Here, $\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}$, so given differential equation is an exact equation.

NOTE :

$\frac{\partial u}{\partial x} = P$

$\frac{\partial u}{\partial y} = Q$.

Now,

$$\frac{\partial u}{\partial x} = 2xy + y - \tan y \quad \text{— (i)}$$

and $$\frac{\partial u}{\partial y} = \sec^2 y - x\tan^2 y + x^2 + 2 \text{ — (ii)}$$

Integrating equ (i) w.r.t 'x'

$$\int \frac{\partial u}{\partial x} = u(x,y) = 2\cdot\frac{x^2}{2}y + y\cdot x - \tan y(x) + c(y)$$

$$\therefore u = x^2y + xy - x\tan y + c(y) \text{ —— (iii)}$$

Differentiating equ (iii) partially with respect to y.

$$\frac{\partial u}{\partial y} = x^2 + x - x\sec^2 y + c'(y) \text{ —— (iv)}$$

from (ii) and (iv)

$$\frac{\partial u}{\partial y} = \sec^2 y - x\tan^2 y + x^2 + 2 = x^2 + x - x\sec^2 y + c'(y)$$

$$c'(y) = \sec^2 y - x\tan^2 y + 2 - x + x\sec^2 y$$
$$= \sec^2 y + x(\sec^2 y - \tan^2 y) + 2 - x$$
$$= \sec^2 y + x + 2 - x$$
$$c'(y) = 2 + \sec^2 y \text{ —— (v)}$$

Integrating (v) w.r.t 'y' we get,

$$c(y) = 2y + \tan y \text{ —— (vi)}$$

Thus, soln is $u = \int P\,dx + c(y)$

$$= \int (2xy + y - \tan y)\,dx + c(y)$$

$x\,dx = -y\,dy$

Integrating both sides,

$$\int x\,dx = \int -y\,dy$$

$$\frac{x^2}{2} = -\frac{y^2}{2} + \frac{c}{2}$$

$$\therefore x^2 + y^2 = c$$

2) $\frac{dy}{dx} = \frac{x^2+1}{y^2+1}$

soln.

$$(y^2+1)\,dy = (x^2+1)\,dx.$$

Integrating both sides,

$$\frac{y^3}{3} + y = \frac{x^3}{3} + x + c$$

or, $\frac{y^3 + 3y}{3} = \frac{x^3+3x}{3} + \frac{c}{3}$

$$\therefore y^3 + 3y - x^3 - 3x = c$$

Page 1

or, $\int\left(1 + \frac{(2t+4)}{t^2+4t+5}\right) dt = 2x + \log c$

or, $\int dt + \int \frac{2t+4}{t^2+4t+5}\, dt = 2x + \log c$

or, $t + \log(t^2+4t+5) = 2x + \log c \quad \left[\int \frac{f'(x)}{f(x)} = \log f\right]$

or, $2x = t + \log(t^2+4t+5) - \log c$

or, $2x - x - y = \log(t^2+4t+5) - \log c \quad [t = x+y]$

or, $x - y = \log\left(\frac{t^2+4t+5}{c}\right)$

or, $ce^{x-y} = t^2 + 4t + 5$ #

A first order differential equation

$$P(x,y)\,dx + Q(x,y)\,dy = 0 \quad \text{— (1)}$$

is said to be exact if the left-hand side is a total derivative of the type

$$du = \frac{\partial u}{\partial x}\,dx + \frac{\partial u}{\partial y}\,dy$$, where u is a function of x and y. In such case eqn 1 becomes $du = 0$,

such that,

$$\boxed{\frac{\partial u}{\partial x} = P \text{ and } \frac{\partial u}{\partial y} = Q}$$

then,

$$\frac{\partial^2 u}{\partial x \partial y} = \frac{\partial P}{\partial y} = \frac{\partial^2 u}{\partial y \partial x} = \frac{\partial Q}{\partial x}$$

$$\boxed{\therefore \frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}}$$

~~$x\,dy + y\,dx = d(xy)$~~

~~Sol^n~~

1) $$\frac{dy}{dx} = \frac{2x - y + 1}{x + 2y - 3}$$

Sol^n.

$(x+2y-3)\,dy = (2x-y+1)\,dx$

or, $(x+2y-3)\,dy - (2x-y+1)\,dx = 0$

or, $(y-2x-1)\,dx + (x+2y-3)\,dy = 0$

Now,

$M = y - 2x - 1$

$N = x + 2y - 3$

$$\frac{\partial M}{\partial y} = 1$$

$$\frac{\partial N}{\partial x} = 1$$

$\therefore$ Hence the equation is exact

Now,

$\int M\,dx$, taking y as constant

$= \int y - 2x - 1\,dx$

$= y \cdot x - \frac{2x^2}{2} - x + c$

$= xy - x^2 - x + c$

Example 1.4

We consider the differential equation

$$x^2(1+y)\frac{dy}{dx} + (1-x)y^2 = 0$$

Note.

$$\int x^n dx = \frac{x^{n+1}}{n+1} + c$$

This can be put into the form

$$\frac{1+y}{y^2}dy = \frac{-1-x}{x^2}dx.$$

where the variables are seperated. Integrating both sides, we get

$$\int\left(\frac{1}{y^2}+\frac{1}{y}\right)dy = -\int\left(\frac{1}{x^2}-\frac{1}{x}\right)dx.$$

which gives,

$$\int y^{-2}dy + \int\frac{1}{y}dy = -\int x^{-2}dx + \int\frac{1}{x}dx$$

or $\frac{y^{-2+1}}{-2+1} + \log_e y = -\frac{x^{-2+1}}{-2+1} + c + \log_e x$

or, $-\frac{1}{y} + \log_e y = \log_e x + \frac{1}{x}$ $\log_e e$

where c is an aribitary constant. Simplification gives

$$\log_e\left(\frac{y}{x}\right) = \frac{1}{x} + \frac{1}{y} + c$$

which is the general solution of the differential equation or, $-v + \log c = \log x + 2\log(v-1)$

or, $-v + \log c = \log x + \log(v-1)^2$

or, $-v + \log c = \log(x \cdot (v-1)^2)$

or,

or, $-\left(v + 2\int \frac{1}{v-1}\, dv\right) = \int \frac{1}{x}\, dx$

or, $v + 2\log(v-1) = -\log x + \log c$

or, $\log x + 2\log(v-1) = -v + \log c$

or, $\log(x) + \log(v-1)^2 = -v + \log c$

or, $\log(x \cdot (v-1)^2) = -v + \log c$

or, $\log(x \cdot (v-1)^2) - \log c = -v$

or, $\log \frac{(x \cdot (v-1)^2)}{c} = -v$

or, Substituting the value of $v = y/x$ we get,

or, $\log \frac{\left(x \cdot \left(\frac{y}{x} - 1\right)^2\right)}{c} = -y/x$

or, $\log \frac{\left(x \cdot \left(\frac{y-x}{x}\right)^2\right)}{c} = -y/x$

or, $\log \frac{(y-x)^2}{cx} = -y/x$

$\Rightarrow \frac{(y-x)^2}{cx} = e^{-y/x}$

or, $\int \frac{2v+1-v}{v(2v+1)}\,dv = -\int \frac{1}{x}\,dx$

or, $\int \frac{2v+1}{v(2v+1)}\,dv - \int \frac{v}{v(2v+1)}\,dv = -\int \frac{1}{x}\,dx$

or, $\int \frac{1}{v}\,dv - \int \frac{1}{2v+1}\,dv = -\int \frac{1}{x}\,dx$

or, $\log v - \frac{1}{2}\int \frac{2}{2v+1}\,dv = -\log x + \log c$

or, $\log v - \frac{1}{2}\log(2v+1) = -\log x + \log c$

or, $2\log v - \log(2v+1) = -2\log x + 2\log c$

or, $\log v^2 - \log(2v+1) = -\log x^2 + \log c^2$

or, $\log\left(\frac{v^2}{2v+1}\right) = \log\left(\frac{c^2}{x^2}\right)$

$\therefore \frac{v^2}{2v+1} = \frac{c^2}{x^2}$

Substituting value $v = y/x$ we get,

$$\frac{(y/x)^2}{2(y/x)+1} = \frac{c^2}{x^2}$$

or, $$\frac{y^2/x^2}{\frac{2y+x}{x}} = \frac{c^2}{x^2}$$

or, $$\frac{y^2}{x(2y+x)} = \frac{c^2}{x^2}$$

7) $(x+y$

Soln

$(x+$

or,

Let

or

or

...equation of all straight lines touching the circle $x^2+y^2=a^2$

Solⁿ.

