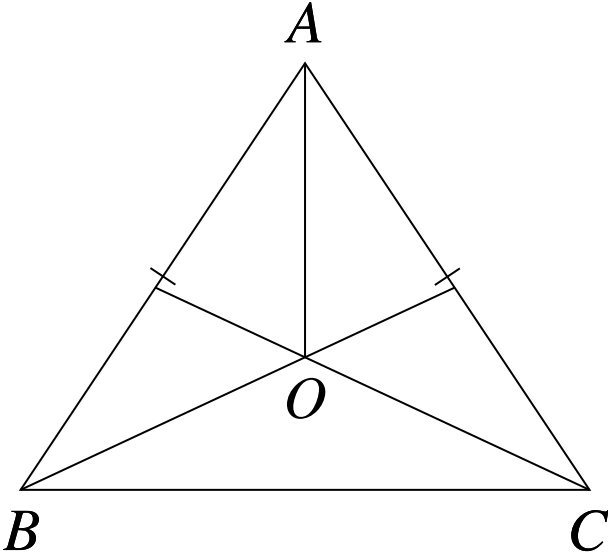

Fig. 1: $\triangle ABC$ with angle bisectors by tikz

Given:

$$\angle ABC = \angle ACB \tag{2.0.1}$$

$$\implies \angle OBC = \angle OCB \tag{2.0.2}$$

$$\implies \cos(OBC) = \cos(OCB) \tag{2.0.3}$$

$$\frac{(\mathbf{B} - \mathbf{O})^\mathbf{T}(\mathbf{B} - \mathbf{C})}{\|\mathbf{B} - \mathbf{O}\| \, \|\mathbf{B} - \mathbf{C}\|} = \frac{(\mathbf{C} - \mathbf{O})^\mathbf{T}(\mathbf{C} - \mathbf{B})}{\|\mathbf{C} - \mathbf{O}\| \, \|\mathbf{C} - \mathbf{B}\|} \tag{2.0.4}$$

$$\implies \frac{(\mathbf{B} - \mathbf{O})^\mathbf{T}(\mathbf{B} - \mathbf{C})}{\|\mathbf{B} - \mathbf{O}\|}$$
$$= \frac{(\mathbf{C} - \mathbf{O})^\mathbf{T}(\mathbf{C} - \mathbf{B})}{\|\mathbf{C} - \mathbf{O}\|} \tag{2.0.5}$$

From the figure, it can be derived that:

$$(\mathbf{B} - \mathbf{O})^\mathbf{T}(\mathbf{B} - \mathbf{C})$$
$$= \left\|\mathbf{O} - \mathbf{B}^2\right\| - (\mathbf{O} - \mathbf{C})^\mathbf{T}(\mathbf{C} - \mathbf{B}) \tag{2.0.6}$$

Similarly,

$$(\mathbf{C} - \mathbf{O})^\mathbf{T}(\mathbf{C} - \mathbf{B})$$
$$= \left\|\mathbf{O} - \mathbf{C}^2\right\| - (\mathbf{O} - \mathbf{C})^\mathbf{T}(\mathbf{C} - \mathbf{B}) \tag{2.0.7}$$

Substituting (2.0.7) and (2.0.6) in (2.0.5), we get:

$$\frac{\|\mathbf{O} - \mathbf{B}\|^2 - (\mathbf{O} - \mathbf{C})^\mathbf{T}(\mathbf{O} - \mathbf{B})}{\|\mathbf{B} - \mathbf{O}\|}$$
$$= \frac{\|\mathbf{O} - \mathbf{C}\|^2 - (\mathbf{O} - \mathbf{C})^\mathbf{T}(\mathbf{O} - \mathbf{B})}{\|\mathbf{B} - \mathbf{O}\|} \tag{2.0.8}$$

$$\implies \|\mathbf{O} - \mathbf{B}\| - \frac{(\mathbf{O} - \mathbf{C})^\mathbf{T}(\mathbf{O} - \mathbf{B})}{\|\mathbf{B} - \mathbf{O}\|}$$
$$= \|\mathbf{O} - \mathbf{C}\| - \frac{(\mathbf{O} - \mathbf{C})^\mathbf{T}(\mathbf{O} - \mathbf{B})}{\|\mathbf{C} - \mathbf{O}\|} \tag{2.0.9}$$

$$\implies \|\mathbf{O} - \mathbf{B}\| - \|\mathbf{O} - \mathbf{C}\| \cos BOC$$
$$= \|\mathbf{O} - \mathbf{C}\| - \|\mathbf{O} - \mathbf{B}\| \cos COB \tag{2.0.10}$$

$$\implies \|\mathbf{O} - \mathbf{B}\| \, (1 + \cos \angle BOC)$$
$$= \|\mathbf{O} - \mathbf{C}\| \, (1 + \cos \angle COB) \tag{2.0.11}$$

$$\implies \|\mathbf{O} - \mathbf{B}\| = \|\mathbf{O} - \mathbf{C}\| \tag{2.0.12}$$

From figure,

$$(\mathbf{O} - \mathbf{B}) = (\mathbf{O} - \mathbf{A}) + (\mathbf{A} - \mathbf{B}) \tag{2.0.13}$$

$$(\mathbf{O} - \mathbf{C}) = (\mathbf{0} - \mathbf{A}) + (\mathbf{A} - \mathbf{C}) \tag{2.0.14}$$

Squaring (2.0.13) and (2.0.14) on both the sides,

we get:

$$\left\|\mathbf{O} - \mathbf{B}^2\right\| = \left\|\mathbf{O} - \mathbf{A}^2\right\| + \left\|\mathbf{A} - \mathbf{B}^2\right\|$$
$$- 2\left\|\mathbf{O} - \mathbf{A}\right\| \left\|\mathbf{A} - \mathbf{B}\right\| \cos OAB \quad (2.0.15)$$

$$\left\|\mathbf{O} - \mathbf{C}^2\right\| = \left\|\mathbf{O} - \mathbf{A}^2\right\| + \left\|\mathbf{A} - \mathbf{C}^2\right\|$$
$$- 2\left\|\mathbf{O} - \mathbf{A}\right\| \left\|\mathbf{A} - \mathbf{C}\right\| \cos OAC \quad (2.0.16)$$

Using (2.0.12), we get

$$\left\|\mathbf{O} - \mathbf{A}^2\right\| + \left\|\mathbf{A} - \mathbf{B}^2\right\|$$
$$- 2\left\|\mathbf{O} - \mathbf{A}\right\| \left\|\mathbf{A} - \mathbf{B}\right\| \cos OAB$$
$$= \left\|\mathbf{O} - \mathbf{A}^2\right\| + \left\|\mathbf{A} - \mathbf{C}^2\right\|$$
$$- 2\left\|\mathbf{O} - \mathbf{A}\right\| \left\|\mathbf{A} - \mathbf{C}\right\| \cos OAC \quad (2.0.17)$$

$$\implies 2\left\|\mathbf{O} - \mathbf{A}\right\| \left\|\mathbf{A} - \mathbf{B}\right\| \cos OAB$$
$$= 2\left\|\mathbf{O} - \mathbf{A}\right\| \left\|\mathbf{A} - \mathbf{C}\right\| \cos OAC \quad (2.0.18)$$

$$\implies \cos OAB = \cos OAC \quad (2.0.19)$$

$$\implies \angle OAB = \angle OAC \quad (2.0.20)$$

Therefore proved that AO bisects $\angle A$