Linear Algebra and its Applications

Assignment 5

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Import all prerequisites

In [1]:
```
from helper_functions import *
import time
import scipy as sp
import pandas as pd
import scipy.linalg
from scipy.linalg import cho_factor, cho_solve, cholesky
import matplotlib.pyplot as plt
```

Initialization

Initialization of all data structures for storing the following:

- Running time of both solvers
- Solutions L, U, P, x or P, G, x and the chosen A and b for both solvers

In [2]:
```
n_list=[2,5,9,10,12,13,14,15,16,30,50,90,100]
scipy_LU_rt={} #Running times by scipy's LU factorization algorithm
my_LU_rt_gepp={}  # Running times by my implementation LU factorization algorithm
my_LU_rt_gerp={}
my_LU_rt_gecp={}
my_cholesky_rt={}
scipy_cholesky_rt={}

scipy_cholesky_rt={}
scipy_LU_subst_rt={} #Running times by scipy's Substitution algorithm
# Running times by my implementation of Substitution algorithm
my_subst_rt_gepp={}
my_subst_rt_gerp={}
my_subst_rt_gecp={}
my_cholesky_subst_rt={}

scipy_LU_solutions={}
scipy_cholesky_solutions={}
my_solutions_gepp={}
my_solutions_gerp={}
my_solutions_gecp={}
my_cholesky_solutions={}
```

Execution

Execute the solvers and record the running times

In [5]:
```
import warnings
warnings.filterwarnings("ignore")
for n in n_list:

    A=np.zeros((n,n))
    for i in range(n):
        for j in range(n):
            A[i,j]=1/(i+j+1)
    b=np.random.rand(n)

    #Partial Pivoting
    start_time = time.time()
    L,U,P,Q=LU_factorize_partial(A)
    my_LU_rt_gepp[n]=time.time()-start_time


    start_time=time.time()
    x=substitution(L,U,P,Q,b)
    my_subst_rt_gepp[n]=time.time()-start_time
    my_solutions_gepp[n]=(L,U,P,Q,x,A,b)

    #Rook Pivoting
    start_time = time.time()
    L,U,P,Q=LU_factorize_rook(A)
    my_LU_rt_gerp[n]=time.time()-start_time


    start_time=time.time()
    x=substitution(L,U,P,Q,b)
    my_subst_rt_gerp[n]=time.time()-start_time
    my_solutions_gerp[n]=(L,U,P,Q,x,A,b)

    #Complete Pivoting
    start_time = time.time()
    L,U,P,Q=LU_factorize_complete(A)
    my_LU_rt_gecp[n]=time.time()-start_time


    start_time=time.time()
    x=substitution(L,U,P,Q,b)
    my_subst_rt_gecp[n]=time.time()-start_time
    my_solutions_gecp[n]=(L,U,P,Q,x,A,b)

    #My Cholesky LU Solver
    start_time = time.time()
    G,P=my_cholesky(A)
    my_cholesky_rt[n]=time.time()-start_time
```

```
    start_time=time.time()
    x=substitution(G,G.transpose(),P,P.transpose(),b)
    my_cholesky_subst_rt[n]=time.time()-start_time
    my_cholesky_solutions[n]=(G,P,x,A,b)


    #Scipy LU Solver
    start_time=time.time()
    p, l, u = sp.linalg.lu(A) # The decomposition is A=plu
    scipy_LU_rt[n]=time.time()-start_time



    lu, piv = sp.linalg.lu_factor(A)
    start_time=time.time()
    x2 = sp.linalg.lu_solve((lu, piv), b)
    scipy_LU_subst_rt[n]=time.time()-start_time


    scipy_LU_solutions[n]=(l,u,p,x2,A,b)


    #Scipy Cholesky Solver
    start_time=time.time()
    try:
        L = cholesky(A, lower=True)
        scipy_cholesky_rt[n]=time.time()-start_time
    except:
        L=np.NaN
    try:
        c, low = cho_factor(A)
        x = cho_solve((c, low), b)
    except:
        x=np.NaN


    scipy_cholesky_solutions[n]=(L,x,A,b)
```

Calculate Norms

Calculate the norms of $PA - LU$ and $Ax_0 - b$ using all solvers.

