

AI20MTECH14010

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Abstract—This document aims to plot a conic given its equation using matrix approach where,

Download all python codes from

[https://github.com/anjanavasudevan/
grad_schoolwork/tree/master/EE5609/
Assignment6/Code](https://github.com/anjanavasudevan/grad_schoolwork/tree/master/EE5609/Assignment6/Code)

and latex-tikz codes from

[https://github.com/anjanavasudevan/
grad_schoolwork/tree/master/EE5609/
Assignment6/Latex](https://github.com/anjanavasudevan/grad_schoolwork/tree/master/EE5609/Assignment6/Latex)

$$\mathbf{V} = \begin{pmatrix} 2 & \frac{3}{2} \\ \frac{3}{2} & -2 \end{pmatrix} \quad (2.0.6)$$

$$\mathbf{u} = \begin{pmatrix} -\frac{7}{2} \\ \frac{1}{2} \end{pmatrix} \quad (2.0.7)$$

$$f = -2 \quad (2.0.8)$$

$$\det(\mathbf{V}) = \begin{vmatrix} 2 & \frac{3}{2} \\ \frac{3}{2} & -2 \end{vmatrix} = -\frac{25}{4} \quad (2.0.9)$$

As $\det(\mathbf{V}) < 0$, the given conic represents a hyperbola.

The characteristic equation of \mathbf{V} is given by the determinant:

$$|\mathbf{V} - \lambda \mathbf{I}| = 0 \quad (2.0.10)$$

$$\begin{vmatrix} 2 - \lambda & \frac{3}{2} \\ \frac{3}{2} & -2 - \lambda \end{vmatrix} = 0 \quad (2.0.11)$$

$$\Rightarrow \lambda^2 - \frac{25}{4} = 0 \quad (2.0.12)$$

1 QUESTION

Trace the following central conic:

$$2x^2 + 3xy - 2y^2 - 7x + y - 2 = 0 \quad (1.0.1)$$

The roots of (2.0.12) (the eigenvalues) are:

$$\lambda_1 = \frac{5}{2}, \lambda_2 = -\frac{5}{2} \quad (2.0.13)$$

2 ANSWER

The eigenvector \mathbf{p} is defined as:

$$\mathbf{V}\mathbf{p} = \lambda\mathbf{p} \quad (2.0.14)$$

$$\Rightarrow (\mathbf{V} - \lambda\mathbf{I})\mathbf{p} = 0 \quad (2.0.15)$$

Any second degree equation of the form:

$$ax^2 + 2bxy + cy^2 + 2dx + 2ey + f = 0 \quad (2.0.1)$$

Evaluating (2.0.15) for $\lambda_1 = \frac{5}{2}$, we get:

Can be represented in matrix / vector form as:

$$\mathbf{x}^T \mathbf{V} \mathbf{x} + 2\mathbf{u}^T \mathbf{x} + f = 0 \quad (2.0.2)$$

$$(\mathbf{V} - \lambda_1 \mathbf{I}) = \begin{pmatrix} -\frac{1}{2} & \frac{3}{2} \\ \frac{3}{2} & -\frac{9}{2} \end{pmatrix} \quad (2.0.16)$$

where,

$$\mathbf{V} = \mathbf{V}^T = \begin{pmatrix} a & b \\ b & c \end{pmatrix} \quad (2.0.3)$$

$$\mathbf{u} = \begin{pmatrix} d & e \end{pmatrix} \quad (2.0.4)$$

Reducing the above equation to row-echelon form, we get:

$$\xleftrightarrow{R_2 \rightarrow R_2 + 3R_1} \begin{pmatrix} -\frac{1}{2} & \frac{3}{2} \\ 0 & 0 \end{pmatrix} \xleftrightarrow{R_1 \rightarrow -2R_1} \begin{pmatrix} 1 & -3 \\ 0 & 0 \end{pmatrix} \quad (2.0.17)$$

Rewriting (1.0.1) in matrix form, we get:

$$\mathbf{x}^T \begin{pmatrix} 2 & \frac{3}{2} \\ \frac{3}{2} & -2 \end{pmatrix} \mathbf{x} + 2 \begin{pmatrix} -\frac{7}{2} & \frac{1}{2} \end{pmatrix} \mathbf{x} - 2 = 0 \quad (2.0.5)$$

Substituting (2.0.17) in (2.0.15), we get:

$$\begin{pmatrix} 1 & -3 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad (2.0.18)$$

where,

$$\mathbf{p} = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} \quad (2.0.19)$$