Let differential equation be $\frac{dy}{dx} = m$

Then, equation of all straight lines touching the given circle may be written as,

$y = mx + c$, tangent so,

$$y = mx + a\sqrt{1+m^2} \qquad \left[\because c = a\sqrt{1+m^2}\right]$$

Then, eliminating m, we get

$$y = x\cdot\frac{dy}{dx} + a\sqrt{1+m^2}$$

$$\left(y - x\cdot\frac{dy}{dx}\right) = a\sqrt{1+m^2}$$

$$\left(y - x\cdot\frac{dy}{dx}\right) = a\sqrt{1+\left(\frac{dy}{dx}\right)^2}$$

Squaring.

$$\left(y - x\cdot\frac{dy}{dx}\right)^2 = a^2\left(1+\left(\frac{dy}{dx}\right)^2\right)$$ is req'd equation.

$\therefore$ $y = mx + c$ will be tangent to the circle $x^2+y^2=a^2$ if $c = a\sqrt{1+m^2}$

Theorem:

For a differential equation of the first order $\frac{dy}{dx} = f(x,y)$ which satisfies the initial condition .. $y(x_0) = y_0$, there exist a unique solution .. $y = \phi(x)$, defined for all values of x in a certain region including x_0.

- Methods to solve differential equation

i) Variable Seperation Method
ii) Exact differential Equation
iii) Homogeneous differential Equation
iv) Linear differential Equation

i) Variable Seperation Method

An equation of the form

$Y\,dy = X\,dx$ where Y is the function of y alone and X that of x alone

$$Y = (ay^n + by^{n-1} + c)$$
$$X = (x^p + ax^{p-1} + ..)$$

ii) Exact differential Equation.

The differential eq^n of the form $M(x,y)\,dx + N(x,y)\,dy$ where M and N are functions of x and y such that the left hand side of the equation can be expressed as a single (or perfect or exact) differential of the form $d f(x,y)$

Integrating both sides.

$$\int \frac{\sin v}{\cos v}\, dv = \int dx.$$

or, $-\int \frac{-\sin v}{\cos v}\, dv = \int x$ $\left[\int \frac{f'(x)}{f(x)} = \log f(x) + c\right]$

or, $-\log(\cos v) + \log c = x.$

or, $x = -\log(\cos v) + \log c$

or, $x = \log(\cos v)^{-1} + \log c$

or, $x = \log \frac{c}{\cos v}$

or, $x = \log \frac{c}{\cos(x+y)}$, which is reqrd eqn.

or, $\frac{-1}{2v^2} + \log(vx) = \log c$

Substituting $\cancel{y=}$ $v = y/x$ we get

or, $\frac{-1}{2(y/x)^2} + \log\left(\frac{y}{x} \cdot x\right) = \log c$

or, $\frac{-x^2}{2y^2} + \log y = \log c$

or, $\log y = \log c + \frac{x^2}{2y^2}$

$\cancel{\therefore y = e^{\log c + \frac{x^2}{2y^2}}}$

or, $\log y - \log c = \frac{x^2}{2y^2}$

or, $\log(y/c) = \frac{x^2}{2y^2}$

or, $y/c = e^{x^2/2y^2}$

$\therefore y = ce^{x^2/2y^2}$, which is the reqrd solⁿ

3) $\frac{dy}{dx} + 4x = 2e^{2x}$

Soln.

$\frac{dy}{dx} = 2e^{2x} - 4x$

$dy = 2e^{2x} - 4x \, dx$

Integrating both sides,

$\int dy = \int 2e^{2x} - 4x \, dx$

or, $y = \frac{2e^{2x}}{2} - \frac{4x^{2x}}{2} + c$

or, $y = e^{2x} - 2x^{2x} + c$

4) $(xy^2 + x)dx + (yx^2 + y)dy = 0$

Soln

$(xy^2 + x)dx = -(yx^2 + y)dy$

or, $x(y^2+1)dx = -y(x^2-1)dy$

or, $\frac{x}{x^2+1} dx = -\frac{y}{y^2+1} dy$

or, $\frac{x}{x} - x$

Dividing by xy on both sides,

26) $y' = \frac{3x^2+4x+2}{2(y-1)}$

Soln

$$\frac{dy}{dx} = \frac{3x^2+4x+2}{2(y-1)}$$

or, $2(y-1)\,dy = (3x^2+4x+2)\,dx$

Integrating both sides,

$$2\int(y-1)\,dy = \int(3x^2+4x+2)\,dx$$

or, $2\left[\frac{y^2}{2} - y\right] = \frac{3x^3}{3} + \frac{4x^2}{2} + 2x + c$

or, $y^2 - 2y = x^3 + 2x^2 + 2x + c$ #

⇒ $(x+y)(dx)^2 + 2xy\,dx\,dy - (dy)^2 = 0$

or, $(x+y)(dx)^2 + 2xy\,dx\,dy = (dy)^2$

or, $(x+y)\left(\frac{dx}{dy}\right)^2 + 2xy \cdot \frac{dx\,dy}{(dy)^2} = 0$

or, dividing by $(dx)^2$ on both sides,

$$\frac{(x+y)(dx)^2 + 2xy\,dx\,dy}{(dx)^2} = \frac{(dy)^2}{(dx)^2}$$

or, $(x+y) + 2xy\left(\frac{dy}{dx}\right) = \left(\frac{dy}{dx}\right)^2$

order - 1

degree - 2

∴ It is first order & second degree diff eqn.

8) $2 \cdot \frac{dy}{dx} \cdot \frac{d^3y}{dx^3} = 5\left(\frac{d^2y}{dx^2}\right)^2$

order - 3

degree - 1

∴ It is third order & first degree diff eqn.

Differential Equation of first order

→ **Definition 1** :- A first order ordinary differential equation (ODE) in the unknown function $y(x)$ of the form $\frac{dy}{dx}$ is a function of (x, y)

i.e. $$\frac{dy}{dx} = f(x,y)$$

where $\frac{dy}{dx}$ derivative appears only on the

Left side of 1 is known as standard form of first order ODE.

Example:

$$y = 2x^2 - x + 3$$

$$\frac{dy}{dx} = 4x - 1 = f(x,y)$$

→ **Order of differential equation:**

(Examples)

$\frac{dy}{dx} + y = 0$ → first order

$\frac{d^2y}{dx^2} + \frac{dy}{dx} - x^2 y = 12$ → second order

$\left(\frac{dy}{dx}\right)^2 + 2xy = 7$

order - 1 , degree - 2

Definition:

The order of a differential equation is the order of the highest derivative it contains, whereas it's degree is the algebric degree of the highest derivative

~ Define order and degree of differential equation and identify the order and degree of differential eqn?