In [6]:
```
my_PA_LU_norms_gepp={}
my_PA_LU_norms_gerp={}
my_PA_LU_norms_gecp={}
my_cholesky_GG_norms={}
scipy_PA_LU_norms={}
scipy_cholesky_GG_norms={}

my_Ax_b_norms_gepp={}
my_Ax_b_norms_gerp={}
my_Ax_b_norms_gecp={}
my_cholesky_AX_b_norms={}
scipy_Ax_b_norms={}
```

```
scipy_cholesky_AX_b_norms={}

for n in n_list:
    #Partial Pivoting Norms
    L,U,P,Q,x,A,b=my_solutions_gepp[n]
    my_PA_LU_norms_gepp[n]=np.linalg.norm(np.matmul(P,A@Q)-np.matmul(L,U),ord=2)
    my_Ax_b_norms_gepp[n]=np.linalg.norm(np.matmul(A,x)-b)


    #Rook Pivoting Norms
    L,U,P,Q,x,A,b=my_solutions_gerp[n]
    my_PA_LU_norms_gerp[n]=np.linalg.norm(np.matmul(P,A@Q)-np.matmul(L,U),ord=2)
    my_Ax_b_norms_gerp[n]=np.linalg.norm(np.matmul(A,x)-b,ord=2)

    #Complete Pivoting Norms
    L,U,P,Q,x,A,b=my_solutions_gecp[n]
    my_PA_LU_norms_gecp[n]=np.linalg.norm(np.matmul(P,A@Q)-np.matmul(L,U),ord=2)
    my_Ax_b_norms_gecp[n]=np.linalg.norm(np.matmul(A,x)-b,ord=2)

    #Cholesky Norms
    G,P,x,A,b=my_cholesky_solutions[n]
    try:
        my_cholesky_GG_norms[n]=np.linalg.norm(G@(G.transpose())-P@A@(P.transpose()
        my_cholesky_AX_b_norms[n]=np.linalg.norm(np.matmul(A,x)-b,ord=2)
    except:
        my_cholesky_GG_norms[n]=np.NaN
        my_cholesky_AX_b_norms[n]=np.NaN
    #Scipy LU Norms
    L,U,P,x,A,b=scipy_LU_solutions[n]
    scipy_PA_LU_norms[n]=np.linalg.norm(A-np.matmul(P,np.matmul(L,U)),ord=2)
    scipy_Ax_b_norms[n]=np.linalg.norm(np.matmul(A,x)-b,ord=2)

    #Scipy Cholesky Norm
    G,x,A,b=scipy_cholesky_solutions[n]
    try:
        scipy_cholesky_GG_norms[n]=np.linalg.norm(G@(G.transpose())-A,ord=2)
        scipy_cholesky_AX_b_norms[n]=np.linalg.norm(np.matmul(A,x)-b,ord=2)
    except:
        scipy_cholesky_GG_norms[n]=np.NaN
        scipy_cholesky_AX_b_norms[n]=np.NaN
```

Norms Table

This table gives us the the matrix norms of $PA - LU$(or $PAP^T - GG^T$) and $Ax_0 - b$ for both solvers.

The columns labels are set as n for an $n \times n$matrix

In [7]:
```
matrix_norms={}
col_names=["PA-LU gepp","PA-LU gerp","PA-LU gecp","Cholesky GG(t)-PAP(t)","PA-LU Sc
           "Ax-b gepp","Ax-b gerp","Ax-b gecp","Ax-b Cholesky","Ax-b SciPy(LU)","Ax
for n in n_list:
    matrix_norms[n]=[my_PA_LU_norms_gepp[n],my_PA_LU_norms_gerp[n],my_PA_LU_norms_g
                     scipy_PA_LU_norms[n],
```

```
                    scipy_cholesky_GG_norms[n],my_Ax_b_norms_gepp[n],my_Ax_b_norms
                    my_Ax_b_norms_gecp[n],my_cholesky_AX_b_norms[n],scipy_Ax_b_nor
                    scipy_cholesky_AX_b_norms[n]]
norm_table=pd.DataFrame(matrix_norms, index=col_names)
```

