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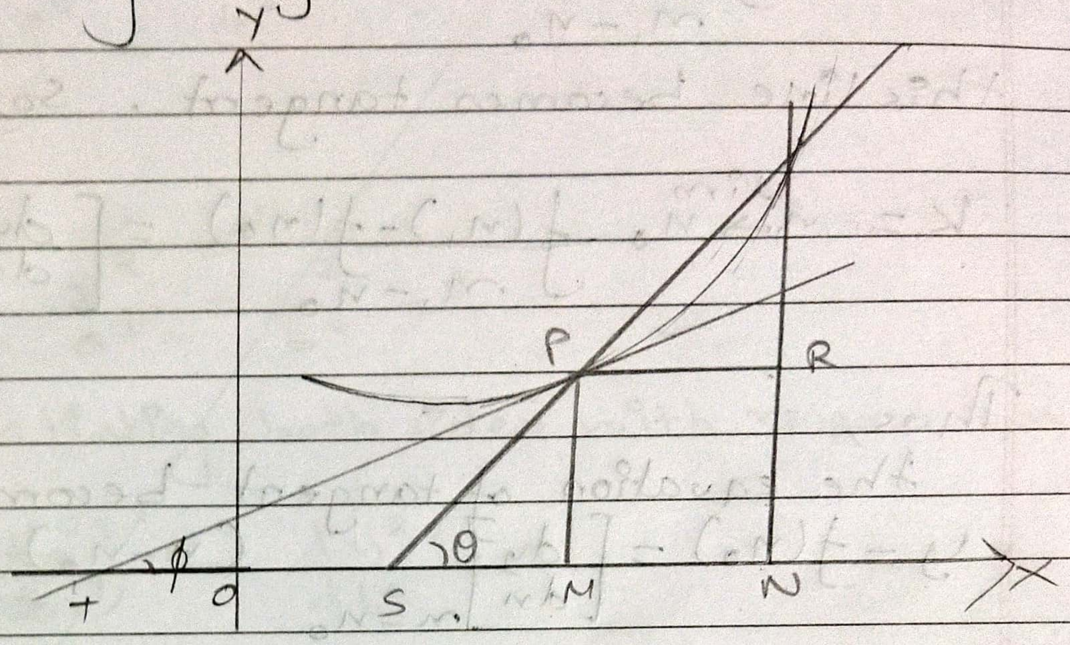
# Differentiation

## # Tangents and Normal

### Equation of Tangent and Normal




Let $y = f(x)$ be the equation of the curve and let $P(x_0, f(x_0))$ be any point on it. let Q $(x_1, f(x_1))$ be another point on the curve close to P. Join QP and produce it to meet the x-axis at S and let $\angle PSX$ and θ. Let PT, where $\angle PTX = \phi$ be the tangent to the curve at P. Let QN be perpendicular to OX, PR perpendicular to NQ, and PM perpendicular to OX. Now,

$PR = ON - OM = x_1 - x_0$ and

$RQ = NQ - MP = f(x_1) - f(x_0)$. Then

equation of the line PQ is given by

$$y - f(x_0) = \frac{f(x_1) - f(x_0)}{x_1 - x_0}(x - x_0).$$ Then

above equation can be written as

$$y - f(x_0) = k(x - x_0), \text{ where}$$

$$k = \frac{f(x_1) - f(x_0)}{x_1 - x_0}.$$ As Q approaches P

this line becomes tangent. So,

$$k = \lim_{x_1 \to x_0} \frac{f(x_1) - f(x_0)}{x_1 - x_0} = \left[\frac{dy}{dx}\right]_{x = x_0}$$

Thus,

the equation of tangent becomes

$$y - f(x_0) = \left[\frac{dy}{dx}\right]_{x = x_0} (x - x_0)$$

Since,

the normal at P perpendicular with tangent. The equation of normal be

$$(x - x_0) + (y - f(x_0)) \left[\frac{dy}{dx}\right]_{x = x_0} = 0$$

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Q. Find the equation of tangent and normal to the curve $x^2+y^2=25$ at $(3,-4)$.

Soln

Given curve, $x^2+y^2=25$

Differentiating both sides with respect to x we get.

$$x^2+y^2=25$$

$$\frac{d(x^2+y^2)}{dx} = \frac{d}{dx}(25)$$

or, $$\frac{dx^2}{dx} + \frac{dy^2}{dx} = 0$$

or, $$2x + \frac{dy^2}{dy} \times \frac{dy}{dx} = 0$$

or $$2x + 2y\frac{dy}{dx} = 0$$

or, $$\frac{dy}{dx} = -\frac{2x}{2y}$$

or $$\frac{dy}{dx} = -\frac{x}{y}$$

So, $$\left[\frac{dy}{dx}\right]_{(3,-4)} = \frac{3}{4}$$

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Thus,

equation of tangent be

$$y - f(x_0) = \left[\frac{dy}{dx}\right]_{x=x_0} (x - x_0)$$

or, $y + 4 = \frac{3}{4}(x - 3)$

or, $4y + 16 = 3x - 9$

or, $3x - 4y - 25 = 0$

or, $3x - 4y = 25$

Again

Equation of normal be

$$(x - x_0) + (y - f(x_0))\left[\frac{dy}{dx}\right]_{x=x_0} = 0$$

or, $(x - 3) + (y + 4)\frac{3}{4} = 0$

or, $\cancel{4x - 12 + 4y + 16}$

or, $4x - 12 + 3y + 12 = 0$

or, $4x + 3y = 0$

Ans #

Q. Find the eauation of tangent and normal to the curve ~~$x^2+y^2=25$ at~~ $x^2+y^2=16$ at [illegible]

Sol$^n$

Given,

$$x^2+y^2=16$$

Differentiating both sides with respect to $x$ we get,

Q. Find the equation of tangent and normal to the curve $x^2+y^2=25$ at $(4,3)$

Sol$^n$

Given,

$$x^2+y^2=25$$

Differentiating both sides with respect to $x$.

$$\frac{dy}{dx} = \frac{-x}{y} \quad \text{so, } \left[\frac{dy}{dx}\right]_{(4,3)} = -\frac{4}{3}$$

Thus equation of tangent be

$$y-f(x_0) = \left[\frac{dy}{dx}\right]_{x=x_0} (x-x_0)$$

or, $y-3 = -\frac{4}{3}(x-3)$

or, $3y-9=-4x+12$

or, $4x+3y=25$

Again, equation of normal be

$$(x-x_0)+(y-f(x_0))\left[\frac{dy}{dx}\right]_{x=x_0}=0$$

or $(x-4)+(y-3)\left[\cancel{\frac{dy}{dx}}\ \frac{-4}{3}\right]=0$

or $3x-12-4y+12=6$

or $3x-4y=0$

Ans #

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Q. Find the equation of tangent and normal to the curve $y^2 = 16x$ at $(1/4, 2)$

Soln

Given,

curve

$$y^2 = 16x$$

Differentiating both sides with respect to x we get

$$\frac{dy^2}{dx} = \frac{d\,16x}{dx}$$

or $$\frac{dy^2}{dy} \times \frac{dy}{dx} = 16$$

or $$2y \cdot \frac{dy}{dx} = 16$$

or $$\frac{dy}{dx} = \frac{\cancel{16}^{8}}{\cancel{2}y}$$

or, $$\frac{dy}{dx} = \frac{8}{y}$$

So, $$\left[\frac{dy}{dx}\right]_{(1/4,2)} = 4$$

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Equation of tangent be

$$y - f(x_0) = \left[\frac{dy}{dx}\right]_{x=x_0} (x - x_0)$$

or $y - 2 = 4\left(x - \frac{1}{4}\right)$

or $y - 2 = 4\left(\frac{4x-1}{4}\right)$

or $y - 2 = 4x - 1$

or $4x - y = -1$

Equation of normal be

$$(x - x_0) + (y - f(x_0))\left[\frac{dy}{dx}\right]_{x=x_0} = 0$$

or $\left(x - \frac{1}{4}\right) + (y - 2)4 = 0$

or $\frac{4x-1}{4} + (4y - 8) = 0$

or $4x - 1 + 16y - 32 = 0$

or $4x + 16y - 33 = 0$

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Q. Find the equation of tangent and normal to the curve $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ at point $(x_1, y_1)$

Sol^n

Given,

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$$

Differentiating both sides with respect to x we get,

$$\frac{dx^2}{a^2} - \frac{dy^2}{b^2} = 0$$

$$\text{or} \quad \frac{2x}{a^2} - \frac{2y}{b^2}\frac{dy}{dx} = 0$$

$$\text{or,} \cancel{-}\frac{dy}{dx} = \cancel{-}\frac{\cancel{2}x}{a^2} \times \frac{b^2}{\cancel{2}y}$$

$$\therefore \frac{dy}{dx} = \frac{b^2 x}{a^2 y}$$

$$\text{So,} \left[\frac{dy}{dx}\right]_{(x_1, y_1)} = \frac{x_1 b^2}{y_1 a^2}$$

The equation of tangent be

$$y - f(x_0) = \left[\frac{dy}{dx}\right]_{x = x_0} (x - x_0)$$

$$\text{or,} \quad y - y_1 = \frac{x_1 b^2}{y_1 a^2}(x - x_1)$$

or, $\frac{xx_1}{a^2} - \frac{yy_1}{b^2} = 1$

Equation of normal be

$$(x - x_0) + (y - f(x_0))\left[\frac{dy}{dx}\right]_{x = x_0} = 0$$

or, $(x - x_1) + \frac{x_1 b^2}{y_1 a^2}(y - y_1) = 0$

or, $\frac{a^2(x - x_1)}{x_1} + \frac{b^2(y - y_1)}{y_1} = 0$

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2) Find the equation of tangent and normal to the curve. $xy = a$ at point $(x_1, y_1)$

Soln

Given,

$$xy = a$$

Differentiating both sides with respect to $x$ we get,

$$\frac{d(xy)}{dx} = 0$$

or $$\frac{dy}{dx} = \frac{-y}{x}$$

So, $$\left[\frac{dy}{dx}\right]_{(x_1, y_1)} = \frac{-y_1}{x_1}$$

The equation of tangent be

$$y - f(x_0) = \left[\frac{dy}{dx}\right]_{x = x_0} (x - x_0)$$

Th or $$y - y_1 = \frac{-y_1}{x_1^2}(x - x_1)$$

or, $$x_1^2 y - x_1^2 y_1 = -y_1 x + x y_1$$

or $$x_1 y - x_1 y_1 + y_1 x - x_1 y_1 = 0$$

or $$\therefore x_1 y - 2x_1 y_1 + y_1 x = 0$$

Equation of normal be

$$(x - x_0) + (y - f(x_0)) \left[\frac{dy}{dx}\right]_{x = x_0} = 0$$

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$$(x-x_1) - \frac{y_1}{x_1}(y-y_1) = 0$$

$$x_1(x-x_1) - y_1(y-y_1) = 0$$

3) Find the equation of tangent and normal to the curve $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ at the point $(x_1, y_1)$

Sol$^n$

Given,

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

Differentiating both sides with respect to $x$ we get,

$$\frac{d\,x^2}{a^2\,dx} + \frac{dy^2}{y} \cdot \frac{1}{b^2} = 0$$

$$\frac{2x}{a^2} + \frac{2y}{b^2} \cdot \frac{dy}{dx} = 0$$

$$\therefore \frac{dy}{dx} = \frac{-2x}{a^2} \times \frac{b^2}{2y} = \frac{-xb^2}{ya^2}$$

$$\text{So, } \left[\frac{dy}{dx}\right]_{(x_1, y_1)} = \frac{-x_1 b^2}{-y_1 a^2}$$

The equation of tangent be

$$y - f(x_0) = \left[\frac{dy}{dx}\right]_{x = x_0} (x - x_0)$$

or, $y - y_1 = \frac{-x_1 b^2}{y_1 a^2}(x - x_1)$

or $\frac{x x_1}{a^2} + \frac{y y_1}{b^2} = 1$.