Exercise: (1.1)

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[Note: Highest order power - degree]

1) $\frac{dy}{dx} = \left(\frac{1+x}{1+y}\right)^{1/3}$

order - 1

degree - 1

It is first order and first degree differential equation.

2) $\frac{d^2y}{dx^2} = 2\frac{dy}{dx} + 3y$

order - 2

degree - 1

It is second order & first degree differential equation

3) $\left(\frac{dy}{dx}\right)^2 = 4xy$

order - 1

degree - 2, It is first order & second degree diff eqn.

$(x^2+2xy^2)\,dx + (2x^2y+y^2)\,dy = 0$

Sol

Soln

$M = x^2+2xy^2$

$N = 2x^2y+y^2$

$\frac{\partial M}{\partial y} = 2x \cdot 2y$ $\qquad\qquad \frac{\partial N}{\partial x} = 4xy$

$= 4xy$

$\therefore$ Hence the equation is exact.

Now,

$\int M\,dx$ taking y as constant.

$\int x^2+2xy^2\,dx$

$= \frac{x^3}{3} + 2y^2 \cdot \frac{x^2}{2} + c$

$= \frac{x^3}{3} + x^2y^2 + c$

Again, the term free x in N is y^2

So,

$\int y^2\,dy$

$\frac{y^3}{3} + c$

Thus reqrd equ is

$\frac{x^3}{3} + x^2y^2 + \frac{y^3}{3} + c$

$\therefore x^3 + y^3 + 3x^2y^2 = c$ #

Integrating we get,

$$e^{xy} + y^2 - c \text{ is the sol}^n$$

Example: (1.11)

Show that the differential equation

$$\frac{dy}{dx} = \frac{\tan y - y - 2xy}{\sec^2 y - x\tan^2 y + x^2 + 2}$$

is an exact equation and solve it.

Sol^n.

Given equation can be written as

$$(\sec^2 y - x\tan^2 y + x^2 + 2)\,dy = (\tan y - y - 2xy)\,dx$$

or, $(\sec^2 y - x\tan^2 y + x^2 + 2)\,dy - (\tan y - y - 2xy)\,dx = 0$

or, $(2xy + y - \tan y)\,dx + (\sec^2 y - x\tan^2 y + x^2 + 2)\,dy = 0$

comparing with $P\,dx + Q\,dy = 0$ we get,

$P = 2xy + y - \tan y$

$Q = \sec^2 y - x\tan^2 y + x^2 + 2$

Now,

$$\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}$$

$$\frac{\partial P}{\partial y} = \frac{\partial}{\partial y}(2xy + y - \tan y) \qquad \frac{\partial Q}{\partial x} = \frac{\partial}{\partial x}(\sec^2 y - x\tan^2 y + x^2 + 2)$$

$$= 2x + 1 - \sec^2 y \qquad = -\tan^2 y + 2x$$

$$= 2x - \tan^2 y \text{ --- (i)} \qquad = 2x - \tan^2 y \text{ --- (ii)}$$

Integrating eqn(i), we get

$$u(x,y) = \frac{ye^{xy}}{y} + c(y) \quad \left[\int y e^{xy} dx\right]$$

$$= e^{xy} + c(y) \quad \text{—} \; (2)$$

Now,

differentiating partially w.r.t 'y'

$$\frac{\partial u}{\partial y} = \frac{\partial}{\partial y}(e^{xy} + c(y))$$

$$= xe^{xy} + c'(y) \quad \text{—} \; (3)$$

But, from (ii)

$$\cancel{\frac{\partial u}{\partial y} = \frac{\partial Q}{\partial x} = e^{xy} +}$$

$$\frac{\partial u}{\partial y} = x.e^{xy} + 2y \quad \text{—} \; (4)$$

from (3) and (4)

$$c'(y) = 2y \quad \left[\frac{\partial u}{\partial y} = \cancel{xe^{xy}} + c'(y) - \cancel{xe^{xy}} - 2y\right.$$

$$\left. 0 = c'(y) = 2y \right]$$

$$\therefore c'(y) = 2y$$

Therefore, Integrating.

$$c(y) = y^2 + c$$

Soln of exact differential equation [fr. (2)]

$$u(x,y) = e^{xy} + y^2 + \text{constant}.$$

$\therefore$ eqn(1) can be written as

$$d(e^{xy} + y^2) = 0$$

Solve the following differential equation

29) $y' = \dfrac{\cos x}{3y^2 + e^y}$

Sol^n:

$$\frac{dy}{dx} = \frac{\cos x}{3y^2 + e^y}$$

or, $(3y^2 + e^y)\,dy = \cos x\,dx$

~~or, $6y + e^y = \sin x$~~

Integrating both sides, we get

$$\int (3y^2 + e^y)\,dy = \int \cos x\,dx. \qquad \left[y^n = \frac{y^{n+1}}{n+1} \right]$$

or, $3 \cdot \dfrac{y^3}{3} + e^y = \sin x + c$

or, $y^3 + e^y = \sin x + c$

$\therefore y^3 + e^y - \sin x = c$ #

9) $x\sin(y/x)\,dy = y\sin(y/x) \ldots$

Solⁿ.

$$\frac{dy}{dx} = \frac{\left(y\sin(y/x) - x\right)}{x\sin(y/x)}$$

Let $y = vx$

$$\frac{d(vx)}{dx} = \frac{vx\sin\left(\frac{vx}{x}\right) - x}{x\sin\left(\frac{vx}{x}\right)}$$

or, $v + x\cdot\frac{dv}{dx} = \frac{vx\sin v - x}{x\sin v}$

or, $v + x\frac{dv}{dx} = \frac{\cancel{x}(v\sin v - 1)}{\cancel{x}\sin v}$

or, $v + x\frac{dv}{dx} = \frac{\sin v^2 - 1}{\sin v}$

or, $x\frac{dv}{dx} = \frac{\sin v^2 - 1}{\sin v} - v$

or, $x\frac{dv}{dx} = \frac{\sin v^2 - 1 - \sin v^2}{\sin v}$

or, $x\frac{dv}{dx} = \frac{-1}{\sin v}$

or, $\sin v\,dv = -\frac{1}{x}\,dx$

Integrating both sides

$$\int \sin v\,dv = -\int \frac{1}{x}\,dx$$

$$-\cos v = -\log x + c$$

substituting value of $v = y/x$, we get

$$-\cos(y/x) = -\log x + c$$

Find the solution of the differential equation

$$(x+1)\,dy + (y-1)\,dx = 0$$

Subject to the condition that $y=3$ and $x=0$

Seperating the variables, we obtain

$$\frac{dx}{x+1} = -\frac{dy}{y-1}$$

Integrating both the sides, we get

$$\log(x+1) = -\log(y-1) + \log c$$

or,

$$(x+1)(y-1) = c \quad \text{—— (i)}$$

Substituting $x=0$, $y=3$ in eqn (i) we obtain

$c = 2$

∴ Hence, the solution of the given initial value problem is $(x+1)(y-1) = 2$

Soln.

$\cancel{e^{x}+e^{-y}}$

$$\frac{e^x}{e^y}\,dx + \frac{e^y}{e^x}\,dy = 0$$

or, Multiplying by e^{x+y}, we get

or, $\frac{e^x}{e^y}\cdot e^{x+y}\,dx + \frac{e^y}{e^x}\cdot e^{x+y}\,dy = 0$

or, $\frac{e^x}{e^y}\cdot e^x\cdot e^y\,dx + \frac{e^y}{e^x}\cdot e^x\cdot e^y\,dy = 0$

or, $e^{2x}\,dx + e^{2y}\,dy = 0$

For, $e^{2x}\,dx = -e^{2y}\,dy$

Integrating on both sides, we get

$$\int e^{2x}\,dx = -\int e^{2y}\,dy$$

or, $\frac{e^{2x}}{2} = -\frac{e^{2y}}{2} + \frac{c}{2}$

or, $e^{2x} + e^{2y} = c$

6) $(e^x + 1)y\,dy = (y+1)e^x\,dx.$

Solⁿ.