In [8]:
```
display(norm_table.iloc[:,:6])
```

	2	5	9	10	12	13
PA-LU gepp	0.000000	2.775558e-17	5.173899e-17	4.913042e-17	6.261478e-17	5.871185e-17
PA-LU gerp	0.000000	2.775558e-17	5.173899e-17	4.913042e-17	6.261478e-17	5.871185e-17
PA-LU gecp	0.000000	4.490946e-17	5.973831e-17	5.093450e-17	6.702340e-17	4.303850e-17
Cholesky GG(t)-PAP(t)	0.000000	1.340158e-16	6.700789e-17	1.190237e-16	2.262175e-16	NaN
PA-LU SciPy	0.000000	2.775558e-17	4.033996e-17	1.962616e-17	5.160349e-17	5.722031e-17
Cholesky Scipy	0.000000	0.000000e+00	2.504924e-17	2.715048e-17	3.359559e-17	3.301474e-17
Ax-b gepp	0.000000	1.901657e-12	3.618864e-06	2.965031e-05	8.466475e-03	9.488734e-01
Ax-b gerp	0.000000	1.901657e-12	3.618864e-06	2.965031e-05	8.466475e-03	9.488734e-01
Ax-b gecp	0.000000	4.623118e-12	5.181760e-06	2.286102e-05	5.828233e-03	1.272643e+00
Ax-b Cholesky	0.105849	8.178088e-02	2.998494e-01	6.461434e-01	7.405266e-01	NaN
Ax-b SciPy(LU)	0.000000	3.631007e-12	6.229130e-06	2.720404e-05	6.879998e-03	7.823553e-01
Ax-b Scipy(Cholesky)	0.000000	4.543342e-12	5.865915e-06	2.327277e-05	4.451865e-03	1.515401e+00

In [9]:
```
display(norm_table.iloc[:,6:])
```

	14	15	16	30	50	
PA-LU gepp	5.330853e-17	6.062295e-17	5.949965e-17	6.744857e-17	7.113867e-17	7.323909
PA-LU gerp	5.330853e-17	6.062295e-17	5.949965e-17	6.720375e-17	7.103216e-17	7.328106
PA-LU gecp	4.908086e-17	4.865046e-17	5.244584e-17	7.322928e-17	7.438936e-17	8.073525
Cholesky GG(t)-PAP(t)	NaN	NaN	NaN	NaN	NaN	
PA-LU SciPy	5.481817e-17	5.934849e-17	6.081652e-17	6.448746e-17	6.896413e-17	7.237879
Cholesky Scipy	3.184169e-17	NaN	NaN	NaN	NaN	
Ax-b gepp	1.418014e+01	1.395668e+00	6.353899e+00	3.087226e+01	3.828336e+01	3.342325
Ax-b gerp	1.418014e+01	1.395668e+00	6.353899e+00	3.991581e+01	3.926930e+01	4.062674
Ax-b gecp	5.594126e+01	2.096839e+00	4.985956e+00	1.357073e+02	6.260920e+01	4.531907
Ax-b Cholesky	NaN	NaN	NaN	NaN	NaN	
Ax-b SciPy(LU)	4.907927e+00	1.332972e+00	2.507493e+00	3.268468e+02	1.179363e+02	6.532960
Ax-b Scipy(Cholesky)	2.841543e+00	NaN	NaN	NaN	NaN	

A is a Hilbert matrix and it ceases to be positive definite after $n = 14$. Thus, after $n = 14$, Cholesky decompositon doesn't work on A and we get NaN values.

