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***Abstract*—This document solves triangle equality using matrices**

1 Problem

In an isosceles $\triangle ABC$ with $AB = AC$, the bisectors of $\angle B$ and $\angle C$ intersect each other at O. Join A to O. Show that : a) $OB = OC$ b) AO bisects $\angle A$

2 Solution

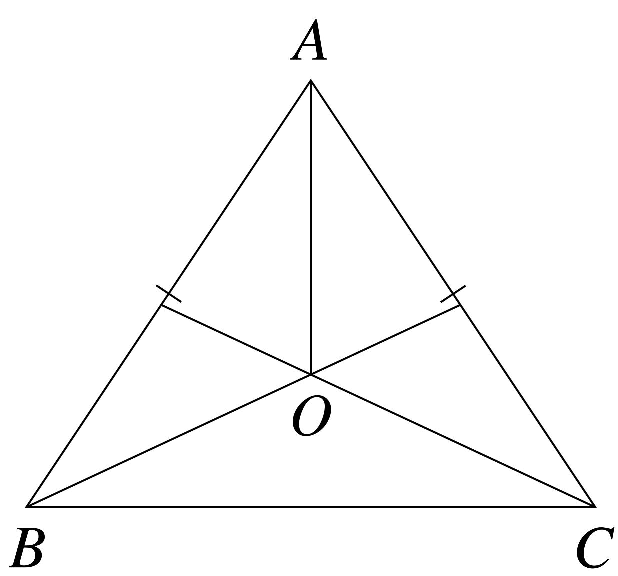

Fig. 1: $\triangle ABC$ with angle bisectors by tikz

Given:

$$\angle ABC = \angle ACB \tag{2.0.1}$$

$$\implies \angle OBC = \angle OCB \tag{2.0.2}$$

$$\implies \cos(OBC) = \cos(OCB) \tag{2.0.3}$$

$$\frac{(\mathbf{B}-\mathbf{O})^{\mathbf{T}}(\mathbf{B}-\mathbf{C})}{\|\mathbf{B}-\mathbf{O}\|\,\|\mathbf{B}-\mathbf{C}\|} = \frac{(\mathbf{C}-\mathbf{O})^{\mathbf{T}}(\mathbf{C}-\mathbf{B})}{\|\mathbf{C}-\mathbf{O}\|\,\|\mathbf{C}-\mathbf{B}\|} \tag{2.0.4}$$

$$\implies \frac{(\mathbf{B}-\mathbf{O})^{\mathbf{T}}(\mathbf{B}-\mathbf{C})}{\|\mathbf{B}-\mathbf{O}\|} = \frac{(\mathbf{C}-\mathbf{O})^{\mathbf{T}}(\mathbf{C}-\mathbf{B})}{\|\mathbf{C}-\mathbf{O}\|} \tag{2.0.5}$$

From the figure, it can be derived that:

$$(\mathbf{B}-\mathbf{O})^{\mathbf{T}}(\mathbf{B}-\mathbf{C}) = \left\|\mathbf{O}-\mathbf{B}^2\right\| - (\mathbf{O}-\mathbf{C})^{\mathbf{T}}(\mathbf{C}-\mathbf{B}) \tag{2.0.6}$$

Similarly,

$$(\mathbf{C}-\mathbf{O})^{\mathbf{T}}(\mathbf{C}-\mathbf{B}) = \left\|\mathbf{O}-\mathbf{C}^2\right\| - (\mathbf{O}-\mathbf{C})^{\mathbf{T}}(\mathbf{C}-\mathbf{B}) \tag{2.0.7}$$