$\underline{e^x/y/1}$

$$\frac{y}{(y+1)}\,dy = \frac{e^x}{(e^x+1)}\,dx.$$

Integrating on both sides,

$$\int \frac{y}{(y+1)}\,dy = \int \frac{e^x}{e^x+1}\,dx.$$

or, $\int \frac{y+1-1}{y+1}\,dy = \int \frac{e^x}{e^x+1}\,dx.$

or, $\int \frac{y+1}{y+1}\,dy - \int \frac{1}{y+1}\,dy = \int \frac{e^x}{e^x+1}\,dx.$

or, $\int 1\,dy - \log(y+1) = \log(e^x+1) + c.$

or, $y = \log(y+1) + \log(e^x+1) + c$

$M = x^2 - ay$ $\quad [(x^2-ay)dx + (y^2-ax)dy] = 0$

$N = \cancel{axx}\, y^2 - ax$

Now, $\frac{\partial M}{\partial y} = -a$

$\frac{\partial N}{\partial x} = -a$

$\therefore \frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$

$\therefore$ Hence the equation is exact.

Now,

$\int M dx$, taking y as constant

$= \int x^2 - ay \, dx$

$= \frac{x^3}{3} - axy + c$

Again the term free x in N is y^2

So,

$\int y^2 dy = \frac{y^3}{3} + c$

Thus reqd eqn is

$\frac{x^3}{3} - axy + \frac{y^3}{3} + c = 0$

$\therefore x^3 - 3axy + y^3 = c$ //

or, $\log\left(\frac{y^c x^{2c}}{c_1}\right) = y$ [NOTE: $\log y = x$ then $y = e^x$]

or, $\frac{y^c x^{2c}}{c_1} = e^y$

$\therefore y^c x^{2c} = c_1 e^y$

when $x = 2c$, $y = c$

$c^c (2c)^{2c} = c_1 e^c$

$\Rightarrow c_1 = \frac{c^c (2c)^{2c}}{e^c}$

$\therefore y^c x^{2c} = \frac{c^c (2c)^{2c}}{e^c} \cdot e^y$ is the reqd solⁿ.

Ex. 1·8·

Solve the differential equation

$$\frac{dy}{dx} = \cot(y+x) - 1 \quad \text{—(1)}$$

put $y + x = v$ —(2)

Diff (2) w.r.t 'x'.

$$\frac{dy}{dx} + 1 = \frac{dv}{dx}$$

$$\therefore \frac{dy}{dx} = \frac{dv}{dx} - 1$$

So eqn (1) can be written as

$$\frac{dv}{dx} - 1 = \cot v - 1$$

$$\therefore \frac{dv}{dx} = \cot v$$

Now, $\frac{dv}{\cot v} = dx$

or, $\frac{\sin v}{\cos v}\, dv = dx$.

Again, the term free x in N is $2y-3$

so,

$$\int 2y-3\,dy$$

$$=\frac{2y^2}{2}-3y$$

$$=y^2-3y+c$$

Thus reqrd equ is:

$$xy-x^2-x+y^2-3y+c=0$$

$$\therefore x^2-y^2-xy-x+3y=c \#$$

1) $x + y\frac{dy}{dx} = 2y$

Solⁿ

$$y\frac{dy}{dx} = 2y - x$$

$$\frac{dy}{dx} = \frac{2y-x}{y}$$

Let $y = vx$

$$\frac{dy}{dx} = \frac{2vx-x}{vx}$$

or, $\frac{dy}{dx} = \frac{d(vx)}{dx}$

or, $\frac{dy}{dx} = v\cdot\frac{dx}{dx} + x\cdot\frac{dv}{dx}$

or, $\frac{dy}{dx} = v + x\cdot\frac{dv}{dx}$

or, Now,

$$\frac{2vx-x}{vx} = v + x\cdot\frac{dv}{dx}.$$

or, $\frac{2vx-x}{vx} - v = x\frac{dv}{dx}$

or, $\frac{2vx-x-v^2x}{vx} = x\frac{dv}{dx}$

or, $\frac{2v-1-v^2}{v} = x\frac{dv}{dx}$

or, $-\frac{(v-1)^2}{v} = x\frac{dv}{dx}$

$\frac{d^2y}{dx^2} \quad \frac{dy}{dx} \quad y = 0, \quad y = x^{1/2}$ and $y = 1/x$

Sol^n

$$2x^2 \frac{d^2y}{dx^2} + 3x \frac{dy}{dx} - y = 0 \longrightarrow i)$$

$$y = x^{1/2} \longrightarrow ii)$$

$$y = 1/x \longrightarrow iii)$$

From ii)

$$y = x^{1/2}$$

$$\frac{dy}{dx} = \frac{1}{2} x^{-1/2}$$

Again differentiating w.r.t 'x'

$$\frac{d^2y}{dx^2} = \frac{1}{2} \times \frac{-1}{2} x^{-1/2 - 1}$$

$$= \frac{-1}{4} x^{-3/2}$$

Using the value of $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$ and y in eq^n (i)

$$2x^2 \times \frac{-1}{4} x^{-3/2} + 3x \times \frac{1}{2} x^{-1/2} - x^{1/2} = 0$$

or, $\frac{-1}{2} x^{1/2} + \frac{3x^{1/2}}{2} - x^{1/2} = 0$

or, $-x^{1/2} + 3x^{1/2} - 2x^{1/2} = 0$

$\therefore 0 = 0$ (T)

$\therefore y = x^{1/2}$ satisfies differential eq^n (1)

$yy = 4x$

Soln.

$$y \cdot \frac{dy}{dx} = 4x$$

or, $y \cdot dy = 4x \cdot dx$

Integrating both sides, we get

$$\int y \, dy = \int 4x \, dx$$

or, $\frac{y^2}{2} = 4\frac{x^2}{2} + c$

or, $\frac{y^2}{2} \cdot 1 = 2x^2 + c$

or, $\frac{y^2}{2} = 2x^2 + c$

or, $y^2 = 4x^2 + c$ #

Substituting the value of $v = y/x$, we get

or, $\tan^{-1}(y/x) - \frac{1}{2}\log\left(1 + \frac{y^2}{x^2}\right) = \log x + c$

or, $\tan^{-1}(y/x) - \frac{1}{2}\log\left(\frac{x^2+y^2}{x^2}\right) = \log x + c$

$$(xe^{xy} + 2y)\frac{dy}{dx} + y \cdot e^{xy} = 0$$

or, $(xe^{xy} + 2y)\,dy + y \cdot e^{xy} \cdot dx = 0$

or, $ye^{xy}\,dx + (xe^{xy} + 2y)\,dy = 0$

$\therefore P(x, y) = ye^{xy}$ and $Q(x, y) = xe^{xy} + 2y$,

Now,

$$\frac{\partial P}{\partial y} = e^{xy} + xy \cdot e^{xy} \qquad \left[y \cdot \frac{\partial e^{xy}}{\partial y} + e^{xy} \cdot \frac{\partial y}{\partial y}\right]$$

$$\frac{\partial Q}{\partial x} = e^{xy} + xye^{xy}$$

Since, $\frac{\partial P}{\partial y}$ equals $\frac{\partial Q}{\partial x}$ it follows that the given differential equation is exact and we have,