Let $v_2 = t$. Then

$$v_1 = 3t \quad (2.0.20)$$

Let $t = 1$. The eigenvector \mathbf{p}_1 is:

$$\mathbf{p}_1 = \begin{pmatrix} 3 \\ 1 \end{pmatrix} \quad (2.0.21)$$

Similarly for $\lambda_2 = -\frac{5}{2}$, we get:

$$(\mathbf{V} - \lambda_2 \mathbf{I}) = \begin{pmatrix} \frac{9}{2} & \frac{3}{2} \\ \frac{3}{2} & \frac{1}{2} \end{pmatrix} \xrightarrow[R_1 \rightarrow \frac{2}{9}R_1]{R_2 \rightarrow 3R_2 - R_1} \begin{pmatrix} 1 & \frac{1}{3} \\ 0 & 0 \end{pmatrix} \quad (2.0.22)$$

Substituting (2.0.22) in (2.0.15), we get:

$$\begin{pmatrix} 1 & \frac{1}{3} \\ 0 & 0 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad (2.0.23)$$

where,

$$\mathbf{p} = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} \quad (2.0.24)$$

Let $v_2 = t$. Then

$$v_1 = \frac{-t}{3} \quad (2.0.25)$$

Let $t = 1$. The eigenvector \mathbf{p}_2 is:

$$\mathbf{p}_2 = \begin{pmatrix} -\frac{1}{3} \\ 1 \end{pmatrix} \quad (2.0.26)$$

As $\mathbf{V} = \mathbf{V}^T$, there exists an orthogonal matrix \mathbf{P} such that:

$$\mathbf{PVP}^T = \mathbf{D} = \text{diag}(\lambda_1, \lambda_2) \quad (2.0.27)$$

\mathbf{V} can be rewritten using the above equation as:

$$\mathbf{V} = \mathbf{PDP}^T \quad (2.0.28)$$

where,

$$\mathbf{P} = (\mathbf{p}_1 \quad \mathbf{p}_2) \quad (2.0.29)$$

$$\mathbf{D} = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} \quad (2.0.30)$$

Substituting the values -

$$\mathbf{P} = \begin{pmatrix} 3 & -\frac{1}{3} \\ 1 & 1 \end{pmatrix} \quad (2.0.31)$$

$$\mathbf{D} = \begin{pmatrix} \frac{5}{2} & 0 \\ 0 & -\frac{5}{2} \end{pmatrix} \quad (2.0.32)$$

The center of hyperbola is given by:

$$\mathbf{c} = -\mathbf{V}^{-1}\mathbf{u} \quad (2.0.33)$$

$$\Rightarrow \mathbf{c} = -\begin{pmatrix} \frac{8}{25} & \frac{6}{25} \\ \frac{6}{25} & -\frac{8}{25} \end{pmatrix} \begin{pmatrix} -\frac{7}{2} \\ \frac{1}{2} \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \quad (2.0.34)$$

As

$$\mathbf{u}^T \mathbf{V}^{-1} \mathbf{u} - f = 5 > 0 \quad (2.0.35)$$

there is no requirement for swapping the axes (which will be evident from the equation below).

The axes of the hyperbola are given by:

$$\text{axes} = \begin{cases} \sqrt{\frac{\mathbf{u}^T \mathbf{V}^{-1} \mathbf{u} - f}{\lambda_1}} \\ \sqrt{\frac{f - \mathbf{u}^T \mathbf{V}^{-1} \mathbf{u}}{\lambda_2}} \end{cases} \quad (2.0.36)$$

$$\Rightarrow \sqrt{\frac{\mathbf{u}^T \mathbf{V}^{-1} \mathbf{u} - f}{\lambda_1}} = \sqrt{2} \quad (2.0.37)$$

$$\Rightarrow \sqrt{\frac{f - \mathbf{u}^T \mathbf{V}^{-1} \mathbf{u}}{\lambda_2}} = \sqrt{2} \quad (2.0.38)$$

The standard form of conic can also be written as:

$$\mathbf{y}^T \mathbf{D} \mathbf{y} = \mathbf{u}^T \mathbf{V}^{-1} \mathbf{u} - f \quad (2.0.39)$$

where,

$$\mathbf{y} = \mathbf{P}^T(\mathbf{x} - \mathbf{c}) \quad (2.0.40)$$

$$\Rightarrow \mathbf{y}^T \begin{pmatrix} \frac{5}{2} & 0 \\ 0 & -\frac{5}{2} \end{pmatrix} \mathbf{y} - 5 = 0 \quad (2.0.41)$$