Equation of normal be

$$(x - x_0) + (y - f(x_0))\left[\frac{dy}{dx}\right]_{x = x_0} = 0$$

$$(x - x_1) - \frac{x_1 b^2}{y_1 a^2}(y - y_1) = 0$$

or, $\frac{a^2(x - x_1)}{x_1} - \frac{b^2(y - y_1)}{y_1} = 0$

4. Find the equation of tangent and normal to the curve $y^2 = 4ax$ at the point $(x_1, y_1)$

Sol$^n$

Given,

$$y^2 = 4ax$$

Differentiating both sides with respect to x we get,

$$\frac{dy^2}{dx} = 4ax \frac{dx}{dx}$$

or, $\frac{dy}{dx} = \frac{4a}{2y} = \frac{2a}{y}$

So, $\left[\frac{dy}{dx}\right]_{(x_1, y_1)} = \frac{2a}{y_1}$

The equation of tangent be

$$y - f(x_0) = \left[\frac{dy}{dx}\right]_{x = x_0} (x - x_0)$$

or $y - y_1 = \frac{2a}{y^2}(x - x_1)$

or, $yy_1 - y_1^2 = 2a(x - x_1)$

or, $yy_1 - 4ax_1 = 2ax - 2ax_1$

or $yy_1 = 2ax + 2ax_1$

or, $yy_1 = 2a(x + x_1)$

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Equation of normal be

$$(x-x_0)+(y-f(x_0))\left[\frac{dy}{dx}\right]_{x=x_0}=0$$

or, $(x-x_1)+(y-y_1)\dfrac{2a}{y_1}=0$

or, $(x-x_1)y_1+2a(y-y_1)=0$

5. Find the equation of tangent and normal to the curve $x^2+y^2=a^2$ at a point $(x_1,y_1)$

Soln

Given, $x^2+y^2=a^2$

Differentiating both sides with respect to $x$ we get,

$$\frac{dx^2}{dx}+\frac{dy^2}{dx}=0$$

or, $\dfrac{dy}{dx}=\dfrac{-x}{y}$

So, $\left[\dfrac{dy}{dx}\right]_{(x_1,y_1)}=-\dfrac{x_1}{y_1}$

The equation of tangent be

$$y-f(x_1)=\left[\frac{dy}{dx}\right]_{x=x_1}(x-x_1)$$

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or, $y - y_1 = \frac{-x_1}{y_1}(x - x_1)$

or, $yy_1 - y_1^2 = -xx_1 + x_1^2$

or, $yy_1 + xx_1 = x_1^2 + y_1^2$

or, $yy_1 + xx_1 = a^2$

~~Eq~~

Equation of normal be

$(x - x_0) + (y - f(x_0)) \left[\frac{dy}{dx}\right]_{x = x_0} = 0$

or, $(x - x_1) + (y - y_1) \cdot \frac{-x_1}{y_1} = 0$

or, $(x - x_1)y_1 - (y - y_1)x_1 = 0$

or, $xy_1 - \cancel{y_1 x_1} - x_1 y + \cancel{x_1 y_1} = 0$

or, $xy_1 - x_1 y = 0$

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Q. Find the equation of the tangent at t on the curve $x = a\sin^3 t$ and $y = b\cos^3 t$

Soln

Given, parametric curve is defined by equations

$$x = a\sin^3 t \quad \text{---} \; (i)$$
$$y = b\cos^3 t \quad \text{---} \; (ii)$$

Differentiating eqn (i) with respect to t we get

$$\frac{dx}{dt} = \frac{d}{dt} a\sin^3 t.$$

$$= a\frac{d\sin^3 t}{d\sin t} \times \frac{d\sin t}{dt}$$

$$\therefore \frac{dx}{dt} = 3a\sin^3 t \cdot \cos t$$

Again, differentiating eqn (ii) with respect to t we get

$$\frac{dy}{dt} = \frac{d\,b\cos^3 t}{dt}$$

$$= 3\frac{d\cos^3 t}{d\cos t} \times \frac{d\cos t}{dt}$$

$$= 3b\cos^2 t \cdot -\sin t$$

$$= -3b\cos^2 t \cdot \sin t$$

Now, $\frac{dy}{dx} = \frac{dy}{dt} \cdot \frac{dt}{dx}$

$$= -\cancel{3}b\cos^{\cancel{2}} t \cdot \cancel{\sin t} \cdot \frac{1}{\cancel{3}a\sin^{\cancel{2}} t \cdot \cancel{\cos t}}$$

$$= -\frac{b}{a} \frac{\cos t}{\sin t}$$

$$= -\frac{b}{a} \cot t$$

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Now, $\left[\frac{dy}{dx}\right]_t = -\frac{b}{a}\cot t$

Then, the equation of tangent be

$$(y - f(x_0)) = \left[\frac{dy}{dx}\right]_{x=x_0} (x - x_0)$$

or, $y - b\cos^3 t = -\frac{b}{a}\cot t\,(x - a\sin^3 t)$

or $ay - ab\cos^3 t = -bx\cot t + ab\sin^3 t \cdot \cot$

or, $bx\cot t + ay = \cancel{+}\,ab\sin^3 t \cdot \frac{\cos t}{\sin t} + ab\cos^3 t$

or, $bx\cot t + ay = ab\sin^2 t \cdot \cos t + ab\cos^3 t$

or, $\frac{\cancel{b}x \frac{\cos t}{\sin t}}{a\cancel{b}\cancel{\cos t}} + \frac{\cancel{a}y}{ab\cos t} = \frac{\cancel{ab}(\sin^2 t \cdot \cos t + \cos^3 t)}{\cancel{ab}\cos t}$

$\therefore \frac{x}{a\sin t} + \frac{y}{b\cos t} = 1$

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Q. Find the equation of the tangent at the point $\theta$ of the curve $x = a(\theta + \sin\theta)$ $y = a(2 - \cos\theta)$

Soln

Given, parametric curve is defined by equation

$$x = a(\theta + \sin\theta) \quad \text{--- (i)}$$
$$y = a(1 - \cos\theta) \quad \text{--- (ii)}$$

Differentiating eqn (i) with respect to $\theta$ we get

$$\frac{dx}{d\theta} = a\frac{d(\theta + \sin\theta)}{d\theta}$$

$$= a\frac{d\theta}{d\theta} + \frac{d\sin\theta}{d\theta}$$

$$\therefore \frac{dx}{d\theta} = a(\cos\theta + 1)$$

Again, differentiating eqn (i) with respect to $\theta$ we get,

$$\frac{dy}{d\theta} = \frac{d\,a(1 - \cos\theta)}{d\theta}$$

$$= a\frac{d1}{d\theta} - \frac{d\cos\theta}{d\theta}$$

$$= a \times 0 - (-\sin\theta)$$

$$\therefore \frac{dy}{d\theta} = a\sin\theta$$

Now,

$$\frac{dy}{dx} = \frac{dy}{d\theta} \cdot \frac{d\theta}{dx}$$

$$= a\sin\theta \cdot \frac{1}{a(1+\cos\theta)}$$

$$= \frac{\sin\theta}{1+\cos\theta}$$

Now,

$$\left[\frac{dy}{dx}\right]_{\theta} = \cancel{1}\ \frac{\sin\theta}{1+\cos\theta}$$

Then, the equation of tangent be

$$y - y_0 = \left[\frac{dy}{dx}\right]_{x=x_0} (x - x_0)$$

or, $y - a(1-\cos\theta) = \frac{\sin\theta}{1+\cos\theta}\{x - a(\theta+\sin\theta)\}$

or, $y(1+\cos\theta) - \cancel{a\sin^2\theta} = x\sin\theta - a\theta\sin\theta - \cancel{a\sin^2\theta}$

or, $y(1+\cos\theta) = x\sin\theta - a\theta\sin\theta$

$y + a\sin\theta = x\sin\theta - y\cos\theta$

Cartesian subtangent and subnormal, length of tangent and normal

8. Find the length of subtangent and subnormal on the parabola $y^2 = 4ax$.

Soln

Given curve, $y^2 = 4ax$

Differentiating both sides with respect to x we get

$$\frac{dy}{dx} = \frac{2a}{y}$$

$$\text{Subtangent} = \frac{y}{y_1} = \frac{y}{\frac{2a}{y}}$$

$$= \frac{y^2}{2a}$$

$$= \frac{4ax}{2a}$$

$$= 2x$$

$$\text{Subnormal} = yy' = y\frac{2a}{y} = 2a$$ Ans

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Q. Find the length of the tangent, length of the normal, sub-tangent and sub normal of the parametric curve.

$$x = a(\theta + \sin\theta),\ y = a(1 - \cos\theta)$$

Sol$^n$

$$x = a(\theta + \sin\theta) \quad \text{---} \ (i)$$
$$y = a(1 - \cos\theta) \quad \text{---} \ (ii)$$

Differentiating both side of eqn (i) with respect to $\theta$ we get,

$$\frac{dx}{d\theta} = a(1 + \cos\theta)$$

Again, differentiating both side of eqn (ii) with respect to $\theta$ we get.

$$\frac{dy}{dx} = a\sin\theta$$

Then,

$$\frac{dy}{dx} = \frac{dy}{d\theta} \cdot \frac{d\theta}{dx}$$

$$= a\sin\theta \cdot \frac{1}{a(1+\cos\theta)}$$

$$= \frac{\sin\theta}{1+\cos\theta}$$

$$= \frac{(\sin\theta/2)}{(1+\cos\theta/2)}$$

$$= \frac{2\sin\theta/2\ \cos\theta/2}{2\cos^2\theta/2}$$

$$= \tan\frac{\theta}{2}$$

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Now,

length of the tangent $= y\sqrt{1+\frac{1}{y'^2}}$

$$= a(1-\cos\theta)\sqrt{1+\frac{1}{\tan^2\theta/2}}$$

$$= a(1-\cos\theta)\sqrt{\frac{\tan^2\theta/2+1}{\tan^2\theta/2}}$$

$$= a(1-\cos\theta)\sqrt{\frac{\sec^2\theta/2}{\tan^2\theta/2}}$$

$$= a(1-\cos\theta)\cdot\operatorname{cosec}\theta/2$$

$$= a\cdot 2\sin^2\theta/2\cdot\operatorname{cosec}\theta/2$$

$$= 2a\sin\theta/2$$

length of the normal $= y\sqrt{1+y'^2}$

$$= a(1-\cos\theta)\sqrt{1+\tan^2\frac{\theta}{2}}$$

$$= a(1-\cos\theta)\cdot\sec\frac{\theta}{2}$$

$$= a\cdot 2\sin^2\frac{\theta}{2}\cdot\cancel{\sec\frac{\theta}{2}}\ \frac{1}{\cos\theta/2}$$

$$= 2a\sin\frac{\theta}{2}\cdot\tan\frac{\theta}{2}$$

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$$\text{subtangent} = \frac{y}{y_1}$$

$$= \frac{a(1-\cos\theta)}{\frac{\sin\theta}{1+\cos\theta}}$$

$$= \frac{a(1-\cos\theta)(1+\cos\theta)}{\sin\theta}$$

$$= \frac{a(1-\cos^2\theta)}{\sin\theta}$$

$$= \frac{a\sin^2\theta}{\cancel{\sin\theta}}$$

$$= a\sin\theta$$

$$\text{Subnormal} = yy'$$

$$= a(1-\cos\theta)\cdot\frac{\sin\theta}{(1+\cos\theta)}$$

$$= a(1-\cos\theta)\cdot\tan\frac{\theta}{2}$$

$$= 2a\sin^2\frac{\theta}{2}\ \tan\frac{\theta}{2}$$

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8. Show that at any point of the hyperbola $xy = c^2$, the subtangent varies as the abscissa and the subnormal varies as the cube of the ordinate of the point of contact.

Soln.

Given,

$$xy = c^2 \quad \text{--- (i)}$$

Then,

$$\frac{dy}{dx} = \frac{-y}{x}$$

Now,

$$\text{subtangent} = \frac{y}{y_1} = \frac{y}{-y/x} = -x$$

Thus, subtangent varies as the abscissa.

Again,

$$\text{subnormal} = yy' = y \cdot \left(-\frac{y}{x}\right)$$

$$= -y^2 \cdot \frac{1}{x}$$

from eqn (i) $x = \frac{c^2}{y}$

Then, the subnormal becomes

$$-y^2 = \frac{1}{\frac{c^2}{y}}$$

$$= \frac{-y^3}{c^2}$$

Thus, the subnormal varies as the cube of the ordinate.

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Q. Show that in the curve $by^2 = (a+x)^3$, the square of the subtangent varies as the sub normal.

Sol$^n$

Given curve is $by^2 = (a+x)^3$

Differentiating both sides with respect to $x$ we get,

$$\frac{d}{dx} b \cdot y^2 \frac{dy}{dx} = \frac{d(a+x)^3}{dx}$$

$$\frac{dy}{dx} = \frac{3(a+x)^2}{2by}$$

Then,

$$\text{Subtangent} = \frac{y}{y_1} = \frac{2by \cdot y^2}{3(a+x^2)}$$

$$= \frac{2(a+x)^3}{3(a+x)^2} = \frac{2}{3}(a+x)$$

$$\text{Subnormal} = yy' = \cancel{y}\,\cancel{(3a} \; \cancel{y}\frac{3(a+x)^2}{2b\cancel{y}}$$

$$= \frac{3(a+x)^2}{2b}$$

$$\frac{(\text{Subtangent})^2}{\text{subnormal}} = \frac{4(a+x)^2}{3} \times \frac{2b}{3(a+x)^2}$$

$$= \frac{8b}{9} = \text{constant.}$$

$$(\text{subtangent})^2 = \text{constant} \times \text{subnormal}$$
$$(\text{subtangent})^2 \propto \text{subnormal.}$$

$\therefore$ The square of tangent varies as the subnormal

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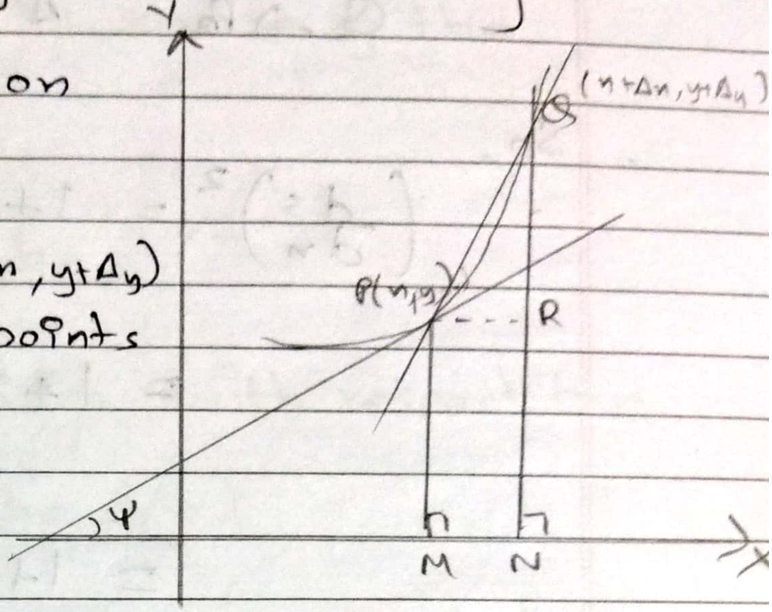
# Derivation of arc-length in cartesian form.

let A be a fixed point on the curve $y = f(x)$




let $P(x, y)$ and $Q(x+\Delta x, y+\Delta y)$ be two neighbouring points on the curve.