NOTE: $P(x,y)\,dx + Q(x,y)\,dy = 0$ is exact then the solⁿ will be

$du = 0$ where $u = \int P\,dx + c(y)$

where, $c(y)$ is obtained

$$\frac{\partial u}{\partial y} = Q.$$

$$\frac{\partial x}{\partial y} \neq \quad \frac{\partial u}{\partial x} = ye^{xy} \text{ —(1) and}$$

$$\frac{\partial u}{\partial y} = xe^{xy} + 2y \text{ —— (2)}$$

'FORMULAS'

List of formula:

1) $\int x^p \, dx = \frac{x^{p+1}}{p+1} + c$, $p \neq -1$

2) $\int \frac{1}{x} \, dx = \log x + c$

3) $\int (ax+b)^p \, dx = \frac{(ax+b)^{p+1}}{a(p+1)} + c$, $p \neq -1$

4) $\int \frac{1}{ax+b} \, dx = \frac{\log(ax+b)}{a} + c$

5) $\int \frac{f'(x)}{f(x)} \, dx = \log f(x) + c$

6) $\int \frac{1}{a^2+x^2} \, dx = \frac{1}{a} \tan^{-1}(x/a) + c$

7) $\int \frac{1}{a^2-x^2} \, dx = \frac{1}{2a} \log \left\{ \frac{a+x}{a-x} \right\} + c$

8) $\int \frac{1}{x^2-a^2} \, dx = \frac{1}{2a} \log \left\{ \frac{x-a}{x+a} \right\} + c$

9) $\int \frac{1}{\sqrt{a^2-x^2}} \, dx = \sin^{-1}(x/a) + c$

10) $\int \frac{1}{\sqrt{a^2+x^2}} \, dx = \sinh^{-1}(x/a) + c$
$= \log(x + \sqrt{x^2+a^2}) + c$

11) $\int \frac{1}{\sqrt{x^2-a^2}} \, dx = \log(x + \sqrt{x^2-a^2}) + c$
$= \cosh^{-1}(x/a) + c$

12) $\int \sin x \, dx = -\cos x + c$

13) $\int \sin ax \, dx = -\frac{\cos ax}{a} + c$

14) $\int \cos x \, dx = \sin x + c$

15) $\int \cos bx \, dx = \frac{\sin bx}{b} + c$

16) $\int \sec^2 ax \, dx = \frac{\tan ax}{a} + c$

17) $\int \operatorname{cosec}^2 bx \, dx = -\frac{\cot bx}{b} + c$

18) $\int \sec ax \cdot \tan ax \, dx = \frac{\sec ax}{a} + c$

19) $\int \operatorname{cosec} bx \cdot \cot bx \, dx = -\frac{\operatorname{cosec} bx}{b}$

20) $\int \sec x \, dx = \log(\sec x + \tan x) + c$

21) $\int e^{ax} \, dx = \frac{e^{ax}}{a} + c$

or, $\frac{dy}{dx} + \frac{d^2y}{dx^2} - 2\frac{dy}{dx} - 1 - 2y + 2x + 2 = 0$

or, $\frac{d^2y}{dx^2} - \frac{dy}{dx} - 2y + 2x + 1 = 0$,

Which is the required differential equation.

(1.3)

Q Find the differential equation of all parabolas with their axes parallel to the x-axis

→ The equation of any parabola with (h,k) as its vertex, the x-axis as it's axis and latus rectum $4a$, is given by

$$(y-k)^2 = 4a(x-h), \quad \text{— i)}$$

where h, k and a are arbitrary constants. Differentiating (i) with respect to x, we get

$$2(y-k)\frac{dy}{dx} = 4a \quad \text{or,} \quad (y-k)\frac{dy}{dx} = 2a \quad \text{— ii)}$$

Differentiating (ii) with respect to x, we obtain (u.v rule)

$$\left(\frac{dy}{dx}\right)^2 + (y-k)\frac{d^2y}{dx^2} = 0 \quad \text{— iii)}$$

Differentiation of (iii) and simplification now gives.

$$3\frac{dy}{dx}\left(\cancel{\frac{dy}{dx}}\right)^2$$

$y = 1/x$

$\frac{dy}{dx} = -1x^{-1-1}$
$= -x^{-2} = -1/x^2$

Again differentiating eqn (w.r.t) 'x'

$\frac{d^2y}{dx^2} = 2x^{-3}$

Using the value of $\frac{dy}{dx}$, $\frac{d^2y}{dx^2}$ and y in eqn (i)

$2x^2 \times 2x^{-3} + 3x \times -x^{-2} - x^{-1} = 0$

or, $4x^{-1} + 3x(-3x^{-1}) - x^{-1} = 0$

or, $4x^{-1} - 3x^{-1} - x^{-1} = 0$

$\therefore 0 = 0$ (T)

$\therefore y = 1/x$ satisfies differential equation (1)

Sol.

$$\frac{dy}{dx} = \frac{y}{x} + \tan\left(\frac{y}{x}\right)$$

or, Let $y = vx$

$$\frac{d(vx)}{dx} = \frac{vx}{x} + \tan\left(\frac{vx}{x}\right)$$

or, $v + x \cdot \frac{dv}{dx} = v + \tan v$

or, $x \cdot \frac{dv}{dx} = v + \tan v - v$

or, $x \frac{dv}{dx} = \tan v$

or, $\frac{dv}{\tan v} = \frac{1}{x} dx$

Integrating both sides.

$$\int \frac{1}{\tan v} dv = \int \frac{1}{x} dx$$

or, $\int \cot v \, dv = \int \frac{1}{x} dx$

or, $\log(\sin v) \cancel{+\log c} = \log x + \log c$

or, $\log(\sin v) = \log(xc)$

$\therefore \sin v = xc$

Substituting $v = y/x$

$\therefore \sin(y/x) = xc$

or, $-\log(1-v$ ...

or, $\log x + \log(1-v^2) = \log c$

or, $x(1-v^2) = c$

Substituting the value of $v = y/x$ we get

$$x\left(1-\left(\frac{y}{x}\right)^2\right) = c$$

or, $x\left(\frac{x^2-y^2}{x^2}\right) = c$

or, $x^2 - y^2 = cx$, which is the reqrd solⁿ.

or, $e^{-x} + 2e^{-x} - 3e^{-x} = 0$

$\therefore 0 = 0 (T)$

$\therefore y = e^{-x}$ satisfies the differential eqn(1)

Again, from (3)

$$y = e^{3x}$$

$$\frac{dy}{dx} = 3e^{3x} \quad \text{—— (6)}$$

Again, differentiating eqn (6) w.r.t 'x'

$$\frac{d^2y}{dx^2} = 9e^{3x} \quad \text{—— (7)}$$

Using the value of $\frac{dy}{dx}$, $\frac{d^2y}{dx^2}$, y in eqn (1)

or, $9e^{3x} - 2 \times 3e^{3x} - 3 \times e^{3x} = 0$

or, $9e^{3x} - 6e^{3x} - 3e^{3x} = 0$

$\therefore 0 = 0 (T)$

$\therefore y = e^{3x}$ satisfies the differential eqn(1)

