AI20MTECH14010

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***Abstract*—This document aims to plot a conic given its equation using matrices**

Download all python codes from

and latex-tikz codes from

https://github.com/anjanavasudevan/grad_schoolwork/tree/master/EE5609/Assignment7/Latex

1 Question

Trace the following central conic:

$$2x^2 + 3xy - 2y^2 - 7x + y - 2 = 0 \tag{1.0.1}$$

2 Answer

Any second degree equation of the form:

$$ax^2 + 2bxy + cy^2 + 2dx + 2ey + f = 0 \tag{2.0.1}$$

Can be represented in matrix / vector form as:

$$\mathbf{x}^T\mathbf{V}\mathbf{x} + 2\mathbf{u}^T\mathbf{x} + f = 0 \tag{2.0.2}$$

where,

$$\mathbf{V} = \mathbf{V}^T = \begin{pmatrix} a & b \\ b & c \end{pmatrix} \tag{2.0.3}$$

$$\mathbf{u} = \begin{pmatrix} d & e \end{pmatrix} \tag{2.0.4}$$

Rewriting (1.0.1) in matrix form, we get:

$$\mathbf{x}^T\begin{pmatrix} 2 & \frac{3}{2} \\ \frac{3}{2} & -2 \end{pmatrix}\mathbf{x} + 2\begin{pmatrix} -\frac{7}{2} & \frac{1}{2} \end{pmatrix} - 2 = 0 \tag{2.0.5}$$

where,

$$\mathbf{V} = \begin{pmatrix} 2 & \frac{3}{2} \\ \frac{3}{2} & -2 \end{pmatrix} \tag{2.0.6}$$

$$\mathbf{u} = \begin{pmatrix} -\frac{7}{2} \\ \frac{1}{2} \end{pmatrix} \tag{2.0.7}$$

$$f = -2 \tag{2.0.8}$$

$$det(\mathbf{V}) = \begin{vmatrix} 2 & \frac{3}{2} \\ \frac{3}{2} & -2 \end{vmatrix} = -\frac{25}{4} \tag{2.0.9}$$

As $det(\mathbf{V}) < 0$, the given conic represents a hyperbola.
The characteristic equation of $\mathbf{V}$ is given by the determinant:

$$\left|\mathbf{V} - \lambda\mathbf{I}\right| = 0 \tag{2.0.10}$$

$$\begin{vmatrix} 2-\lambda & \frac{3}{2} \\ \frac{3}{2} & -2-\lambda \end{vmatrix} = 0 \tag{2.0.11}$$

$$\implies \lambda^2 - \frac{25}{4} = 0 \tag{2.0.12}$$

The roots of (2.0.12) (the eigenvalues) are:

$$\lambda_1 = \frac{5}{2}, \lambda_2 = -\frac{5}{2} \tag{2.0.13}$$

The eigenvector $\mathbf{p}$ is defined as:

$$\mathbf{V}\mathbf{p} = \lambda\mathbf{p} \tag{2.0.14}$$

$$\implies (\mathbf{V} - \lambda\mathbf{I})\mathbf{p} = 0 \tag{2.0.15}$$

Evaluating (2.0.15) for $\lambda_1 = \frac{5}{2}$, we get:

$$(\mathbf{V} - \lambda_1\mathbf{I}) = \begin{pmatrix} -\frac{1}{2} & \frac{3}{2} \\ \frac{3}{2} & -\frac{9}{2} \end{pmatrix} \tag{2.0.16}$$

Reducing the above equation to row-echelon form, we get:

$$\xleftrightarrow{R_2 \to R_2 + 3R_1} \begin{pmatrix} -\frac{1}{2} & \frac{3}{2} \\ 0 & 0 \end{pmatrix} \xleftrightarrow{R_1 \to -2R_1} \begin{pmatrix} 1 & -3 \\ 0 & 0 \end{pmatrix} \tag{2.0.17}$$

Substituting (2.0.17) in (2.0.15), we get:

$$\begin{pmatrix} 1 & -3 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \tag{2.0.18}$$

where,

$$\mathbf{p} = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} \tag{2.0.19}$$