$AP = s$

$AQ = s + \Delta s$

Join PQ

$PM \perp OX$, $QN \perp OX$

$PR \perp QN$

$OM = x$, $PM = y$, $ON = x + \Delta x$, $MN = \Delta x$, $QN = y + \Delta y$

$QR = \Delta y$

When Q approaches P $(Q \to P)$

$$\lim_{Q \to P} \frac{\text{chord } PQ}{\text{arc } PQ} = 1$$

From rt angled $\Delta$ PRQ

$$(\text{Chord } PQ)^2 = (PR)^2 + (QR)^2$$
$$= (\Delta x)^2 + (\Delta y)^2$$

$$\left(\frac{\text{Chord } PQ}{\Delta s} \times \frac{\Delta s}{\Delta x}\right)^2 = \left(\frac{\Delta x}{\Delta x}\right)^2 + \left(\frac{\Delta y}{\Delta x}\right)^2$$

when

$Q \to P_2 \quad \Delta x \to 0$

So,

$$\left(\frac{ds}{dx}\right)^2 = 1+\left(\frac{\Delta y}{\Delta x}\right)^2$$

$$= 1+ \lim_{\Delta x \to 0}\left(\frac{\Delta y}{\Delta x}\right)^2$$

$$= 1+\left(\frac{dy}{dx}\right)^2$$

$$\boxed{\frac{ds}{dx} = \sqrt{1+\left(\frac{dy}{dx}\right)^2}}$$

similarly,

$$\frac{ds}{dy} = \sqrt{1+\left(\frac{dx}{dy}\right)^2}$$

Values of $\sin \cancel{\phi}\, \psi$ and $\cos\psi$

$$\sin\psi = \frac{dy}{ds}$$

~~cos~~

$$\cos\psi = \frac{dx}{ds}$$

$$\tan\psi = \frac{dy}{dx}$$

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* Show that in the curve $y = be^{-a/x}$, the tangent varies as the square of the abscissa.

Sol^n

Given,

$$y = \cancel{de^{-a/x}}\ be^{-a/x}$$

Differentiating both sides with respect to x we get,

$$\frac{dy}{dx} = \frac{d}{dx}(be^{-a/x})$$

$$= b\frac{de^{-a/x}}{d-a/x} \times \frac{d(-a/x)}{dx}$$

$$= be^{-a/x} \cdot (-a)\frac{dx^{-1}}{dx}$$

$$= be^{-a/x}(-a) \cdot (-x^{-2})$$

$$= be^{-a/x} \cdot ax^{-2}$$

$$= \frac{abe^{-a/x}}{x^2}$$

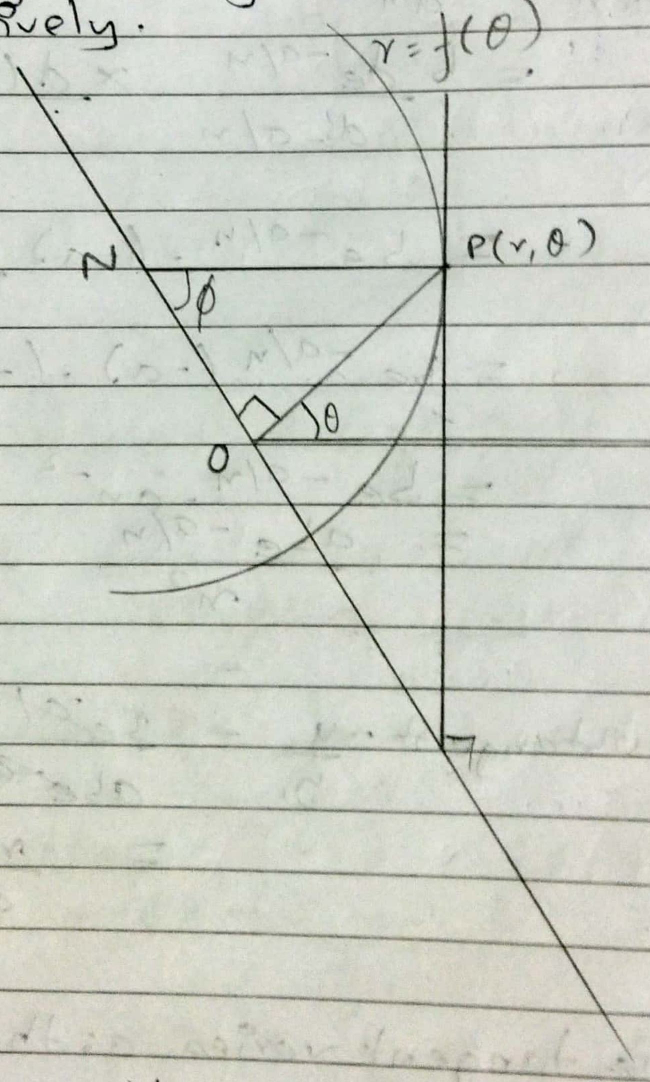
Now, subtangent $= \frac{y}{y_1} = \frac{\cancel{be^{-a/x}}}{a\cancel{be^{-a/x}}} \times x^2$

$$= \frac{x^2}{a}$$

Thus, the tangent varies as the square of absicca.

# Polar subtangent and Polar subnormal.

let $P(r,\theta)$ be a point on the curve whose polar equation is $r=f(\theta)$. let O be the pole and OX, the fixed line. From O draw a line perpendicular to OP meeting the tangent and normal to the curve $r=f(\theta)$ at a point $P(r,\theta)$ in two points T and N respectively. Then OT and ON are the polar of subtangent and polar subnormals respectively.




Now,

$$\text{Polar subtangent} = OT = \frac{r^2 d\theta}{dr}$$

$$\text{Polar subnormal} = ON = \frac{dr}{d\theta}$$

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Q Show that for the curve $r\theta = a$, the polar subtangent is constant and for the curve $r = a\theta$, the polar subnormal is constant.

Soln

Given curve $r\theta = a$

Differentiating with respect to $\theta$ we get,

$$\frac{r\,d\theta}{d\theta} + \theta\frac{dr}{d\theta} = 0$$

or $$r + \theta\frac{dr}{d\theta} = 0$$

or, $$\frac{dr}{d\theta} = -\frac{r}{\theta}$$

or $$\frac{dr}{d\theta} = -\frac{r^2}{a} \quad \left[\because \theta = \frac{a}{r}\right]$$

Now, polar subtangent $= r^2\frac{d\theta}{dr} = r^2\left(-\frac{a}{r^2}\right)$

$$\therefore \frac{d\theta}{dr} = -a = \text{constant}$$

Again, for curve $r = a\theta$

Differentiating with respect to $\theta$ we get,

$$\frac{dr}{d\theta} = \frac{d\,a\theta}{d\theta}$$

$$\therefore \frac{dr}{d\theta} = 0$$

$\therefore$ polar sub normal $= \frac{dr}{d\theta} = a =$ constant.

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Q. Find the polar subtangent of the curve $r = a(1-\cos\theta)$

Sol$^n$

Given,

curve $r = a(1-\cos\theta)$

Differentiating both sides with respect to $\theta$ we get,

$$\frac{dr}{d\theta} = a\frac{d(1-\cos\theta)}{d\theta}$$

$$= a\left(\frac{d1}{d\theta} - \frac{d\cos\theta}{d\theta}\right)$$

$$= a\cancel{\times}\{0-(-\sin\theta)\}$$

$$= a\sin\theta$$

Now,

$$\text{polar subtangent} = r^2\frac{d\theta}{dr}$$

$$= r^2\frac{1}{a\sin\theta}$$

$$= \frac{r^2}{a\sin\theta} = \frac{\{a(1-\cos\theta)\}^2}{a\sin\theta}$$

$$= \frac{a(4\sin^4\theta/2)}{2\sin\theta/2\cdot\cos\theta/2} = \frac{a^{\cancel{2}}(1-\cos\theta)^2}{\cancel{a}\sin\theta}$$

$$= \frac{a(1-2\cos\theta+\cos^2\theta)}{\sin\theta}$$

$$= a\left(\frac{1}{\sin\theta} - \frac{2\cos\theta}{\sin\theta} + \frac{\cos^2\theta}{\sin\theta}\right)$$

$$= a(\operatorname{cosec}\theta - 2\cot\theta + \cot\theta.$$

Related rate

velocity $v = \frac{ds}{dt}$

Acceleration $a = \frac{d^2s}{dt^2}$

Q. A point in motion obeys the law $s = a\cos\omega t$. Determine its velocity and acceleration at time $t$ and show that the acceleration $\frac{d^2s}{dt^2}$ and displacement $s$ are related by the equation $\frac{d^2s}{dt^2} + \omega^2 s = 0$

Sol$^n$

Given $s = a\cos\omega t$ --- (i)

Differentiating with respect to $t$ we get

$$\frac{ds}{dt} = -a\omega \cdot \sin\omega t \quad --- (ii)$$

Again differentiating eqn (ii) with respect to $t$ we get

$$\frac{d^2s}{dt} = -a\omega^2\cos\omega t \quad --- (iii)$$

From eqn (i) & (iii)

$$\frac{d^2s}{dt^2} = -\omega^2 s$$

$$\frac{d^2s}{dt^2} + \omega^2 s = 0.$$

Q. The relation $x = 3te^{-2t}$ defines the displacement of a certain mechanism at a time t sec. show that this function satisfies the differential equation

$$\frac{d^2x}{dt^2} + 4\frac{dx}{dt} + 4x = 0$$

Sol$^n$

we have,

$$x = 3te^{-2t}$$

Differentiating both sides with respect to t we get,

$$\frac{dx}{dt} = 3e^{-2t} + 3t(-2e^{-2t})$$

or $$\frac{dx}{dt} = 3e^{-2t} - 6te^{-2t}$$

and $$\frac{d^2x}{dt^2} = -6e^{-2t} - 6[e^{-2t} + te^{-2t}(-2)]$$

$$= -6e^{-2t} - 6e^{-2t} + 12te^{-2t}$$

$$= -12e^{-2t} + 12e^{-2t}$$

Hence, $$\frac{d^2x}{dt^2} + 4\frac{dx}{dt} + 4x$$

$$= -12e^{-2t} + 12te^{-2t} + 4(3e^{-2t} - 6te^{-2t}) + 4(3te^{-2t})$$

$$= 0$$

which shows that the function (i) satisfies the given differential equation.

2. The radius of sphere in is increasing at a variable rate and is equal to 1 cm/sec, when the radius is 3 cm. find the rate of change of volume at this time.

Soln

The volume V of the sphere is given by

$$v = \frac{4}{3}\pi r^3$$

and so, $\frac{dv}{dt} = \frac{4}{3}\pi\, 3r^2 \frac{dr}{dt} = 4\pi r^2 \frac{dr}{dt}$

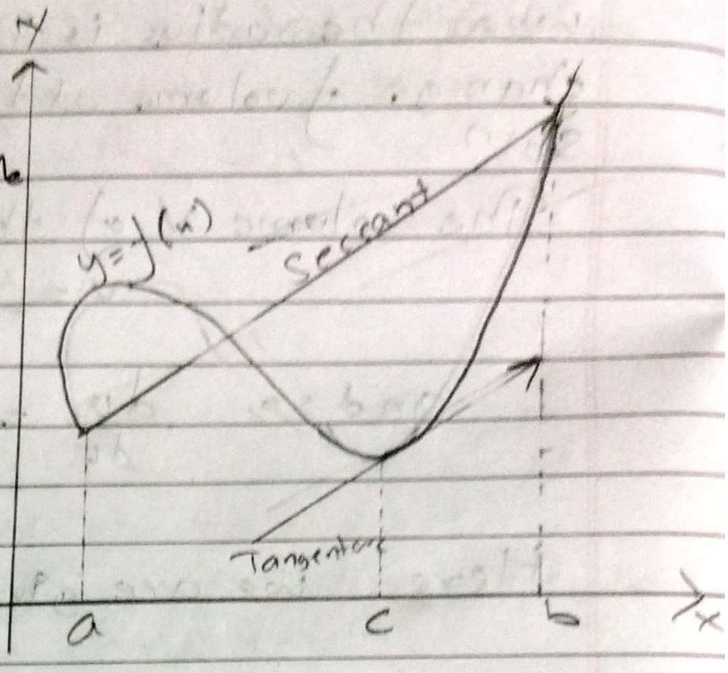
Here, we are given $\frac{dr}{dt} = 1$ when $r=3$

Hence, $\frac{dv}{dt} = 36\pi$ cc/sec

Which ~~me~~ means that the volume is changing at the rate of $36\pi$ cc per second.