Q. Find the differential equation whose general solution is

$$y = Ae^{2x} + Be^{-x} + x$$

Let

$$y = Ae^{2x} + Be^{-x} + x \quad \text{—— i>}$$

We then have,

$$\frac{dy}{dx} = 2Ae^{2x} - Be^{-x} + 1 \quad \text{—— ii>}$$

$$\frac{d^2y}{dx^2} = 4Ae^{2x} + Be^{-x} \quad \text{—— iii>}$$

Adding i> and ii>, we obtain

$$y + \frac{dy}{dx} = \cancel{\tfrac{1}{2}}\, 3Ae^{2x} + Be^{-x} - Be^{-x} + x + 1$$

$$= 3Ae^{2x} + x + 1 \quad \text{—— iv>}$$

Adding ii> and iii>, we obtain

$$\frac{dy}{dx} + \frac{d^2y}{dx^2} = 6Ae^{2x} - Be^{-x} + Be^{-x} + 1$$

$$= 6Ae^{2x} + 1 \quad \text{—— v>}$$

$$Ae^{2x} = 1/6\left(\frac{dy}{dx} + \frac{d^2y}{dx^2} - 1\right)$$

Eliminating Ae^{2x} from iv> and v> we get,

$$y + \frac{dy}{dx} = 3 \cdot 1/6\left(\frac{dy}{dx} + \frac{d^2y}{dx^2} - 1\right) + x + 1$$

$$= 1/2\left(\frac{dy}{dx} + \frac{d^2y}{dx^2} - 1\right) + x + 1$$

or, $\cancel{\frac{d^2y}{dx^2}} + \cancel{\frac{dy}{dx}} - 2y - 2\frac{dy}{dx} - \cancel{\frac{1}{2}} \cancel{+x+}$

or, $y + \frac{dy}{dx} - x - 1 = \frac{1}{2}\left(\frac{dy}{dx} + \frac{d^2y}{dx^2} - 1\right)$

5) $\frac{dy}{dx} = \frac{2x-y+1}{x+2y-3}$

Soln.

HOMEWORK

1) $\sqrt{1-x^2}\,dy + \sqrt{1-y^2}\,dx = 0$

Soln.

$$\frac{dy}{\sqrt{1-y^2}} = -\frac{dx}{\sqrt{1-x^2}}$$

Integrating both sides,

$$\int \frac{dy}{\sqrt{1-y^2}} = -\int \frac{dx}{\sqrt{1-x^2}}$$

or, $\sin^{-1}(y) = -\sin^{-1}(x) + c$

$\therefore \sin^{-1}(y) + \sin^{-1}(x) = c$ #

2) $(\sin x + \cos x)\,dy = (\cos x - \sin x)\,dx$

Soln

$$dy = \frac{\cos x - \sin x}{\sin x + \cos x}\,dx$$

Integrating both sides,

$$\int dy = \int \frac{\cos x - \sin x}{\sin x + \cos x}\,dx$$

$$y = \log(\sin x + \cos x) + c \text{ #}$$

5) $\frac{dy}{dx} = \frac{2x-y+1}{x+2y-3}$

Soln.

HOMEWORK

1) $\sqrt{1-x^2}\,dy + \sqrt{1-y^2}\,dx = 0$

Soln.

$$\frac{dy}{\sqrt{1-y^2}} = -\frac{dx}{\sqrt{1-x^2}}$$

Integrating both sides,

$$\int \frac{dy}{\sqrt{1-y^2}} = -\int \frac{dx}{\sqrt{1-x^2}}$$

or, $\sin^{-1}(y) = -\sin^{-1}(x) + c$

$\therefore \sin^{-1}(y) + \sin^{-1}(x) = c$ #

2) $(\sin x + \cos x)\,dy = (\cos x - \sin x)\,dx$

Soln

$$dy = \frac{\cos x - \sin x}{\sin x + \cos x}\,dx$$

Integrating both sides,

$$\int dy = \int \frac{\cos x - \sin x}{\sin x + \cos x}\,dx$$

$$y = \log(\sin x + \cos x) + c$$ #

Soln.

$$\frac{dy}{dx} = \cancel{x}\ \frac{-y^2}{(xy+x^2)}$$

Let $y = vx$

$$\frac{d(vx)}{dx} = \frac{-(vx)^2}{x(vx)+x^2}$$

or, $v + x\frac{dv}{dx} = \frac{-v^2x^2}{vx^2+x^2}$

or, $v + x\frac{dv}{dx} = \frac{-\cancel{x^2}v^2}{\cancel{x^2}(v+1)}$

or, $v + x\frac{dv}{dx} = \frac{-v^2}{(v+1)}$

or, $\cancel{v+} x\frac{dv}{dx} = \frac{-v^2}{v+1} - v$

or $x\frac{dv}{dx} = \frac{-v^2-v^2-v}{v+1}$

or, $x\frac{dv}{dx} = \frac{-2v^2-v}{v+1}$

or, $\cancel{x\frac{dv}{dx}} =$ or, $x\frac{dv}{dx} = -\left(\frac{2v^2+v}{v+1}\right)$

or, $\frac{v+1}{2v^2+v}\,dv = \frac{-1}{x}\,dx$

or, Integrating both sides,

$$\int \frac{v+1}{2v^2+v}\,dv = -\int \frac{1}{x}\,dx$$

or, $\int \frac{v+1}{v(2v+1)}\,dv = -\int \frac{1}{x}\,dx$

The differential equation of the form $\frac{dy}{dx} = f(y/x)$.
where $f(y/x)$ is a function of y/x is called Homogeneous differential equation

iv) Linear differential equation.
An equation of the form

$\frac{dy}{dx} + Py = Q$, where P and Q are function of

x only is known as Linear differential equation.

Soln.

Let ~~$y = vx$~~

$$\frac{dy}{dx} = \frac{x^2+y^2}{(x^2+xy)}$$

Let $y = vx$

or, $$\frac{d(vx)}{dx} = \frac{x^2+v^2x^2}{(x^2+x\cdot vx)}$$

or, $$v + x\cdot\frac{dv}{dx} = \frac{x^2(1+v^2)}{x^2+vx^2}$$

or, $$v + x\frac{dv}{dx} = \frac{\cancel{x^2}(1+v^2)}{\cancel{x^2}(1+v)}$$

or, $$x\frac{dv}{dx} = \frac{1+v^2}{(1+v)} - v$$

or, $$x\frac{dv}{dx} = \frac{1+\cancel{v^2}-v-\cancel{v^2}}{1+v}$$

or, $$x\cdot\frac{dv}{dx} = \frac{1-v}{1+v}$$

or, $$\frac{1+v}{1-v}\,dv = \frac{1}{x}\,dx$$

Integrating on both sides, we get,

$$-\int\frac{1+v}{v-1}\,dv = \int\frac{1}{x}\,dx$$

or, $$-\int\frac{v+1-1+1}{v-1}\,dv = \int\frac{1}{x}\,dx$$

or, $$-\int\frac{v-1}{v-1} + \frac{2}{v-1}\,dv = \int\frac{1}{x}\,dx$$

or, $$-\int\frac{v-1}{v-1}\,dv - \int\frac{2}{v-1}\,dv = \int\frac{1}{x}\,dx$$

or, $x \cdot \frac{dv}{dx} = \frac{v-4-v(1-v)}{1-v}$

or, $x \cdot \frac{dv}{dx} = \frac{v-4-v+v^2}{1-v}$

or, $x \cdot \frac{dv}{dx} = \frac{v^2-4}{1-v}$

or, $\frac{1-v}{(v+2)(v-2)} dv = \frac{1}{x} dx$

Integrating both sides,

$$\int \frac{1-v}{(v+2)(v-2)} dv = \int \frac{1}{x} dx$$

or, $\int \frac{1}{v^2-4} dv - \int \frac{v}{v^2-4} dv = \log x + \log c$

or, $\frac{1}{2\times 2} \log\left(\frac{v-2}{v+2}\right) - \frac{1}{2}\int \frac{2v}{v^2-4} dv = \log x + \log c$

or, $\frac{1}{4} \log\left(\frac{v-2}{v+2}\right) - \frac{1}{2} \log(v^2-4) = \log x + \log c$.