Let $v_2 = t$. Then

$$v_1 = 3t \tag{2.0.20}$$

Let $t = 1$. The eigenvector $\mathbf{p_1}$ is:

$$\mathbf{p_1} = \begin{pmatrix} 3 \\ 1 \end{pmatrix} \tag{2.0.21}$$

Similarly for $\lambda_2 = -\frac{5}{2}$, we get:

$$(\mathbf{V} - \lambda_2 \mathbf{I}) = \begin{pmatrix} \frac{9}{2} & \frac{3}{2} \\ \frac{3}{2} & \frac{1}{2} \end{pmatrix} \xleftrightarrow[R_1 \to \frac{2}{9}R_1]{R_2 \to 3R_2 - R_1} \begin{pmatrix} 1 & \frac{1}{3} \\ 0 & 0 \end{pmatrix} \tag{2.0.22}$$

Substituting (2.0.22) in (2.0.15), we get:

$$\begin{pmatrix} 1 & \frac{1}{3} \\ 0 & 0 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \tag{2.0.23}$$

where,

$$\mathbf{p} = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} \tag{2.0.24}$$

Let $v_2 = t$. Then

$$v_1 = \frac{-t}{3} \tag{2.0.25}$$

Let $t = 1$. The eigenvector $\mathbf{p_2}$ is:

$$\mathbf{p_2} = \begin{pmatrix} \frac{-1}{3} \\ 1 \end{pmatrix} \tag{2.0.26}$$

As $\mathbf{V} = \mathbf{V}^T$, there exists an orthogonal matrix P such that:

$$\mathbf{P} \mathbf{V} \mathbf{P}^T = \mathbf{D} = diag(\lambda_1, \lambda_2) \tag{2.0.27}$$

$\mathbf{V}$ can be rewritten using the above equation as:

$$\mathbf{V} = \mathbf{P} \mathbf{D} \mathbf{P}^T \tag{2.0.28}$$

where,

$$\mathbf{P} = \begin{pmatrix} \mathbf{p_1} & \mathbf{p_2} \end{pmatrix} \tag{2.0.29}$$

$$\mathbf{D} = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} \tag{2.0.30}$$

Substituting the values -

$$\mathbf{P} = \begin{pmatrix} 3 & \frac{-1}{3} \\ 1 & 1 \end{pmatrix} \tag{2.0.31}$$

$$\mathbf{D} = \begin{pmatrix} \frac{5}{2} & 0 \\ 0 & \frac{-5}{2} \end{pmatrix} \tag{2.0.32}$$

The center of hyperbola is given by:

$$\mathbf{c} = -\mathbf{V}^{-1} \mathbf{u} \tag{2.0.33}$$

$$\implies \mathbf{c} = -\begin{pmatrix} \frac{8}{25} & \frac{6}{25} \\ \frac{6}{25} & \frac{-8}{25} \end{pmatrix} \begin{pmatrix} \frac{-7}{2} \\ \frac{1}{2} \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \tag{2.0.34}$$

As

$$\mathbf{u}^T \mathbf{V}^{-1} \mathbf{u} - f = 5 > 0 \tag{2.0.35}$$

there is no requirement for swapping the axes (which will be evident from the equation below). The axes of the hyperbola are given by:

$$axes = \begin{cases} \sqrt{\frac{\mathbf{u}^T \mathbf{V}^{-1} \mathbf{u} - f}{\lambda_1}} \\ \sqrt{\frac{f - \mathbf{u}^T \mathbf{V}^{-1} \mathbf{u}}{\lambda_2}} \end{cases} \tag{2.0.36}$$

$$\implies \sqrt{\frac{\mathbf{u}^T \mathbf{V}^{-1} \mathbf{u} - f}{\lambda_1}} = \sqrt{2} \tag{2.0.37}$$

$$\implies \sqrt{\frac{f - \mathbf{u}^T \mathbf{V}^{-1} \mathbf{u}}{\lambda_2}} = \sqrt{2} \tag{2.0.38}$$

The standard form of conic can also be written as:

$$\mathbf{y}^T \mathbf{D} \mathbf{y} = \mathbf{u}^T \mathbf{V}^{-1} \mathbf{u} - f \tag{2.0.39}$$

where,

$$\mathbf{y} = \mathbf{P}^T (\mathbf{x} - \mathbf{c}) \tag{2.0.40}$$

$$\implies \mathbf{y}^T \begin{pmatrix} \frac{5}{2} & 0 \\ 0 & \frac{-5}{2} \end{pmatrix} \mathbf{y} - 5 = 0 \tag{2.0.41}$$