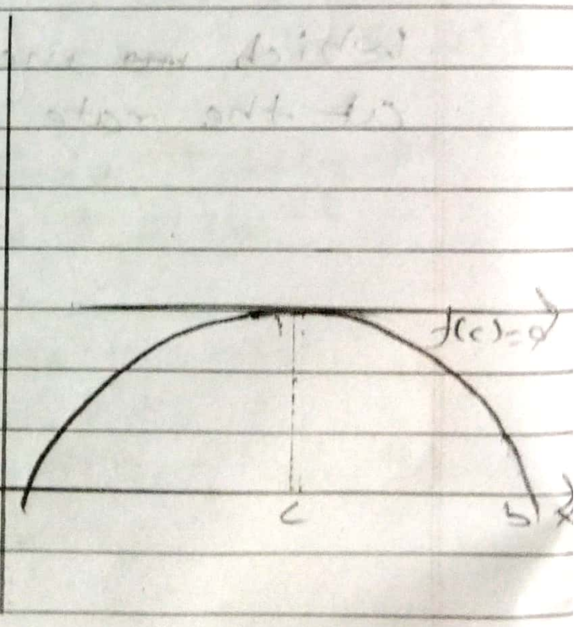
# Mean Value Theorem

For any function that is continuous on $[a,b]$ and differentiable on $(a,b)$ there exists some $c$ in the interval $(a,b)$ such that the secant joining the end points of the interval $[a,b]$ is parallel to the tangent arc.




# Rolle's Theorem

If a real-valued function $f$ is continuous on a closed interval $[a,b]$, differentiable on the open interval $(a,b)$ and $f(a)=f(b)$, then there exists a $c$ in the open interval $(a,b)$ such that $f'(c)=0$




OR

If $f(x)$ is continuous in a closed interval $a \le x \le b$, if $f'(x)$ ~~exist~~ exists in the open interval $a<x<b$ if $f(a)=f(b)$, then $f'(x)$ vanishes for at least one value of $x$ in the interval $a<x<b$.

Geometrically, If $f(x)$ holds the all three conditions of Rolle's theorem then there exist at least one tangent parallel to x-axis on the interval $(a,b)$

Q. If $f(x)$ and $g(x)$ are differentiable functions for $0 \le x \le 1$ such that $f(0) = 5$, $g(0) = 1$, $f(1) = 8$, $g(1) = 2$, then show that there exists c satisfying $0 < c < 1$ and $f'(c) = 3g'(c)$

Soln

let function $h(x) = f(x) - 3g(x)$ since $f(x)$ and $g(x)$ are differentiable function, $h(x)$ is also differentiable in $0 \le x \le 1$.

So,

$$h(0) = f(0) - 3g(0)$$
$$= 5 - 3$$
$$= 2$$

and
$$h(1) = f(1) - 3g(1)$$
$$= 8 - 3 \times 2$$
$$= 2$$

$\therefore$ $h(x)$ satisfies the condition of Rolle's theorem. Hence, there exists a number c $0 < c < 1$, such that $h'(x) = 0$

Thus

$$f'(c) - 3g'(c) = 0 \longrightarrow f'(c) = 3g(c)$$

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Q. Use Rolle's theorem to prove that between any two real roots of $e^{-x} = \sin x$, there lies at least one real root of $e^{-x} = -\cos x$.

Soln

let $f(x) = e^{-x} - \sin x$

let two real roots of $f(x) = 0$ be given by $x = a$ and $x = b$ so that $f(a) = f(b) = 0$

Now, $f'(x) = -e^{-x} - \cos x$, which exists in $a < x < b$.

Hence, $f(x)$ satisfies all conditions of Rolle's theorem. It follows that $f'(x) = 0$ for at least one value of $x$ in $a < x < b$.

Thus

$$-e^{-x} - \cos x = 0$$

$$\therefore e^{-x} = -\cos x$$

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1. If $f(x) = x^3 + x$, $a = 0, b = 1$ determine $\xi$ such that $\cancel{f(b) =} \frac{f(b) - f(a)}{b - a} = f'(\xi), a < \xi < b)$

Soln

$f(x) = x^3 + x$

i) $f(x)$ is continuous on the $[0, 1]$ being polynomial.

ii) $f'(x) = 3x^2 + 1$, which is well defined for all real $x$. $\therefore f(x)$ is differentiable on $(0, 1)$

Thus, $f(x)$ holds condition of mean value theorem

$\therefore$ There exist $\xi \in (0, 1)$ such that

$$f'(\xi) = \frac{f(1) - f(0)}{1 - 0}$$

Then,

$$3\xi^2 + 1 = \frac{2 - 0}{1}$$

$$\therefore 3\xi^2 + 1 = 2$$

$$\text{or } \xi^2 = \frac{1}{3}$$

$$\therefore \xi = \frac{1}{\sqrt{3}}.$$

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2. Verify that $f(x) = x^2+x+4$ satisfies the conditions of Rolle's theorem on the interval $[-3, 2]$ and find the numbers such that $f'(c) = 0$

Soln

$f(x) = x^2+x+4$

i) $f(x)$ is continuous on the $[-3,2]$ being polynomial

ii) $f'(x) = 2x+1$, which is well defined for all real.

$\therefore f'(x)$ exists on the $(-3,2)$

iii) $f(-3) = (-3)^2 + (-3) + 4$
$= 10$

$f(2) = (2)^2 + 2 + 4$
$= 10$

$\therefore f(-3) = f(2)$

Thus, all the condition of Rolle's theorem holds

Thus, $\exists\, c \in (-3, 2)$ such that $f'(c) = 0$

$2c+1 = 0$

$c = -\frac{1}{2}$

$\therefore \in [-3, 2)$

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3. Apply Rolle's theorem to the function $f(n) = n^3 - 12n$ in $[0, 2\sqrt{3}]$

Sol$^n$

$f(n) = n^3 - 12n$ is given.

i) $f(n)$ is continuous in $[0, 2\sqrt{3}]$ being polynomial

ii) $f'(n) = 3n^2 - 12$ being differentiable in $(0, 2\sqrt{3})$ and well defined for all real $x$.

iii) $f(0) = 0^3 - 12 \times 0 = 0$

And,

$$f(2\sqrt{3}) = (2\sqrt{3})^2 - 12(2\sqrt{3})$$
$$= 0$$
$$\therefore f(0) = f(2\sqrt{3})$$

Thus, all the conditions of Rolle's theorem holds. Hence, there exists $c \in (0, 2\sqrt{3})$ such that

$$f'(c) = 0$$

and

$$f'(c) = 3c^2 - 12 = 0$$
$$\text{or, } c^2 = \frac{12}{3}$$

$$c = \sqrt{\frac{12}{3}} \in (0, 2\sqrt{3})$$

4. Explain why Rolle's theorem is inapplicable to the function $f(x) = 1-(x-1)^{2/3}$ on the interval $[-1, 1]$

Soln

Given

$$f(x) = 1-(x-1)^{2/3}$$

i) $f(x)$ is continuous on the interval $[-1, 1]$ being polynomial

ii) $f(x)$ is differentiable on the interval $(-1, 1)$ and was defined for all real $x$.

$$f(-1) = 1-(-1-1)^{2/3}$$
$$= 1-(-2)^{2/3}$$

And,

$$f(1) = 1-(1-1)^{2/3}$$
$$= 1-(0)^{2/3}$$
$$= 1$$

Since, $f(-1) \neq f(1)$ Rolle's theorem is inapplicable to the function $f(x) = 1-(x-1)^{2/3}$

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5. What point on the parabola $y = x^2 + x + 1$ satisfies the mean value theorem in the range $-1 \leq x \leq 3$

Soln

Given $f(x) = x^2 + x + 1$

i) $f(x)$ is continuous on interval $[-1, 3]$ being polynomial.

Since $f(x)$ satisfies mean value theorem, There exists a point $c \in (-1, 3)$ such that

$$f'(c) = \frac{f(b) - f(a)}{b - a}$$

$$= \frac{f(3) - f(-1)}{3 - 1}$$

or $2c + 1 = \frac{13 - 1}{4}$

or $2c + 1 = 3$

or $c = 1$

$\therefore c = 1 \in (-1, 3)$

Hence, at $c = 1$, the parabola $y = x^2 + x + 1$ satisfies the mean value theorem in the range $-1 \leq x \leq 3$.

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6. If $f(x) = x^4 + x^3 - 5x - 7$, show that there is a point $\xi$ in the interval $[0, 1]$ such that $f'(\xi) = 0$

Soln

Given $f(x) = x^4 + x^3 - 5x - 7$

i) $f(x)$ is continuous in $[0,1]$ being polynomial

ii) $f'(x) = 4x^3 + 3x^2 - 5$ being differentiable in $(0,1)$ and defined for all real $x$

iii) $f(0) = 0^4 + 0^3 - 5\times 0 - 7$
$= -7$

and

$f(1) = 1 + 1 - 5 - 7$
$= 2 - 5 - 7 = -10$

$f'(x) = 4x^3 + 3x^2 - 5$

$f'(\xi) = 0$

or $4\xi^3 + 3\xi^2 - 5 = 0$

or, $4\xi^3 + 3\xi^2 = 5$

L' Hospital rule

Statement: If $f(x)$ and $g(x)$ as also their ~~dere~~ derivatives $f'(x)$ and $g'(x)$ are continuous at $x=a$, and if $f(a)=g(a)=0$, then,

$$\lim_{x\to a} \frac{f(x)}{g(x)} = \lim_{x\to a} \frac{f'(x)}{g'(x)}$$

provided that $g'(x) \neq 0$.

Evaluate: $\lim_{x\to 0} \frac{\log(1-x^2)}{\log(\cos x)}$ $\left(\frac{0}{0}\text{ form}\right)$

using L'Hospitals rule,

$$\lim_{x\to 0} \frac{\cancel{-}2x(1-x^2)}{\cancel{-}\tan x}$$

$$\lim_{x\to 0} \frac{2x}{(1-x^2)\tan x} \quad \left(\frac{0}{0}\right)\text{ form}\}$$

$$= \lim_{x\to 0} \frac{x}{\tan x} \times \lim_{x\to 0} \frac{2}{1-x^2}$$

$$= \lim_{x\to 0} \frac{x/x}{\left(\frac{\tan x}{x}\right)} \times 2$$

$$= 2\times 1$$

$$= 2$$

$$\lim_{x\to 0} \frac{5\sin x - 7\sin 2x + 3\sin 3x}{\tan x - x} \left(\frac{0}{0}\text{ form}\right)$$

Using L' Hospital rule.

$$\lim_{x\to 0} \frac{5\cos x - 14\cos 2x + 9\cos 3x}{\sec^2 x - 1} \left(\frac{0}{0}\text{ form}\right)$$

Again using L' Hospital rule

$$\lim_{x\to 0} \frac{-5\sin x + 28\sin 2x - 27\sin 3x}{2\sec^2 x \cdot \tan x} \left(\frac{0}{0}\text{ form}\right)$$

Again using L' Hospital rule

$$\lim_{x\to 0} \frac{-5\cos x - 56\cos 2x - 81\cos 3x}{2\sec^4 x + 4\sec^2 x \cdot \tan^2 x}$$

$$= \frac{-5 + 56 - 81}{2 + 4\times 0}$$

$$= \frac{-30}{2}$$

$$= -15$$

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# Evaluate:

$$\lim_{x\to 0} \frac{a^x-1}{x} \quad \left(\frac{0}{0} \text{ form}\right)$$

$$= \lim_{x\to 0} \frac{e^{x\log_e a}-1}{x}$$

Using L'Hospital rule,

$$= \lim_{x\to 0} \frac{\log_e a \, e^{x\log_e a}}{1}$$

$$= \log_e a \, e^0$$

$$= \log_e a$$

$$\therefore \lim_{x\to 0} \frac{a^x-1}{x} = \log_e a.$$

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\# Evaluate :

$$\lim_{n \to 0} \frac{2^n - 1}{(1+n)^{1/2} - 1}$$

We have,

$$\frac{2^n - 1}{\sqrt{1+n} - 1} = \frac{2^n - 1}{\sqrt{1+n} - 1} \times \frac{\sqrt{1+n} + 1}{\sqrt{1+n} + 1}$$

$$= \frac{(2^n - 1)(\sqrt{1+n} + 1)}{n}$$

$$\therefore \lim_{n \to 0} \frac{2^n - 1}{\sqrt{1+n} - 1}$$

$$= \lim_{n \to 0} \frac{(2^n - 1)(\sqrt{1+n} + 1)}{n} \cdot \left(\frac{0}{0} \text{ form}\right)$$

Using L'Hospital Rule,

$$\lim_{n \to 0} \frac{2^n - 1}{n} \times \lim_{n \to 0} \sqrt{1+n} + 1$$

$$= \log_e 2 \times (\sqrt{1+1} + 1)$$

$$= 2 \log_e 2$$

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# Evaluete :

$$\lim_{n\to 0} \frac{n^n}{e^n} \left(\frac{\infty}{\infty} \text{ form}\right)$$

Using L Hospital rule

$$\lim_{n\to\infty} \frac{n n^{n-1}}{e^n} \left(\frac{\infty}{\infty} \text{ form}\right)$$

or, $$\lim_{n\to\infty} \frac{n(n-1)n^{n-2}}{e^n} \left(\frac{\infty}{\infty} \text{ form}\right)$$

continuously applying L' Hospital rule we get,

$$\lim_{n\to 0} \frac{n(n-1)(n-2)\,.\,\ldots\,1}{e^n}$$

$$= \frac{n(n-1)\ (n-2)\ \ldots\,1}{\infty}$$

$$= 0$$

$$\therefore \lim_{n\to 0} \frac{n^n}{e^n} = 0$$

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# Evaluate: $\lim_{x\to 0} \frac{\log x}{\operatorname{cosec} x}$ $\left(\frac{\infty}{\infty} \text{ form}\right)$

$$= \lim_{x\to 0} \frac{1/x}{-\operatorname{cosec} x . \cot x}$$

$$= \lim_{x\to 0} \frac{\sin x}{x} \tan x$$

$$= -1 \tan 0$$

$$= -1 \times 0 = 0$$

# Evaluate: $\lim_{x\to 0} \sin x \log x^2$ [$0\times\infty$ form]

$$= \lim_{x\to 0} \frac{2 \log x}{\operatorname{cosec} x} \left(\frac{\infty}{\infty} \text{ form}\right)$$