or, $\frac{1}{4} \log\left(\frac{\frac{y}{x}-2}{\frac{y}{x}+2}\right) - \frac{1}{2} \log\left(\left(\frac{y}{x}\right)^2-4\right) = \log x + \log c$

or, $\frac{1}{4} \log\left(\frac{y-2x}{y+2x}\right) - \frac{1}{2} \log\left(\left(\frac{y}{x}\right)^2-4\right) = \log x + \log c$

Solve the di

$$c\left(x\frac{dy}{dx}+2y\right)=xy\frac{dy}{dx}$$

Sol^n.

$$c\left(x\frac{dy}{dx}+2y\right)=xy\frac{dy}{dx}.$$

or, $cx\frac{dy}{dx}+2cy=xy\frac{dy}{dx}.$

or, $cx\frac{dy}{dx}-xy\frac{dy}{dx}=-2cy$

or, $\cancel{xc}\ x(c-y)\frac{dy}{dx}=-2cy$

or, $\cancel{\frac{x(c-y)}{y}dy=-2yc}$

or, $\frac{c-y}{y}dy=\frac{-2c}{x}dx$

Integrating both sides,

$$\int\frac{c-y}{y}dy=\int\frac{-2c}{x}dx$$

or, $\int\frac{c}{y}dy-\int dy=-2c\int\frac{dx}{x}$

or, $\cancel{c-y}\ c\int\frac{1}{y}dy-\int dy=-2c\int\frac{1}{x}dx$

or, $c.\log_e y-y=-2c\log_e x+c$

or, $c\log_e y-y=-2c\log_e x+\log c_1$

or, $\log y^c-\log x^{2c}-\log c_1=y$ $\quad[\log x^p=p\log x]$

Solve the di

$$c\left(x\frac{dy}{dx}+2y\right)=xy\frac{dy}{dx}$$

Sol^n.

$$c\left(x\frac{dy}{dx}+2y\right)=xy\frac{dy}{dx}.$$

or, $cx\frac{dy}{dx}+2cy=xy\frac{dy}{dx}.$

or, $cx\frac{dy}{dx}-xy\frac{dy}{dx}=-2cy$

or, $x(c-y)\frac{dy}{dx}=-2cy$

or, $\cancel{\frac{x(c-y)}{y}dy=-\frac{2y}{c}}$

or, $\frac{c-y}{y}dy=\frac{-2c}{x}dx$

Integrating both sides,

$$\int\frac{c-y}{y}dy=\int\frac{-2c}{x}dx$$

or, $\int\frac{c}{y}dy-\int dy=-2c\int\frac{dx}{x}$

or, $\cancel{c-y}\; c\int\frac{1}{y}dy-\int dy=-2c\int\frac{1}{x}dx$

or, $c.\log_e y-y=-2c\log_e x+c$

or, $c\log_e y-y=-2c\log_e x+\log c_1$

or, $\log y^c-\log x^{2c}-\log c_1=y$ $\quad[\log x^p=p\log x]$

$\therefore u(x,y) = x^2y + xy + (1-x)\tan y + 2y = c$

$\therefore$ Thus, the sol^n is $x^2y + xy + (1-x)\tan y + 2y = c$ //

Solution By Inspection

1) $x\,dy + y\,dx = d(xy)$

2) $x\,dx + y\,dy = \frac{1}{2}\,d(x^2+y^2)$

3) $\frac{x\,dy - y\,dx}{x^2} = d(y/x)$

4) $\frac{y\,dx - x\,dy}{y^2} = d(x/y)$

5) $\frac{y\,dx - x\,dy}{xy} = \frac{dy}{y} - \frac{dx}{x} = d\log(y/x)$

6) $\frac{2xy\,dx - x^2\,dy}{y^2} = d\left(\frac{x^2}{y}\right)$

7) $\frac{2xy\,dy - y^2\,dx}{x^2} = d\left(\frac{y^2}{x}\right)$

Exercise (9-12)

9) $\frac{d^2y}{dx^2} + y = 0$, $y = \cos x$ and $y = \sin x$

Soln.

Let

$$\frac{d^2y}{dx^2} + y = 0 \quad \text{--- (1)}$$

$$y = \cos x \quad \text{--- (2)}$$

$$y = \sin x \quad \text{--- (3)}$$

From (2),

$$y = \cos x$$

$$\frac{dy}{dx} = -\sin x$$

Again, differentiating w.r.t 'x'

$$\frac{d^2y}{dx^2} = -\cos x$$

Now, using the value of $\frac{d^2y}{dx^2}$ and y in eqn (1)

$$-\cos x + \cos x = 0$$

$$\therefore \quad 0 = 0 \text{ (True)}$$

$\therefore$ $y = \cos x$ satisfies the differential eqn (1)

From (3)

$$y = \sin x$$

$$\frac{dy}{dx} = \cos x$$

Again differentiating w.r.t 'x'

Example: 1.9.

Solve

$$\frac{dy}{dx} = \left(\frac{x+y+1}{x+y+3}\right)^2 \quad \text{— (1)}$$

Let $x+y=t$ then $1+\frac{dy}{dx} = \frac{dt}{dx}$

$$\therefore \frac{dy}{dx} = \frac{dt}{dx} - 1$$

So, eqn (1) becomes

$$\frac{dt}{dx} - 1 = \left(\frac{t+1}{t+3}\right)^2$$

or, $\frac{dt}{dx} = \left(\frac{t+1}{t+3}\right)^2 + 1$

or, $\frac{dt}{dx} = \frac{t^2+2t+1}{t^2+6t+9} + 1$

or, $\frac{dt}{dx} = \frac{t^2+2t+1+t^2+6t+9}{(t+3)^2}$

or, $\frac{dt}{dx} = \frac{2t^2+8t+10}{(t+3)^2}$

or, $\frac{dt}{dx} = \frac{2(t^2+4t+5)}{(t+3)^2}$

or, $\frac{(t+3)^2}{t^2+4t+5}\, dt = 2dx.$

Integrating both sides, we get.

$$\int \frac{(t+3)^2}{t^2+4t+5}\, dt = 2\int dx$$

or, $\int \frac{t^2+6t+9}{t^2+4t+5}\, dt = 2x + \log c$

or, $\int \frac{(t^2+4t+5)+(2t+4)}{\quad} dt = 2x + \log c$

$$3\frac{dy}{dx}\cdot\frac{d^2y}{dx^2} + (y-k)\frac{d^3y}{dx^3} = 0$$

Eliminating $(y-k)$ from (iii) and (iv), we obtain

$$3\cdot\frac{dy}{dx}\frac{d^2y}{dx^2} + \frac{d^3y}{dx^3}\left[-\frac{(dy/dx)^2}{d^2y/dx^2}\right] = 0$$

or, $3\frac{d^2y}{dx^2} -$

- Solve:

$$(3x-2y+1)dx+(3y-2x-1)dy=0$$

Soln.

$$M=(3x-2y+1)$$
$$N=(3y-2x-1)$$

$$\frac{\partial M}{\partial y}=-2$$

$$\frac{\partial N}{\partial x}=-2$$

$$\therefore \frac{\partial M}{\partial y}=\frac{\partial N}{\partial x}$$

$\therefore$ Hence the equation is exact.

Now,

$\int M dx$ taking y as constant

$$=\int 3x-2y+1\,dx.$$

$$=\frac{3x^2}{2}-2xy+x$$

Again the term free x in N is $3y-1$.