Substituting (2.0.7) and (2.0.6) in (2.0.5), we get:

$$\frac{\|\mathbf{O}-\mathbf{B}\|^2 - (\mathbf{O}-\mathbf{C})^{\mathbf{T}}(\mathbf{O}-\mathbf{B})}{\|\mathbf{B}-\mathbf{O}\|} = \frac{\|\mathbf{O}-\mathbf{C}\|^2 - (\mathbf{O}-\mathbf{C})^{\mathbf{T}}(\mathbf{O}-\mathbf{B})}{\|\mathbf{B}-\mathbf{O}\|} \tag{2.0.8}$$

$$\implies \|\mathbf{O}-\mathbf{B}\| - \frac{(\mathbf{O}-\mathbf{C})^{\mathbf{T}}(\mathbf{O}-\mathbf{B})}{\|\mathbf{B}-\mathbf{O}\|} = \|\mathbf{O}-\mathbf{C}\| - \frac{(\mathbf{O}-\mathbf{C})^{\mathbf{T}}(\mathbf{O}-\mathbf{B})}{\|\mathbf{C}-\mathbf{O}\|} \tag{2.0.9}$$

$$\implies \|\mathbf{O}-\mathbf{B}\| - \|\mathbf{O}-\mathbf{C}\|\cos BOC = \|\mathbf{O}-\mathbf{C}\| - \|\mathbf{O}-\mathbf{B}\|\cos COB \tag{2.0.10}$$

$$\implies \|\mathbf{O}-\mathbf{B}\|(1+\cos\angle BOC) = \|\mathbf{O}-\mathbf{C}\|(1+\cos\angle COB) \tag{2.0.11}$$

$$\implies \|\mathbf{O}-\mathbf{B}\| = \|\mathbf{O}-\mathbf{C}\| \tag{2.0.12}$$

From figure,

$$(\mathbf{O}-\mathbf{B}) = (\mathbf{O}-\mathbf{A}) + (\mathbf{A}-\mathbf{B}) \tag{2.0.13}$$

$$(\mathbf{O}-\mathbf{C}) = (\mathbf{0}-\mathbf{A}) + (\mathbf{A}-\mathbf{C}) \tag{2.0.14}$$

Squaring (2.0.13) and (2.0.14) on both the sides, we get:

$$\left\|\mathbf{O}-\mathbf{B}^2\right\| = \left\|\mathbf{O}-\mathbf{A}^2\right\| + \left\|\mathbf{A}-\mathbf{B}^2\right\| \\ - 2\left\|\mathbf{O}-\mathbf{A}\right\|\left\|\mathbf{A}-\mathbf{B}\right\|\cos OAB \quad (2.0.15)$$

$$\left\|\mathbf{O}-\mathbf{C}^2\right\| = \left\|\mathbf{O}-\mathbf{A}^2\right\| + \left\|\mathbf{A}-\mathbf{C}^2\right\| \\ - 2\left\|\mathbf{O}-\mathbf{A}\right\|\left\|\mathbf{A}-\mathbf{C}\right\|\cos OAC \quad (2.0.16)$$

Using (2.0.12), we get

$$\left\|\mathbf{O}-\mathbf{A}^2\right\| + \left\|\mathbf{A}-\mathbf{B}^2\right\| \\ - 2\left\|\mathbf{O}-\mathbf{A}\right\|\left\|\mathbf{A}-\mathbf{B}\right\|\cos OAB \\ = \left\|\mathbf{O}-\mathbf{A}^2\right\| + \left\|\mathbf{A}-\mathbf{C}^2\right\| \\ - 2\left\|\mathbf{O}-\mathbf{A}\right\|\left\|\mathbf{A}-\mathbf{C}\right\|\cos OAC \quad (2.0.17)$$

$$\implies 2\left\|\mathbf{O}-\mathbf{A}\right\|\left\|\mathbf{A}-\mathbf{B}\right\|\cos OAB \\ = 2\left\|\mathbf{O}-\mathbf{A}\right\|\left\|\mathbf{A}-\mathbf{C}\right\|\cos OAC \quad (2.0.18)$$

$$\implies \cos OAB = \cos OAC \quad (2.0.19)$$

$$\implies \angle OAB = \angle OAC \quad (2.0.20)$$

Therefore proved that AO bisects $\angle A$