Applying L' Hospital Rule,

$$= \lim_{x\to 0} \frac{2/x}{-\operatorname{cosec} x . \cot x} \left(\frac{\infty}{\infty} \text{ form}\right)$$

$$= -\lim_{x\to 0} \frac{2}{x \operatorname{cosec} x . \cot x} \left(\frac{\infty}{\infty} \text{ form}\right)$$

$$= -\lim_{x\to 0} \frac{2 \sin x . \sin x}{x \cos x}$$

$$= -\lim_{x\to 0} \frac{\sin x}{x} . \lim_{x\to 0} 2 \tan x$$

$$= -1 \times 2.0$$

$$= 0.$$

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* $\lim_{x\to 1} (1-x)\tan\left(\frac{\pi x}{2}\right)$ · ( $0 \times \infty$ form)

$= \lim_{x\to 1} \frac{1-x}{\cot(\pi x/2)}$ $\left(\frac{0}{0} \text{ form}\right)$

Applying L Hospital rule,

$= \lim_{x\to 1} \frac{-1}{-\frac{\pi}{2} \cdot \operatorname{cosec}^2\left(\frac{\pi x}{2}\right)}$

$= \frac{1}{\pi/2 \cdot 1^2}$

$= \frac{2}{\pi}$

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# Evaluate $\lim_{x \to 0} x \log x$ ($0 \times \infty$ form)

$= \lim_{x \to 0} \dfrac{\log x}{1/x}$ $\left(\dfrac{\infty}{\infty} \text{ form}\right)$

= Using L'Hospital rule

$= \lim_{x \to 0} \dfrac{\frac{1}{x}}{-1/x^2}$

$= \lim_{x \to 0} (-x)$

$= 0$

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# $\lim_{n\to 0}\left(\frac{1}{n^2}-\frac{1}{\sin^2 n}\right)$ ($\infty-\infty$ form)

$$= \lim_{n\to 0} \frac{\sin^2 n - n^2}{n^2 \sin^2 n} \left(\frac{0}{0} \text{ form}\right)$$

Applying L'Hospital rule

$$= \lim_{n\to 0} \frac{\sin 2n - 2n}{2n\sin^2 n + n^2 \sin 2n} \left(\frac{0}{0} \text{ form}\right)$$

Again, Applying L'Hospital's rule,

$$= \lim_{n\to 0} \frac{2\cos 2n - 2}{2\sin^2 n + 2n\sin 2n + 2n^2\cos 2n + 2n\sin 2n}$$

$$= \lim_{n\to 0} \frac{2(\cos 2n - 1)}{2(\sin^2 n + n\sin 2n + n^2\cos 2n + n\sin 2n)}$$

$$= \lim_{n\to 0} \frac{\cos 2n - 1}{\sin^2 n + 2n\sin 2n + n^2\cos 2n} \left(\frac{0}{0} \text{ form}\right)$$

Again, Applying L-Hospital rule

$$= \lim_{n\to 0} \frac{-2\sin 2n}{\sin 2n + 4n\cos 2n + 2\sin 2n - 2n^2\sin 2n + 2x\cos 2n}$$

$$= \lim_{n\to 0} \frac{-2\sin 2n}{3\sin 2n + 6n\cos 2n - 2n^2\sin 2n} \left(\frac{0}{0} \text{ form}\right)$$

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Again,

Applying L'Hospital rule

$$= \lim_{x \to 0} \frac{-4\cos 2x}{6\cos 2x - 12x\sin 2x + 6\cos^2 x - 4x^2\cos 2x - 4x\sin 2x}$$

$$= \frac{-4}{6.1 - 12.0 + 6.1 - 4.0 - 4.0}$$

$$= \frac{-4}{6+6}$$

$$= \frac{-4}{12}$$

$$= \frac{-1}{3}$$ Ans

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# Evaluate $\lim_{x\to 0} (\cot x)^{\sin 2x}$ ($\infty^0$ form)

let $y = \lim_{x\to 0} (\cot x)^{\sin 2x}$

taking natural log on both sides

$$\log_e y = \lim_{x\to 0} \log_e (\cot x)^{\sin 2x}$$

$$= \lim_{x\to 0} \sin 2x . \log_e(\cot x) \quad (0\times\infty \text{ form})$$

$$= \lim_{x\to 0} \frac{\log_e(\cot x)}{\operatorname{cosec} 2x} \quad \left(\frac{\infty}{\infty} \text{ form}\right)$$

Applying l'Hospital rule,

$$= \lim_{x\to 0} \frac{\frac{-1}{\cot x} \operatorname{cosec}^2 x}{-2 \operatorname{cosec} 2x . \cot 2x}$$

$$= \lim_{x\to 0} \frac{\sin 2x}{\cos 2x}$$

$$= \lim_{x\to 0} \tan 2x$$

$$= 0$$

$$\log_e y = 0$$

$$\therefore y = e^0 = 1$$

$$\therefore \lim_{x\to 0}\left((0+x)^{\sin 2x}\right) = 1.$$

Ans//

# Evaluate

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# Evaluate

$$\lim_{x \to \frac{\pi}{2}} (\sin x)^{\tan x} \quad (1^{\infty} \text{ form})$$

let $y = \lim_{x \to \frac{\pi}{2}} (\sin x)^{\tan x}$

Taking natural log on both sides we get,

$$\log_e y = \lim_{x \to \frac{\pi}{2}} \tan x \log_e \sin x \quad [\infty \times 0 \text{ form}]$$

$$= \lim_{x \to \frac{\pi}{2}} \frac{\log_e (\sin x)}{\cot x} \quad \left(\frac{0}{0} \text{ form}\right)$$

Applying L'Hospital's Rule we get,

$$\Rightarrow \lim_{x \to \frac{\pi}{2}} \frac{\frac{1}{\sin x} \cdot \cos x}{-\operatorname{cosec}^2 x}$$

$$= \lim_{x \to \frac{\pi}{2}} (-\cos x . \sin x)$$

$$= 0.1 = 0$$

$\therefore \log_e y = 0 \Rightarrow y = 1$

$\therefore \lim_{x \to \frac{\pi}{2}} (\sin x)^{\tan x} = 1$

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# Evaluate

$$y = \lim_{x \to 1} \frac{1}{x^{1-x}} \quad (1^\infty \text{ form})$$

$$\log_e y = \lim_{x \to 1} \frac{1}{1-x} \log_e x \quad \left(\frac{0}{0} \text{ form}\right)$$

Applying L'Hospital rule,

$$\lim_{x \to 1} \frac{\frac{1}{x}}{-1}$$

$$= \lim_{x \to 1} \frac{-1}{x} = -1$$

$$\therefore \log_e y = -1$$

$$y = e^{-1}$$

$$\therefore \lim_{x \to 1} \cancel{x^{1-x}} \cdot x^{\frac{1}{1-x}} = \frac{1}{e} \ \#$$

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# $\lim_{x \to 0} \frac{\log(1+kx^2)}{1-\cos x}$ $\left(\frac{0}{0}\text{ form}\right)$

Applying L'Hospital's rule,

$$= \lim_{x \to 0} \frac{\frac{1}{(1+kx^2)} \times 2kx}{\sin x}$$

$$= \lim_{x \to 0} \frac{x}{\sin x} \times \lim_{x \to 0} \frac{2kx}{1+kx^2}$$

$$= \lim_{x \to 0} \frac{1}{\cos x} \times \frac{2k}{1}$$

$$= 2k$$ #

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# $\frac{xe^x - \log(1+x)}{x^2}$ $\left(\frac{0}{0} \text{ form}\right)$

Using L'Hospital's rule

$$= \lim_{x\to 0} xe^x + e^x - \frac{1}{(1+x)} \quad \left(\frac{0}{0} \text{ form}\right)$$

Again,
applying L'Hospital's rule

$$= \lim_{x\to 0} \frac{xe^x + e^x + e^x + \frac{1}{(1+x)^2}}{2}$$

$$= \frac{0\times 1 + 1 + 1\cancel{+}1}{2}$$

$$= \frac{3}{2} \#$$

# Maxima and Minima

Theorem 1 : If $f(x)$ is differentiable at $x=c$ and has a maximum or minimum there, then $f'(c)=0$

Note:-

$f'(x)=0$, $x=c$ (stationary point)

Theorem 2: $f(c)$ is an extreme value of $f(x)$ if and only if $f'(x)$ changes sign as $x$ passes through $c$.

From Theorem ②: $y=f(x)$ have a maximum value at $x=a$ if

(i) $\frac{dy}{dx}=0$ at $x=0$

(ii) $\frac{dy}{dx}$ changes sign from positive to negative at $x=a$

i.e, $\frac{d^2y}{dx^2}<0$ at $x=a$

From Therom (2) $y = f(x)$ have a minimum value at $x = a$ if

(i) $\frac{dy}{dx} = 0$ at $x = 0$

(ii) $\frac{dy}{dx}$ changes sign from negative to positive at $x = a$

i.e, $\frac{d^2y}{dx^2} > 0$ at $x = a$

Q. Find the maximum and minimum values of the function defined by $y=(x+2)^2(x-4)$

Sol$^n$

Given,

$$y=(x+2)^2(x-4)$$

~~i.e. y=~~

Then, $\frac{dy}{dx}=2(x+2)(x-4)+(x+2)^2\cdot 1$

$$=3(x^2-4)$$

Now,

$$\frac{dy}{dx}=0$$

i.e. $3(x^2-4)=0$

$\therefore x=\pm 2$ $(2,-2)$ [stationery point]

$$\frac{d^2y}{dx^2}=6x$$

Now,

at $x=2$, $\frac{d^2y}{dx^2}=12>0$ (minima) [positive]

$\therefore$ y has minimum value at $x=2$ and

minimum value $=(2+2)^2(2-4)$

$=16(-2)$

$=-32$ Ans

Now at $x = -2$,

$$\frac{d^2y}{dx^2} = -12 < 0 \text{ (negative)}$$

$\therefore$ ~~y has minimum value~~

So y has maximum value at $x = -2$

and maximum value $= (-2+2)^2 + (-2-4)$
$= 0$

Ans#

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8. $f(n) = 2n^3 - 9n^2 - 24n + 3$

Sol$^n$

Given,

$$f(n) = 2n^3 - 9n^2 - 24n + 3$$

Then, $f'(n) = 6n^2 - 18n - 24$

Now,

$$f'(n) = 0$$

or, $6n^2 - 18n - 24 = 0$

or, $n^2 - 3n - 4 = 0$

or, $n^2 - 4n + n - 4 = 0$

or, $n(n-4) + 1(n-4) = 0$

$(n-4)(n+1) = 0$

$\therefore n = (-1, 4)$ [stationery point]

~~At n = -1~~

$$f''(n) = 12n - 18$$

At $n = -1$, $f''(-1) = 12(-1) - 18$

$= -30 < 0$ (negative)

$\therefore f(n)$ has maximum value at $n = -1$

and maximum value $= 2(-1)^3 - 9(-1)^2 - 24(-1) + 3$

$= 16$ Ans

At $x = 4$,

$$f''(4) = 12(4) - 18$$
$$= 30 > 0 \text{ (positive)}$$

$\therefore f(x)$ has minimum value at $x = 4$

and maximum value $= 2(4)^3 - 9(4)^2 - 24(4) + 3$
$= -109$

Ans

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8. Find the maximum and minimum value of the function : $2x^3 - 21x^2 + 36x - 20$

Sol^n

let $f(x) = 2x^3 - 21x^2 + 36x - 20$

$f'(x) = 6x^2 - 42x + 36$

Now,

$f'(x) = 0$

$6x^2 - 42x + 36 = 0$

or, $x^2 - 7x + 6 = 0$

or, $x^2 - (6+1)x + 6 = 0$

or $x^2 - 6x - x + 6 = 0$

or $x(x-6) - 1(x-6) = 0$

or, $(x-6)(x-1) = 0$

$\therefore x = (1, 6)$ [stationery point]

$f''(x) = 12x - 42$

Now at $x = 1$.

