AI20MTECH14010

Anjana Vasudevan

***Abstract*—This document aims to identify subspaces of a given vector space.**

Download all latex-tikz codes from

https://github.com/anjanavasudevan/grad_schoolwork/tree/master/EE5609/Assignment8

1 Question

Which of the following are subspaces of the vector space $\mathbb{R}^3$?

1) $(x, y, z) : x + y = 0$
2) $(x, y, z) : x - y = 0$
3) $(x, y, z) : x + y = 1$
4) $(x, y, z) : x - y = 1$

2 Answer

A subspace $\mathbf{S}$ of a vector space is defined as a non-empty subset that is closed under addition and scalar multiplication, i.e

1) All possible linear combinations of the vectors in $\mathbf{S}$ lie in the subspace.
2) Any vector in $\mathbf{S}$ scaled by a scalar c lies in the subspace.

We define a vector $\mathbf{V} \in \mathbf{S}$ for each of the subspaces defined in the options as:

$$\mathbf{V} = \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} \mathbf{x} \\ z_1 \end{pmatrix} \tag{2.0.1}$$

2.1 Option 1:

Let $\mathbf{A} = \begin{pmatrix} \mathbf{x_1} \\ z_1 \end{pmatrix}$ and $\mathbf{B} = \begin{pmatrix} \mathbf{x_2} \\ z_2 \end{pmatrix} \in \mathbf{S}$, and k_1 and k_2 be some scalars.

$$\implies \begin{pmatrix} 1 & 1 \end{pmatrix} \mathbf{x_1} = \begin{pmatrix} 1 & 1 \end{pmatrix} \mathbf{x_2} = 0 \tag{2.1.1}$$

The linear combination of $\mathbf{A}$ and $\mathbf{B}$ is:

$$k_1\mathbf{A} + k_2\mathbf{B} = k_1 \begin{pmatrix} \mathbf{x_1} \\ z_1 \end{pmatrix} + k_2 \begin{pmatrix} \mathbf{x_2} \\ z_2 \end{pmatrix} = \begin{pmatrix} k_1\mathbf{x_1} + k_2\mathbf{x_2} \\ k_1 z_1 + k_2 z_2 \end{pmatrix} \tag{2.1.2}$$

Verifying the property of the subspace:

$$\begin{pmatrix} 1 & 1 \end{pmatrix} \{k_1\mathbf{x_1} + k_2\mathbf{x_2}\} = \begin{pmatrix} 1 & 1 \end{pmatrix} k_1\mathbf{x_1} + \begin{pmatrix} 1 & 1 \end{pmatrix} k_2\mathbf{x_2} \tag{2.1.3}$$

$$\implies k_1 \begin{pmatrix} 1 & 1 \end{pmatrix} \mathbf{x_1} + k_2 \begin{pmatrix} 1 & 1 \end{pmatrix} \mathbf{x_2} = 0 \tag{2.1.4}$$

It is also evident from above that

$$c\mathbf{A} = \begin{pmatrix} c\mathbf{x_1} \\ cz_1 \end{pmatrix} \tag{2.1.5}$$

$$\implies \begin{pmatrix} 1 & 1 \end{pmatrix} c\mathbf{x} = 0 \tag{2.1.6}$$

for some scalar c. Therefore, option 1 is a subspace of $\mathbb{R}^3$.
It can also be proven that option 2 is also a valid subspace of $\mathbb{R}^3$ as:

$$c\mathbf{A} = \begin{pmatrix} c\mathbf{x_1} \\ cz_1 \end{pmatrix} \tag{2.1.7}$$

$$\implies \begin{pmatrix} 1 & -1 \end{pmatrix} (c\mathbf{x_1}) = c \begin{pmatrix} 1 & -1 \end{pmatrix} \mathbf{x_1} = 0 \tag{2.1.8}$$

$$\implies k_1\mathbf{A} + k_2\mathbf{B} = \begin{pmatrix} k_1\mathbf{x_1} + k_1\mathbf{x_2} \\ k_1 z_1 + k_2 z_2 \end{pmatrix} \tag{2.1.9}$$

$$\implies \begin{pmatrix} 1 & -1 \end{pmatrix} \{k_1\mathbf{x_1} + k_2\mathbf{x_2}\} = \begin{pmatrix} 1 & -1 \end{pmatrix} (k_1\mathbf{x_1}) + \begin{pmatrix} 1 & -1 \end{pmatrix} (k_2\mathbf{x_2}) = 0 \in \mathbf{S} \tag{2.1.10}$$

for some scalars c, k_1 and k_2 and vectors $\mathbf{A}$ and $\mathbf{B} \in \mathbf{S}$.

2.2 Option 3:

Option 3 is not a valid subspace of $\mathbb{R}^3$ as it can be shown that for some scalars k_1 and k_2, $\mathbf{A}$ and $\mathbf{B} \in \mathbf{S}$:

$$k_1\mathbf{A} + k_2\mathbf{B} = \begin{pmatrix} k_1\mathbf{x_1} + k_2\mathbf{x_2} \\ k_1 z_1 + k_2 z_2 \end{pmatrix} \tag{2.2.1}$$

$$\implies \begin{pmatrix} 1 & 1 \end{pmatrix} \{k_1\mathbf{x_1} + k_2\mathbf{x_2}\} = k_1 \begin{pmatrix} 1 & 1 \end{pmatrix} \mathbf{x_1} + k_2 \begin{pmatrix} 1 & 1 \end{pmatrix} \mathbf{x_2} = k_1 + k_2 \neq 1 \tag{2.2.2}$$

Because

$$\begin{pmatrix}1 & 1\end{pmatrix}\mathbf{x_1} = \begin{pmatrix}1 & 1\end{pmatrix}\mathbf{x_2} = 1 \tag{2.2.3}$$

Similarly option 4 is also not a valid subspace of $\mathbb{R}^3$ as it can be be shown in similar manner that

$$k_1\mathbf{A} + k_2\mathbf{B} = \begin{pmatrix}k_1\mathbf{x_1} + k_2\mathbf{x_2}\\ k_1z_1 + k_2z_2\end{pmatrix} \tag{2.2.4}$$

$$\implies \begin{pmatrix}1 & -1\end{pmatrix}\{k_1\mathbf{x_1} + k_2\mathbf{x_2}\} = \begin{pmatrix}1 & -1\end{pmatrix}(k_1\mathbf{x_1}) + \begin{pmatrix}1 & -1\end{pmatrix}(k_2\mathbf{x_2}) = k_1 + k_2 \neq 1 \tag{2.2.5}$$

Therefore, Options 1 and 2 are valid subspces of the vector space $\mathbb{R}^3$