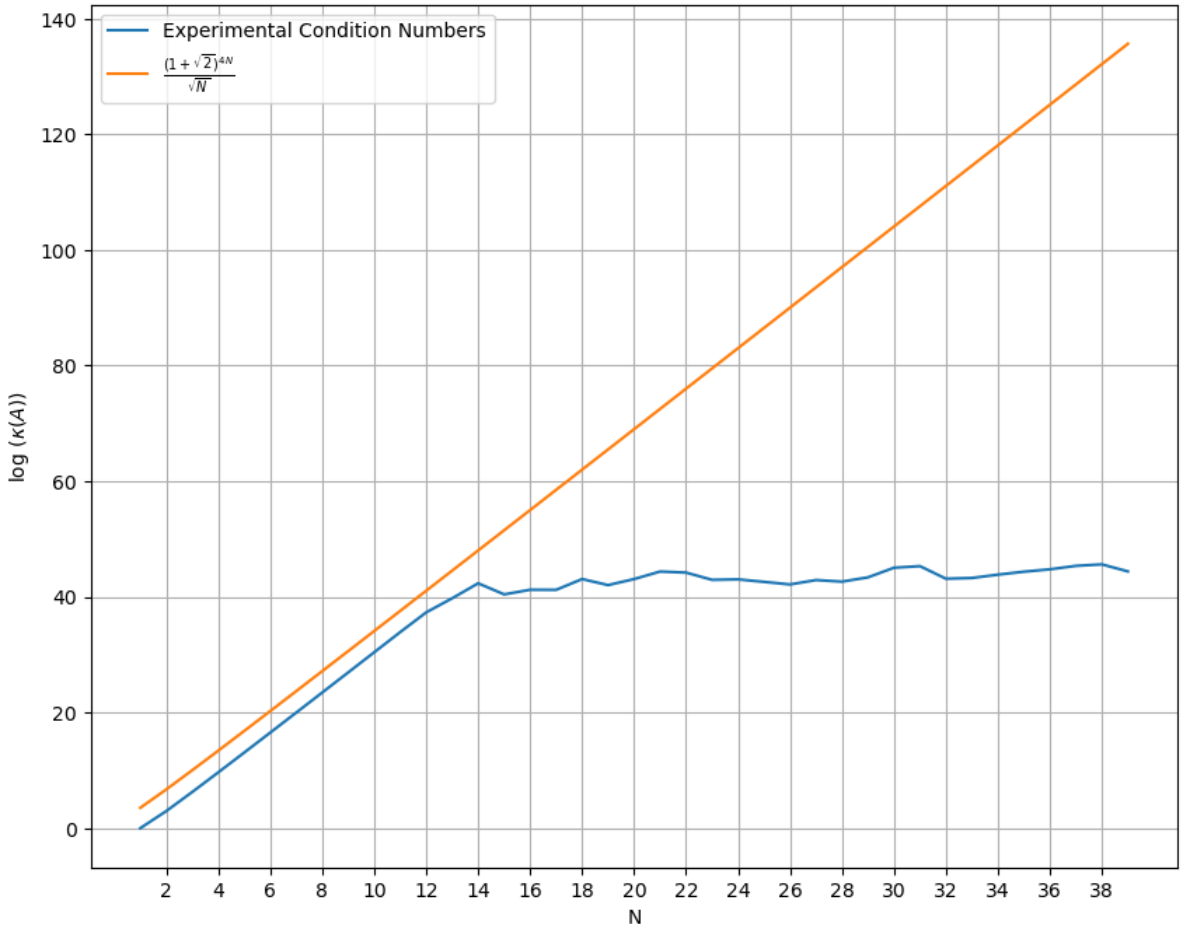
Condition Number

In [10]:
```
conds=[]
bounds=[]
n_list=np.arange(1,40)
for n in n_list:
    A=np.zeros((n,n))
    for i in range(n):
        for j in range(n):
            A[i,j]=1/(i+j+1)
    cond=np.linalg.norm(A)*np.linalg.norm(np.linalg.inv(A))
    conds.append(cond)
    bounds.append(((1+np.sqrt(2))**(4*n))/np.sqrt(n))

import matplotlib.pyplot as plt

plt.figure(figsize=(10, 8))

plt.plot(n_list,np.log(conds),label ='Experimental Condition Numbers')
plt.plot(n_list,np.log(bounds),label =r'$\frac{(1+\sqrt{2})^{4N}}{\sqrt{N}}$')
plt.xlabel('N')
plt.ylabel('log ($\kappa (A)$)')
plt.xticks(2*n_list[:int(n/2)])
plt.legend()
plt.grid()
```



In the above graph, we plot the experimental condition number along with the value $\frac{(1+\sqrt{2})^{4N}}{\sqrt{N}}$.

We can say that the for upto $n = 14$, the condition number has the bound $O\left(\frac{(1+\sqrt{2})^{4N}}{\sqrt{N}}\right)$ but beyond $n = 14$ calculating the condition numbers involves a lot of round off errors and thus stagnates.

Still we can say, that $O\left(\frac{(1+\sqrt{2})^{4N}}{\sqrt{N}}\right)$ is a resaonable bound on the condition number as the experimental values are below the curve.

References

1. *Beckermann, B. (2000). The condition number of real Vandermonde, Krylov and positive definite Hankel matrices. Numerische Mathematik, 85(4), 553–577. doi:10.1007/pl00005392*
2. *Todd, J. "The condition of the finite segments of the Hilbert matrix, NBS Appl." Math. Ser 39 (1954): 109-116.*