$f''(1) = 12(1) - 42$

$= 12 - 42$

$= -30 < 0$ (negative)

$\therefore f(x)$ has maximum value at $x = 1$

and maximum value $= 2(1)^3 - 21(1)^2 + 36(1) - 20$

$= -3$

Ans

Now At $x = 6$,

$$f''(6) = 12(6) - 42$$
$$= 30 > 0 \text{ (positive}$$

$\therefore f(x)$ has minimum value at $x = 6$
and minimum value $=$
$$2(6)^3 - 21(6)^2 - 36(6) - $$
$$= -128$$
Ans ##

Q. $f(x) = x^5 - 5x^4 + 5x^3 - 10$

Soln
Given,
$$f(x) = x^5 - 5x^4 + 5x^3 - 10$$

or, $f'(x) = 5x^4 - 20x^3 + 15x^2$
or, $f''(x) = 20x^3 - 60x^2 + 30x$

Now,
$$f'(x) = 0$$
$$5x^4 - 20x^3 + 15x^2 = 0$$
or, $5x^2(x^2 - 4x + 3) = 0$
or, $5x^2(x^2 - 3x - x + 3) = 0$
or, $5x^2[x(x-3) - 1(x-3)] = 0$
or, $5x^2(x-3)(x-1) = 0$

$\therefore x = 0, 1, 3$ [stationery point]

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Now, at $x = 0$,

$$f''(0) = 20(0)^3 - 60(0)^2 + 30(0)$$
$$= 0$$

$\therefore$ at $x = 0$, $f(x)$ is neither maximum nor minimum Ans#

at $x = 1$

$$f''(1) = 20(1)^3 - 60(1)^2 + 30(1)$$
$$= -10 < 0 \text{ (negative)}$$

$\therefore f(x)$ is maximum value at $x = 1$
and maximum value $= (1)^5 - 5(1)^4 + 5(1)^3 - 10$
$= -9$ Ans#

at $x = 3$

$$f''(3) = 20(3)^3 - 60(3)^2 + 30(3)$$
$$= 90 > 0 \text{ (positive)}$$

$\therefore f(x)$ is minimum value at $x = 3$
and minimum value $= (3)^5 - 5(3)^4 + 5(3)^3 - 10$
$= -37$
Ans#

$f(x) = x+\frac{1}{x}$

$x + x^{-1}$

$f'(x) = 1 - 1x^{-1-1}$
$= 1-x^{-2}$
$= 1-\frac{1}{x^2}$

$f''(x) = 0-(-2)x^{-2-1}$
$= 2x^{-3}$

Q. Let $x$ and $y$ be two real variable such that $x>0$ and $xy=1$. show that minimum value of $x+y$ is 2.

Soln

Given

$x>0$, $xy=1$

let $f(x) = x+y$
$= x+\frac{1}{x}$ --- (i)

Now, $f'(x) = 1-\frac{1}{x^2}$

$f'(x)=0$

i.e, $1-\frac{1}{x^2} = 0$

$x^2-1=0$

$x = (1,-1)$ [stationery point]

According to the question $x>0$ so, only valid value of $x=1$

And $f''(x) = 0+\frac{2}{x^3}$

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At $x=1$, $f''(1) = \frac{2}{1^3} = 2$ (positive)

So $f(x)$ has minimum value at $x=1$
minimum value $= 1 + \frac{1}{1} = 2$

So minimum value of $x+y = 2$
proved #

Q. Find the maxima and minima of the function defined by $f(x) = \frac{40}{3x^4+8x^3-18x^2+60}$

Soln
Given,
$$f(x) = \frac{40}{3x^4+8x^3-18x^2+60}$$

let $F(x) = 3x^4 + 8x^3 - 18x^2 + 60$
$F'(x) = 12x^3 + 24x^2 - 36x = 0$

$\therefore x = (0, 1, -3)$ [stationery point]

$f''(x) = 36x^2 + 48x - 36$

And
$F''(0) < 0$, $F''(1) > 0$, $F''(-3) > 0$

$F(x)$ has minimum value at $x(1, -3)$

maximum value at $x=0$
Ans #

Let $V = \frac{1}{3}\pi r^2 h$

$S = \pi r^2 + \pi r l$

# Find the semivertical angle of a right circular cone of maximum value volume and a given surface area.

Solⁿ

Let $h$, $r$ and $l$ be the height, radius of the base and slant height of a right circular cone of semivertical angle $\alpha$. Then the volume $V$ of the cone is given by.

$$V = \frac{1}{3}\pi r^2 h, \quad \text{--- (i)}$$

and the surface areas is

$$S = \pi r^2 + \pi r l$$

Since $S$ is given we can rewrite (ii) as

$$l = \frac{S - \pi r^2}{\pi r} = \frac{S}{\pi r} - r$$

We have,

$$h^2 = l^2 - r^2$$

$$= \left(\frac{S}{\pi r} - r\right)^2 - r^2$$

$$= \frac{S^2}{\pi^2 r^2} + r^2 - \frac{2S}{\pi} - r^2$$

$$= \frac{S^2}{\pi^2 r^2} - \frac{2S}{\pi}$$

Now from eqn (i)

$$V^2 = \frac{1}{9}\pi^2 r^4 h^2 = \frac{1}{9}\pi^2 r^4\left(\frac{S}{\pi^2 r^2} - \frac{2S}{\pi}\right)$$

$$= \frac{1}{9} r^2 S^2 - \frac{2}{9}\pi S r^4$$

We have thus expressed V in terms of a single variable r. Hence, differentiation with respect to r yields.

$$2V\frac{dV}{dr} = \frac{1}{9}.S^2 2r - \frac{2\pi}{9}S.4r^3$$

$$V\frac{dV}{dr} = \frac{1}{9}S^2 r - \frac{4}{9}\pi S.r^3 \quad \text{(iv)}$$

Differentiating (iv) with respect to r, we obtain

$$V\frac{d^2V}{dr^2} + \left(\frac{dV}{dr}\right)^2 = \frac{1}{9}S^2 - \frac{4}{9}\pi S.3r^2 \quad \text{(v)}$$

For maxima or minima, $\frac{dV}{dr} = 0$. Hence (iv) gives.

$$S^2 r = 4\pi S r^3$$

i.e, $$r^2 = S/4\pi$$

or, $$r = \frac{1}{2}\sqrt{\frac{S}{\pi}}$$

For this value of r, we obtain from (v)

$\cancel{V = d^2}$

$$V\frac{d^2V}{dr^2} = \frac{1}{9}S^2 - \frac{4}{3}\pi S \cdot \frac{S}{4\pi}$$

$$= \frac{1}{9}S^2 - \frac{1}{3}S^2$$

$$= -\frac{2}{9}S^2 < 0.$$

Hence V is a maximum for $r^2 = S/4\pi$.

The slant height $l$ is then given by

$$l^2 = \frac{S^2}{\pi r^2} + r^2 - \frac{2S}{\pi}$$

$$= \frac{S^2}{\pi^2} \cdot \frac{4\pi}{S} + \frac{S}{4\pi} - \frac{2S}{\pi}$$

$$= \frac{4S}{\pi} + \frac{S}{4\pi} - \frac{2S}{\pi}$$

$$= \frac{9}{4}\frac{S}{\pi}$$

It therefore follows that

$$\sin^2\alpha = \frac{r^2}{l^2} = \frac{S}{4\pi} \cdot \frac{4\pi}{9S} = \frac{1}{9},$$

hence, $\alpha = \sin^{-1}\frac{1}{3}$

# Taylor's Theorem

statement: If $f(x)$ is differentiable n times in the interval $a \leq x \leq b$ and $f^n(x)$ is

i. differentiable in open interval $a < x < b$

ii. Continuous in the closed interval $a \leq x \leq b$ then there exist $\xi$ in the interval $a < \xi < b$ such that

$$f(b) = f(a) + f(b-a) f'(a) + \frac{(b-a)^2}{2!} f''(a) + \frac{(b-a)^3}{3!} f'''(a) + \frac{(b-a)^4}{4!} + \cdots + \frac{(b-a)^n}{n!} f^n(a) + \frac{(b-a)^{n+1}}{(n+1)!} f^{n+1}(\xi) \quad a < \xi < b$$

Note: If we replace b by x we get

$$f(x) = f(a) + f(x-a) f'(x) + \frac{(x-a)^2}{2!} f''(a) + \frac{(x-a)^3}{3!} f'''(a) + \frac{(x-a)^4}{4!} + \cdots \frac{(x-a)^n}{n!} f^n(a) + \frac{(x-a)^{n+1}}{(n+1)!} f^{(n+1)}(\xi) \quad a < \xi < x$$

Note (ii) : In equation --- (i) if we put $x = a + h$

$$f(a+h) = f(a) + hf'(a) + \frac{h^2}{2!} f''(a) + \frac{h^3}{3!} f'''(a) + \frac{(h)^4}{4!} + \cdots \frac{(h)^n}{n!} f^n(a) + \frac{(h)^{n+1}}{(n+1)!} f^{n+1}(\xi_1) \quad a < \xi_1 < a+h.$$

$$R = \frac{1}{n!\, f^{n+1}} (\xi_1)(b-a)(b-\xi_1)^n$$

Maclaurin's Theorem : If the function $f(x)$ can be expanded in a convergent series of positive integral powers of $x$ then such an expansion is given by.

$$f(x) = f(0) + xf'(0) + \frac{x^2}{2!} f''(0) + \frac{x^3}{3!} f'''(0) + \cdots \frac{(x)^n}{n!} f^n(0)$$

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Q. State Maclaurin's theorem. Using Maclaurin's theorem show that

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots + \frac{x^n}{n!}$$

$$\sin n\frac{\pi}{2} \cdots\cdots\cdots$$

Soln

Let $f(x) = \sin x$

$f'(x) = \cos x = \sin\left(\frac{\pi}{2} + x\right)$

$f''(x) = \cos\left(\frac{\pi}{2} + x\right) = \sin\left(\frac{2\pi}{2} + x\right)$

$f'''(x) = \cos\left(\frac{2\pi}{2} + x\right) = \sin\left(\frac{3\pi}{2} + x\right)$

$f^{iv}(x) = \cos\left(\frac{3\pi}{2} + x\right) = \sin\left(\frac{4\pi}{2} + x\right)$

$f^{v}(x) = \sin\left(\frac{5\pi}{2} + x\right) \cdots\cdots f^{n}(x)$

$= \sin\left(\frac{n\pi}{2} + x\right)$

We know,

$$f(x) = f(0) + xf'(0) + \frac{x^2}{2}f''(0) + \frac{x^3}{3!}f'''(0) + \frac{x^4}{4!}f^{iv}(0) + \frac{x^5}{5!}f^{v}(0) + \cdots +$$

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$$\frac{x^n}{n!} f^n(0) + \ldots$$

$$= 0 + x \cdot 1 + \frac{x^2}{2} \times 0 + \frac{x^3}{3!}(-1) + \frac{x^4}{4!}(0) + \frac{x^5}{5!}(1) + \ldots + \frac{x^n}{n!} \sin n\frac{\pi}{2} + \ldots$$

$$\therefore \sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \ldots + \frac{x^n}{n!} \sin\left(n\frac{\pi}{2}\right) + \ldots$$

# $e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots + \frac{x^n}{n!} + \dots$

Sol$^n$

let $f(x) = e^x$

Differentiating

$f(x) = e^x$

or $f'(x) = e^x$

$f''(x) = e^x$

$f'''(x) = e^x$

$\vdots$

$f^n(x) = e^x$

So, $f'(0) = e^0 = 1 = f''(0) = f'''(0) \dots$

by maclaurins theorem

$f(x) = f(0) + x f'(0) + \frac{x^2}{2!} f''(0) + \frac{x^3}{3!}$

$f'''(0) + \dots \frac{x^n}{n!} f^n(0) + \dots$

$= 1 + x.1 + \frac{x^2}{2!} + \frac{x^3}{3!}.1 + \dots \frac{x^n}{n!}.1 + \dots$

$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots + \frac{x^n}{n!} + \dots$

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# $\log(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} + \frac{x^4}{4} + \cdots + (-1)\frac{x^n}{n} +$

Soln

let

$$f(x) = \log(1+x)$$

$$f'(x) = \frac{1}{1+x} = (1+x)^{-1}$$

$$f''(x) = -1(1+x)^{-2}$$

$$f'''(x) = 2(1+x)^{-3}$$

$$f^{iv}(x) = -6(1+x)^{-4}$$

$$f^{v}(x) = 24(1+x)^{-5}$$

$$f^{n}(x) = \cancel{(-1)(n-1)(n-1)!(1+x)^{-n}} \quad (-1)^{(n-1)}(n-1)!(1+x)^{-n}$$

Now, $f(0) = \log(1+0) = 0$

$$f'(0) = (1+0)^{-1} = 1$$

$$f''(0) = -1(1+0)^{-2} = -1$$

$$f'''(0) = 2(1+0)^{-3} = 2$$

$$f^{iv}(0) = -6(1+0)^{-4} = -6$$

$$f^{v}(0) = 24(1+0)^{-5} = 24$$

$$f^{n}(0) = (-1)^{(n-1)}(n-1)!$$

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Maclaurins theorem

$$f(x) = f(0) + x f'(0) + \frac{x^2}{2!} f''(0) + \frac{x^3}{3!} f'''(0)$$

$$\cdots \frac{x^n}{n!} f^n(0)$$

$$\log(1+x) = 0 + x.1 + \frac{x^2}{2!} + \frac{x^3}{3!} 2 - \cdots \frac{x^n}{n!}$$

$$(-1)^{(n-1)} (n-1)! + \cdots$$

$$\log(1+x) = x + \frac{x^2}{2!} + \frac{2x^3}{3!} - \cdots \frac{x^n}{n!} (-1)^{(n-1)}$$

$$(n-1)! + \cdots$$

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Q. Expand $e^{\sin x}$ by Maclaurin's theorem as far as the term containing $x^4$.

Soln

let $y = f(x) = e^{\sin x}$ --- (i)

when $x = 0$, then

Differentiating with respect to $x$, we obtain

$y_1 = f'(x) = e^{\sin x} \cdot \cos x = y \cos x$,
then $f'(0) = 1$

$y_2 = f''(x) = y_1 \cos x - y \sin x$. then $f''(0) = 1$

$y_3 = f'''(x) = y_2 \cos x - y_1 \sin x - y_1 \sin x - y \cos x$
$= y_2 \cos x - 2y_1 \sin x - y \cos x$

then $f'''(0) = 0$

$y_4 = f^{iv}(x) = y_3 \cos x - y_2 \sin x - 2y_2 \sin x - 2y_1 \cos x \; y_1 \cos x + y \sin x$
$= y_3 \cos x - 3y_2 \sin x - 3y_1 \cos x + y \sin x$

then $f^{iv}(0) = -3$

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8. Expand $e^{\sin x}$ by Maclaurin's theorem

Soln

let $y = f(x) = e^{\sin x}$ ---- (i)
when $x=0$,

And.

$y_1 = f'(x) = e^{\sin x} \cdot \cos x = y \cos x$

$y_2 = f''(x) = y_1 \cos x - y \sin x$

$y_3 = f'''(x) = y_2 \cos x - y_1 \sin x - y_1 \sin x - y \cos x$

$= y_2 \cos x - 2y_1 \sin x - y \cos x$

$y_4 = f^{iv}(x) = y_3 \cos x - y_2 \sin x - 2y_2 \sin x - 2y_1 \cos x - y_1 \cos x + y \sin x$

$= y_3 \cos x - 3y_2 \sin x - 3y_1 \cos x + y \sin x$

$f'(0) = 1 \cdot \cos 0 = 1 \cdot 1 = 1$ [$y_1(0) = 1$]

$f''(0) = 1 \cdot \cos 0 - 1 \cdot \sin 0 = 1$ [$1 = y_2$]

when $x=0$, $y = f(0) = 1$

$f'''(0) = 1 \cdot \cos 0 - 2 \cdot 1 \sin 0 - 1 \cdot \cos 0$
$= 1 - 0 - 1 = 0 = y_3$

$f^{iv}(0) = 0 \cdot \cos 0 - 3 \cdot 1 \sin 0 - 3 \cdot 1 \cos 0 + 1 \sin 0$
$= -3$

We know,

$$f(x) = f(0) + x f'(0) + \frac{x^2 f''(0)}{2!} + \frac{x^3 f'''(0)}{3!} + \frac{x^4 f^{iv}(0)}{4!} + \cdots$$

$$= 1 + x.1 + \frac{x^2.1}{2} + \frac{x^3.0}{3!} + \frac{x^4(-3)}{4!} + \cdots$$

$$e^{\sin x} = 1 + x + \frac{x^2}{2} - \frac{1}{8}x^4 + \cdots$$

Ans

8. Expand $e^{\cos x}$ by Maclaurin's theorem

Soln

let $y = f(x) = e^{\cos x}$

And

$$y_1 = e^{\cos x}(-\sin x) = -y\sin x$$

$$y_2 = -y_1 \sin x - y\cos x$$

$$y_3 = -y_2 \sin x - y_1 \cos x - y_1 \cos x + y \sin x$$

$$= -y_2 \sin x - 2y_1 \cos x + y \sin x$$

$$y_4 = -y_3 \sin x - y_2 \cos x - 2y_2 \cos x + 2y_1 \sin x + y_1 \sin x + y \cos x$$

$$= -y_3 \sin x - 3y_2 \cos x + 3y_1 \sin x + y \cos x$$

Now,

$y = f(x) = e^{\cos x} \Rightarrow f(0) = e$

$y_1 = e^{\cos x}(-\sin x) = -y\sin x \Rightarrow f'(0) = 0$

$y_2 = -e$

$y_3 = 0$

$y_4 = 3e + e$

$= 4e$

Again,

$f(x) = f(0) + xf'(0) + \frac{x^2 f''(0)}{2!} + \frac{x^3 f'''(0)}{3!} + \frac{x^4 f^{iv}(0)}{4!} + \dots$

$= e + x \cdot 0 + \frac{x^2(-e)}{2!} + \frac{x^3 \cdot 0}{3!} + \frac{x^4 \cdot 4e}{4!}$

$= e - \frac{ex^2}{2} + \frac{x^4 e}{6} \dots$

$e^{\cos x} = e\left(1 - \frac{x^2}{2} + \frac{x^4}{6} \dots\right)$

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Q. Using maclaurins theorem expand $e^{x}\cos x$.

Sol$^n$

let $y = f(x) = e^{x}\cos x \quad \therefore f(0) = 1$

$$f'(x) = e^{x}\cos x - e^{x}\sin x$$
$$= y - e^{x}\sin x$$

$$\therefore f'(0) = 1 = y_1$$

$$f''(x) = y_1 - e^{x}\sin x - e^{x}\cos x$$
$$= y_1 - e^{x}\sin x - y$$

$$\therefore f''(0) = 1 - 0 - 1$$
$$= 0 = y_2$$

$$f'''(x) = y_2 - e^{x}\sin x - e^{x}\cos x - y_1$$

$$= y_2 - y_1 - y - e^{x}\sin x$$

$$= 0 - 1 - 1 = 0$$

$$\therefore f'''(0) = -2$$

$$f^{iv}(x) = y_3 - y_2 - y_1 - e^{x}\sin x - e^{x}\cos x$$
$$= y_3 - y_2 - y_1 - y - e^{x}\sin x$$
$$= -2 - 0 - 1 - 1 - 0$$
$$= -4$$

$$f(x) = f(0) + x f'(0) + x^2 \frac{f''(0)}{2!} + x^3 \frac{f'''(0)}{3!} + x^4 \frac{f^{iv}(0)}{4!} + \cdots$$

$$= 1 + x \cdot 1 + x^2 \frac{0}{2!} + x^3 \frac{(-2)}{3!} + x^4 \frac{(-4)}{4!} + \cdots$$

$$\therefore \; e^x \cos x = 1 + x - \frac{x^3}{3} - \frac{x^4}{6} + \cdots$$

# Curvature

- Curvature at a point.

The curvature at a point P to the curve $y=f(x)$ is the limiting value of $\frac{\Delta\psi}{\Delta s}$

when the point $Q \to P$ along the curve

ie, when $\Delta s \to 0$

Thus, the curvature is the rate of change of direction of the curve with respect to the arc

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## Radius of curvature at a point

The reciprocal of the curvature at any point P of a curve is called the radius of curvature at P and is denoted by $\rho$ (rho)

$$\therefore \rho = \cancel{\left|\frac{1}{d\psi}\right|} \quad \left|\frac{d\psi}{ds}\right| = \frac{ds}{d\psi}$$

$$\text{curvature } (P) = \lim_{\Delta s \to 0} \frac{\Delta \psi}{\Delta s} = \frac{d\psi}{ds}$$

$$\text{radius of curvature } (\rho) = \left|\frac{ds}{d\psi}\right|$$

### Some formula for radius of curvature

(1) For the Cartesian Equation $y = f(x)$;

$$\rho = \frac{(1+y_1^2)^{3/2}}{y_2} \quad \text{where, } y_1 = \frac{dy}{dx}$$

$$y_2 = \frac{d^2y}{dx^2}$$

(2) For the cartesian equation $x = f(y)$;

$$\rho = \frac{(1+x_1^2)^{3/2}}{x_2} \quad \text{where, } x_1 = \frac{dx}{dy}$$

$$x_2 = \frac{d^2x}{dy^2}$$

For the parametric eqn :-

$$x = \phi(t), \; y = \psi(t)$$

Then,

$$\rho = \cancel{(x')} \frac{(x'^2 + y'^2)^{3/2}}{x'y'' - y'x''} \qquad \rho = \frac{\left(1 + \frac{y'^2}{x'^2}\right)^{3/2}}{x'y'' - y'x''}$$

$$x' = \frac{dx}{dt}, \; y' = \frac{dy}{dt}, \; x'' = \frac{d^2x}{dt^2}, \; y'' = \frac{d^2y}{dt^2}$$

For the implicit function : $f(x,y) = 0$

$$\rho = \frac{(f_x^2 + f_y^2)^{3/2}}{f_{xx}(f_y)^2 - 2f_{xy}f_x f_y + f_{yy}(f_x)^2}$$

For the polar eqn $[r = f(\theta)]$

$$\rho = \frac{(r^2 + r_1^2)}{r^2 + 2r_1^2 - rr_2}$$

Where, $r_1 = \frac{dr}{d\theta}$

$r_2 = \frac{d^2r}{d\theta^2}$

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Q. Find the radius of curvature at any point of the parabola $y^2 = 4ax$. and hence show that the radius of curvature at its vertex is equal to its semi latus rectum.

Soln

Given curve

$$y^2 = 4ax \quad \text{--- (i)}$$

or, $2y \frac{dy}{dx} = 4a$ $\left[\frac{dy}{dx} = y_1\right]$

or, $2yy_1 = 4a$

or, $y_1 = \frac{\cancel{2}4a}{2y}$

or, $y_1 = \frac{2a}{y}$

and

$$y_2 = -2ay^{-2}$$

or, $y_2 = \frac{-2a}{y^2} \frac{dy}{dx} = -\frac{2a}{y^2} \cdot \frac{2a}{y}$

$\therefore y_2 = -\frac{4a^2}{y^3}$

we know,

$$\rho = \frac{(1+y_1^2)^{3/2}}{y_2}$$

$$= \frac{\left[1 + \frac{4a^2}{y^2}\right]^{3/2}}{-\frac{4a^2}{y^3}}$$

$$= -\left[1 + \frac{4a^2}{y^2}\right]^{3/2} \times \frac{y^3}{4a^2}$$

$$= -\left[\frac{y^2+4a^2}{y^2}\right]^{3/2} \times \frac{y^3}{4a^2}$$

$$= -\frac{[y^2+4a^2]^{3/2}}{\cancel{y^3}} \times \frac{\cancel{y^3}}{4a^2}$$

$$= -\frac{[y^2+4a^2]^{3/2}}{4a^2}$$

$$= -\frac{(4ax+4a^2)^{3/2}}{4a^2}$$

$$= -\frac{\{(4a)^{3/2}(x+a)^{3/2}\}}{4a^2}$$

$$= -\frac{8a^{3/2}}{4a^2}(x+a)^{3/2}$$

$$= -2a^{-1/2}(x+a)^{3/2}$$

So, $|\rho| = 2a^{-1/2}(x+a)^{3/2}$

radius of curvature.

Now at vertex (0,0)

$$|\rho|_{(0,0)} = 2a^{-1/2} \cdot a^{3/2}$$

$$= 2a = \frac{1}{2}(4a) = \cancel{SR} \text{ semilatus rectrum}$$

(i) vertical parallel to $x$-axis
(ii) horizontal

Asymptote (VVImp)

A line is called asymptotic with curve $y=f(x)$ if they meet at infinity.

* Oblique asymptote

$$y = mx + c$$

Note (1): If in an algebraic curve of degree $n$, $y^n$ is absent, asymptotes parallel to $y$-axis are obtained by equating to zero the coefficient of the highest degree terms in $y$.

Similarly. If $x^n$ is absent, the coefficient of highest degree terms in $x$ equated to zero gives asymptote parallel to $x$-axis.

Note (2): The number of asymptotes of an algebraic curves doesnot exceed the degree of the equation.

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8. Find the asymptote of the curve.

$$y^3 - x^2y + 2y^2 + 4y + x$$

Soln

Given,

curve $y^3 - x^2y + 2y^2 + 4y + x = 0$ — (i)

let $y = mx + c$ --- (ii) is the asymptote

Now,

Solving eqn (i) and eqn (ii)

$(mx+c)^3 - x^2(mx+c) + 2(mx+c)^2 + 4(mx+c) + x = 0$

or, $m^3x^3 + 3m^2x^2c + 3mxc^2 + c^3 - mx^3 - cx^2 + 2m^2x^2 + 4mxb + 2c^2 + 4mx + 4c + x = 0$

or, $x^3(m^3 - m) + x^2(3m^2c - c + 2m^2) + x(3mc^2 + 4mc + 4m + 1) + (c^3 + 2c^2 + 4c) = 0$ --- (iii)

Now,

Equating to zero the coefficient of the two highest powers of $x$ in eqn (iii)

i.e, $m^3 - m = 0$ — (iv)

$3m^2c - c + 2m^2 = 0$ — (v)

From eqn (iv) we get $m^3 - m = 0$

$m = 0, 1, -1$

when, $m=0$. from (iv) we get $c = 0$

when, $m=1$. from (iv) we get $c = -1$

when $m = -1$ from (iv) we get $c = -1$

$y = mx + c$

Thus, asymptotes are

$$\left.\begin{aligned} y &= 0 \\ y &= x - 1 \\ y &= -x - 1 \end{aligned}\right\}$$

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# Find the asymptotes of the curve:

$y^3 - 6xy^2 + 11x^2y - 6x^3 + x + y = 0$ —— (i)

Solⁿ

let $y = mx + c$ — (ii) be the assymptote

Solving (i) & (ii)

$(mx+c)^3 - 6x(mx+c)^2 + 11x^2(mx+c) - 6x^3 + x + (mx+c) = 0$

or, $m^3x^3 + 3m^2x^2c + 3mxc^2 + c^3 - 6m^2x^3 - 12mx^2c - 6xc^2 + 11mx^3 + 11x^2c - 6x^3 + x + mx + c = 0$

or, $(m^3 - 6m^2 + 11m - 6)x^3 + (3m^2c - 12mc + 11c)x^2 + x(3mc^2 - 6c^2 + m + 1) + (c^3 + c) = 0$ --- (iii)

Equating coefficient of $x^3$ and $x^2$ to zero we get,

$m^3 - 6m^2 + 11m - 6 = 0$ —— (iv)

$3x^2c - 12mc + 11c = 0$ —— (v)

From (iv) we get $m = 1, 2, 3$

When, $m = 1$ from (iv) to (v) we get $c = 0$,

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when $m=1$

$$3(1)^2c - 12c + 11c = 0$$

$$3c - 12c + 11c = 0$$

$$2c = 0$$

$$c = 0$$

when $m = 2$

$$3(2)^2c - 12(2)c + 11c$$

$$= 12c - 24c + 11c$$

$$= -1c = 0$$

$$c = 0$$

$\therefore$ $y = x$ — (a)

$y = 2x$ — (b)

$y = 3x$ — (c)

when $m = 3$

$$3(3)^2c - 12(3)c + 11c$$

$$= 27c - 36c + 11c$$

$$= 2c = 0$$

$$c = 0$$

Thus, asymptotes are

$$\left.\begin{aligned} y &= x \\ y &= 2x \\ y &= 3x \end{aligned}\right\}$$

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$m(m^2+2m+1)$ 100

$m^2+(1+1)m+1=0$

$m^2+1m+1m+1=0$

$m(m+1)+1(m+1)=0$

$(m+1)(m+1)=0$, $m=-1$

168. Find the asymptotes of the curve.

Sol$^n$

$y^3+2xy^2+x^2y-y+1=0$ — (i)

let $y=mx+c$ — (ii)

Now,

$(mx+c)^3+2x(mx+c)^2+x^2(mx+c)-(mx+c)+1=0$

or $m^3x^3+3m^2x^2c+3mxc^2+c^3+2m^2x^3+4mx^2c+2xc^2+mx^3+x^2c-mx-c+1=0$

or, $(m^3+2m^2+m)x^3+(3m^2c+4mc+c)x^2+(3mc^2+2c^2-m)x+(c^3-c)+1=0$ — (iii)

Equating coefficient of $x^3$ and $x^2$, to zero we get,

$m^3+2m^2+m=0$ — (iv)

$3m^2c+4mc+c=0$ — (v)

From (iv) we get,

$m=0,-1$ (repeated)

When, $m=0$ in eqn (v) we get

$c=0$

when $m = -1$ ean (v)

$$3c - 4c + c = 0$$

or $3(-1)c^2 + 2c^2 + 1 = 0$

or, $-3c^2 + 2c^2 + 1 = 0$

or, $-c^2 + 1 = 0$

or, $-c^2 = -1$

or, $c = \pm 1$

So, $3mc^2 + 2c^2 - m = 0$

$$c = 1, -1$$

Thus the asymptote are:-

$$\left.\begin{array}{l} y = 0 \\ y = -x + 1 \\ y = -x - 1 \end{array}\right\} \text{A\#}$$

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Unit - 6

## Application of Definite Integral

### Indefinite integrals.

(1) $\int x^n dx = \frac{x^{n+1}}{n+1} + c \quad [n = -1]$

Eg :- $\int x^2 dx = \frac{x^3}{3} + c$

(2) $\int x^{-1} dx = \int \frac{1}{x} dx = \log x$

Eg :-

$$\int x^{-3} dx = \frac{x^{-3+1}}{-3+1} + c$$

$$= \frac{x^{-2}}{-2} + c$$

$$= -\frac{x^{-2}}{2} + c$$

#

(3) $\int e^x dx = e^x$

(4) $\int e^{ax} dx = \frac{e^{ax}}{a}$

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Eg:- $\int e^{3x}dx = \frac{e^{3x}}{3}$

(5) $\int \sin x\,dx = -\cos x$

(6) $\int \sin ax = \frac{-\cos ax}{a}$

(7) $\int \cos x\,dx = \sin x$

(8) $\int \cos bx\,dx = \frac{\sin bx}{b}$

(9) $\int \sec^2 x\,dx = \tan x$

$\int \sec^2 bx\,dx = \frac{\tan bx}{b}$

(10) $\int \operatorname{cosec}^2 x\,dx = -\cot x$

$\int \operatorname{cosec}^2 ax\,dx = -\frac{\cot ax}{a}$

(11) $\int \sec x \tan x \, dx = \sec x$

$\int \sec ax \tan ax \, dx = \frac{\sec ax}{a}$

(12) $\int \operatorname{cosec} x \cot x \, dx = -\operatorname{cosec} x$

$\int \operatorname{cosec} bx \cdot \cot bx \, dx = -\frac{\operatorname{cosec} bx}{b}$

(13) $\int \sinh x \, dx = \cosh x$

(14) $\int \cosh x \, dx = \sinh x$

(15) $\int \operatorname{sech}^2 x = \tanh x$

(16) $\int \operatorname{cosech}^2 x = -\coth x$

(17) $\int \frac{1}{\sqrt{a^2 - x^2}} \, dx = \sin^{-1} \left( \frac{x}{a} \right)$

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(18) $\int \frac{1}{\sqrt{n^2-a^2}}\,dn = \cosh^{-1}$

(19) $\int \frac{1}{\sqrt{n^2+a^2}}\,dn = \sinh^{-1}\left(\frac{n}{a}\right)$

(20) $\int \frac{1}{n^2+a^2}\,dn = \frac{1}{a}\tan^{-1}\left(\frac{n}{a}\right)$

(21) $\int \frac{1}{a^2-n^2}\,dn = \frac{1}{2a}\log\left(\frac{a+n}{a-n}\right)$

(22) $\int \frac{1}{n^2-a^2}\,dn = \frac{1}{2a}\log\left(\frac{n-a}{n+a}\right)$

(23) $\int \sqrt{a^2-n^2}\,dn = \frac{a^2}{2}\sin^{-1}\left(\frac{n}{a}\right) + \frac{n}{2}\sqrt{a^2-n^2}$

(24) $\int \sqrt{a^2+n^2}\,dn = \frac{a^2}{2}\log\left(n+\sqrt{a^2+n^2}\right) + \frac{n}{2}\sqrt{a^2+n^2}$

(25) $\int \sqrt{n^2-a^2}\,dn = \frac{-a^2}{2}\log\left(n+\sqrt{n^2-a^2}\right) + \frac{n}{2}\sqrt{n^2-a^2}$

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# Integration by parts

$$\int uv\,dx = u\int v\,dx - \int\left[\frac{du}{dx}\int v\,dx\right]dx$$

I - Inverse function ↑ u
L - log function
A - Algebraic constant
T - Trigonometric
E - Exponential ↓ v

Eg:-

$$\int \overset{v}{x^2}\,\overset{u}{\log x}\,dx$$

# Properties of definite integral

① $\int_a^b [mf(x) + ng(x)]\,dx$ ($b \rightarrow$ upper limit, $a \rightarrow$ lower limit)

$$= m\int_a^b f(x)\,dx + n\int_a^b g(x)\,dx$$

② $\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx \; (a < c < b)$

③ $\int_a^b f(x)\,dx = -\int_b^a f(x)\,dx$

(4) $$\int_0^{\pi} f(n)\,dn = \int_0^{\pi} f(\pi - n)\,dn$$

For example :-

$$\int_0^1 n^2\,dn = \frac{n^3}{3}\Big|_0^1$$

$$= \left(\frac{1}{3}\right) - \left(\frac{0^3}{3}\right)$$

$$= \frac{1}{3}$$

$$\int_1^2 (n^3 + n)\,dn = \frac{n^3}{3} + \frac{n^2}{2}\Big|_1^3$$

$$= \left(\frac{2^3}{3} + \frac{2^2}{2}\right) - \left(\frac{1^3}{3} + \frac{1^2}{2}\right)$$

$$= \frac{14}{3} - \frac{5}{6}$$

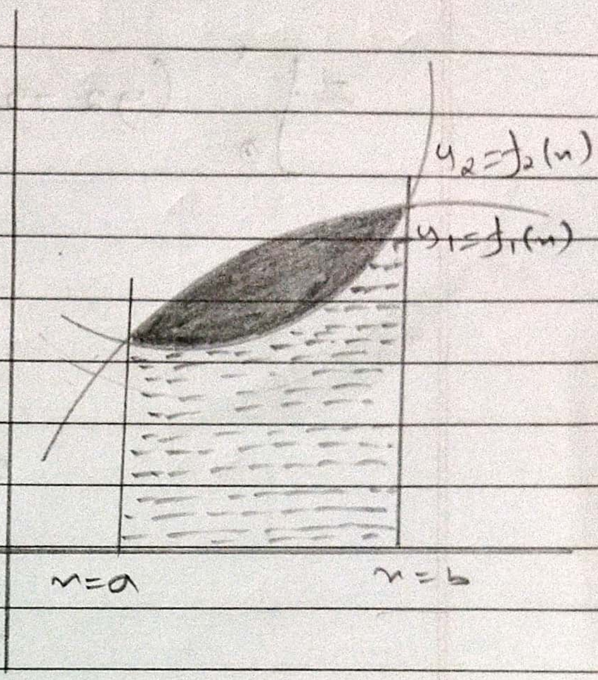
$$= \frac{23}{6}$$

# APPLICATION OF THE DEFINITE INTEGRAL

$A_1 \int_a^b y_1 \, dx$

$A_2 \int_a^b y_2 \, dx$

$\int_a^b (y_1 - y_2) \, dx$




Q. Find the area bounded by the parabola $y^2 = 9x$ and the line $y = 3x$

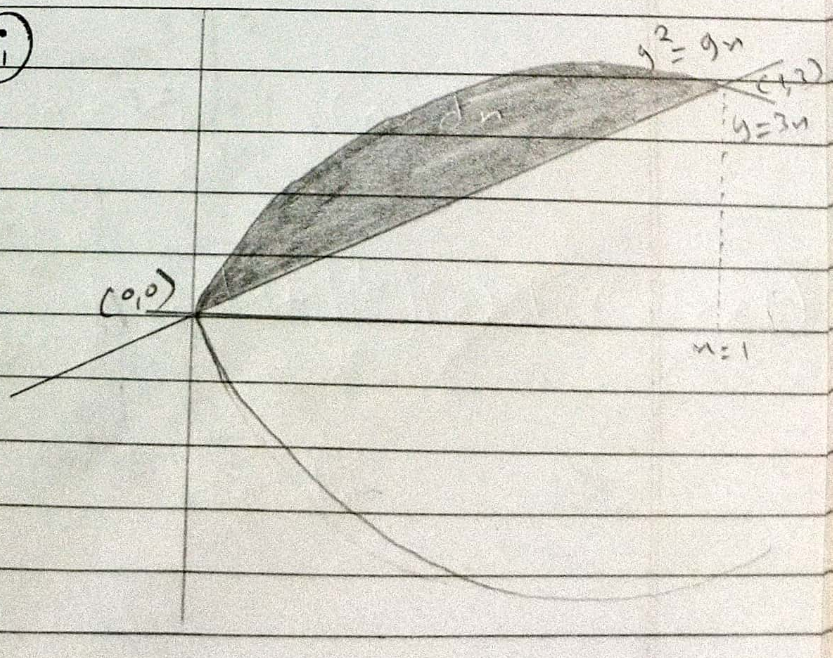
Soln

$y^2 = 9x$ — (i) $\rightarrow y_1 = 3\sqrt{x}$

$y = 3x$ — (ii) $\rightarrow y_2 = 3x$

Solving eqn (i) and (ii)

$x = 0, 1$



Now,
the reavired area

$$= \int_0^1 (y_2 - y_1)\,dx = \int_0^1 (3\sqrt{x} - 3x)\,dx$$

$$= 3\left[\int_0^1 \sqrt{x}\,dx - \int_0^1 x\,dx\right]$$

$$= 3\left\{\left(\frac{x^{3/2}}{3/2}\Big|_0^1\right) - \left(\frac{x^2}{2}\Big|_0^1\right)\right\}$$

$$= 3\left\{\frac{2}{3} - \frac{1}{2}\right\}$$

$$= 3 \cdot \frac{1}{6}$$

$$= \frac{1}{2} \text{ Savore unit}$$

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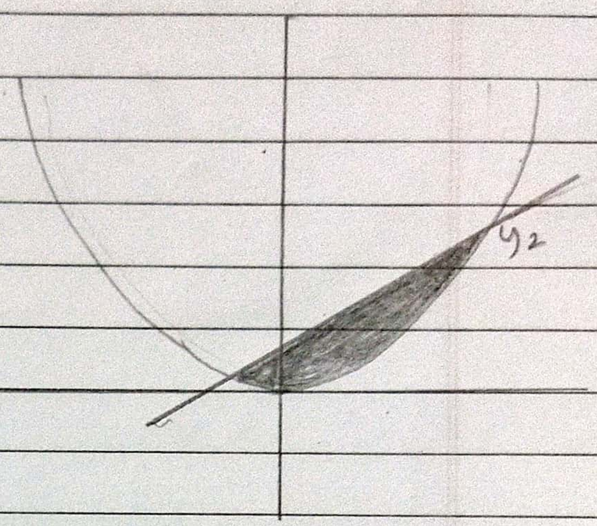
Q. Find the area bounded by the curve $x^2 = 4y$ and the straight line $x = 4y - 2$.

$x^2 = 4y$ — (i)

$x = 4y - 2$ — (ii)



Solving eqn (i) and (ii)

$x^2 = 4\left(\frac{x+2}{4}\right)$

or, $x^2 = x + 2$

or, $x^2 - x - 2 = 0$

or, $x^2 - (2-1)x - 2 = 0$

or, $x^2 - 2x + 1x - 2 = 0$

or, $x(x-2) + 1(x-2) = 0$

or, $(x+1)(x-2) = 0$

$x = -1$ or $x = 2$

$$\text{Area} = \int_{-1}^{2} \left( \frac{x+2}{4} - \frac{x^2}{4} \right) dx$$

$$= \left[ \frac{(x+2)^2}{8} - \frac{x^3}{12} \right]_{-1}^{2}$$

$$= \left[ \frac{4}{3} - \frac{5}{24} \right]$$

$$= \frac{27}{24} = \frac{9}{8}$$

Ans#