So, $\int 3y-1\,dy$

$$=\frac{3y^2}{2}-y$$

Thus required equation is,

$$\frac{3x^2}{2}-2xy+x+\frac{3y^2}{2}-y=k$$

$$\Rightarrow 3x^2-4xy+2x+3y^2-2y=c$$

Sol:

$$2xy\,dy = -(x^2+y^2)\,dx$$

or, $$\frac{dy}{dx} = \frac{-(x^2+y^2)}{2xy}$$

Let $y = vx$.

or, $$\frac{d(vx)}{dx} = -\frac{(x^2+(vx)^2)}{2x(vx)}$$

or, $$v + x\frac{dv}{dx} = -\frac{(x^2+v^2x^2)}{2vx^2}$$

or, $$v + x\frac{dv}{dx} = \frac{-\cancel{x^2}(1+v^2)}{2v\cancel{x^2}}$$

or, $$v + x\frac{dv}{dx} = -\frac{(1+v^2)}{2v}$$

or, $$\cancel{v+}\, x\frac{dv}{dx} = -\frac{(1+v^2)}{2v} - v$$

or, $$x\frac{dv}{dx} = \frac{-(1+v^2)-2v^2}{2v}$$

or, $$x\frac{dv}{dx} = \frac{-1-v^2+2v^2}{2v}$$

or $$x\frac{dv}{dx} = \frac{-1+v^2}{2v}$$

$$\cancel{\text{or, } x\frac{dv}{dx} = \frac{v^2-1}{2v}}$$

or, $$\frac{2v}{-1+v^2}\,dv = \frac{1}{x}\,dx$$

Integrating on both sides,

$$-\int\frac{2v}{1+v^2}\,dv = \int\frac{1}{x}\,dx$$

$$(x+y)dx = -(y-x)dy$$

or, $$\frac{dy}{dx} = \frac{(x+y)}{-(y-x)}$$

Let $y = vx$

$$\frac{d(vx)}{dx} = \frac{x+vx}{-(vx-x)}$$

$$v + x\frac{dv}{dx} = \frac{\cancel{x}(1+v)}{-\cancel{x}(v-1)}$$

or, $$v + x\frac{dv}{dx} = \frac{1+v}{1-v}$$

or, $$x\frac{dv}{dx} = \frac{1+v}{1-v} - v$$

or, $$x\frac{dv}{dx} = \frac{1+v-v(1-v)}{1-v}$$

or, $$x\frac{dv}{dx} = \frac{1+\cancel{v}-\cancel{v}+v^2}{1-v}$$

or, $$\frac{1-v}{1+v^2}dv = \frac{1}{x}dx$$

or, $$\frac{1}{1+v^2} - \frac{v}{1+v^2}dv = \frac{1}{x}dx$$

Integrating on both sides,

$$\int\left(\frac{1}{1+v^2} - \frac{v}{1+v^2}\right)dv = \int\frac{dx}{x}$$

or, $$\tan^{-1}v - \frac{1}{2}\log(1+v^2) = \log x + c$$

Soln.

$$y' = \frac{\sqrt{1+y^2}}{\cos^2 x}$$

or, $\frac{dy}{dx} = \frac{\sqrt{1+y^2}}{\cos^2 x}$

or, $\frac{1}{\sqrt{1+y^2}}\,dy = \frac{1}{\cos^2 x}\,dx.$

Integrating both sides, we get.

$$\int \frac{1}{\sqrt{1+y^2}}\,dy = \int \frac{1}{\cos^2 x}\,dx.$$

or, $\cancel{\sinh^{-1}(y)} =$

or, $\log(y + \sqrt{1+y^2}) + \log c = \int \sec^2 x\,dx$

or, $\log(y + \sqrt{1+y^2}) + \log c = \tan x$

Alternative.

or, $\sinh^{-1}(y) = \tan x + c$

$\therefore \sinh^{-1}(y) - \tan x = c$ #

$y\,dx = (e^x + 1)\,dy$

Soln

$$\frac{dx}{(e^x+1)} = \frac{1}{y}\,dy$$

Integrating both sides, we get

$$\int \frac{dx}{e^x+1} = \int \frac{1}{y}\,dy$$

or, $\displaystyle\int \frac{1}{e^x+1}\,dx = \int \frac{1}{y}\,dy$

or, $\displaystyle\int \frac{1+e^x-e^x}{1+e^x}\,dx = \int \frac{1}{y}\,dy$

or, $\displaystyle\int \frac{1+e^x}{1+e^x}\,dx - \int \frac{e^x}{1+e^x}\,dx = \log y + \log c.$

or, $\displaystyle\int 1\,dx - \int \frac{e^x}{1+e^x}\,dx = \log y + \log c$

or, $x - \log(1+e^x) = \log y + \log c$

$\therefore x = \log(1+e^x) + \log y + \log c$ #

or, $\frac{-1}{3v^3} + \log v + \log x = c$

or, $\frac{-1}{3v^3} + \log(vx) = c$

Substituting $v = \frac{y}{x}$ we get,

or, $\frac{-1}{\frac{3y^3}{x^3}} + \log(y) = c$

or, $-\frac{x^3}{3y^3} + \log y = c$, which is the req^rd sol^n

dx^2

Now, using the value of $\frac{d^2y}{dx^2}$ and y in equⁿ ①

$$-\sin x + \sin x = 0$$
$$\therefore 0 = 0 (T)$$

$\therefore$ $y = \sin x$ satisfies the differential equ (1).

10) $\frac{d^2y}{dx^2} - 2\frac{dy}{dx} - 3y = 0$, $y = e^{-x}$ and $y = e^{3x}$.

Solⁿ

Let

$$\frac{d^2y}{dx^2} - 2\frac{dy}{dx} - 3y = 0 \quad \text{—} ①$$

$$y = e^{-x} \quad \text{—} ②$$

$$y = e^{3x} \quad \text{—} ③$$

From ②

$$y = e^{-x}$$
$$\frac{dy}{dx} = \cancel{x} - e^{-x} \quad \text{—} ④$$

Again, differentiating w.r.t 'x', we get

$$\frac{d^2y}{dx^2} = e^{-x} \quad \text{—} ⑤$$

Using the value of $\frac{d^2y}{dx^2}$, $\frac{dy}{dx}$, y in eqn ①

Integrating both sides.

$$\int \frac{1}{x}\, dx = \int \frac{-v}{(v-1)^2}\, dv.$$

$$\log x = -\int \frac{v}{}$$

$$y' = \frac{y-4x}{x-y}$$

Sol.

$$\frac{dy}{dx} = \frac{y-4x}{x-y} \quad \text{—①}$$

or, dividing by x in eqn ①

$$\frac{dy}{dx} = \frac{\frac{y}{x} - \frac{4x}{x}}{\frac{x}{x} - \frac{y}{x}}$$

$$\frac{dy}{dx} = \frac{\frac{y}{x} - 4}{1 - y/x}$$

Let $v = y/x \quad \therefore y = vx$.

$$\frac{dy}{dx} = \frac{v-4}{1-v}$$

$$\frac{dy}{dx} = \frac{d}{dx}(vx)$$

$$= v + x \cdot \frac{dv}{dx}$$

We know,

$$v + x\frac{dv}{dx} = \frac{dy}{dx} = \frac{y/x - 4}{1 - y/x} = \frac{v-4}{1-v}$$

Now,

$$v + x\frac{dv}{dx} = \frac{v-4}{1-v}$$

$$x \cdot \frac{dv}{dx} = \frac{v-4}{1-v} - v$$

7) $(1+x)(1+y^2)\,dx + (1+y)(1+x^2)\,dy = 0$

Sol^n.

$$(1+x)(1+y^2)\,dx = -(1+y)(1+x^2)\,dy$$

or, $\frac{(1+y^2)}{1+y} \cdot \frac{1+x}{(1+x^2)}\,dx = -\frac{1+y}{(1+y^2)}\,dy$

or, $\frac{1}{1+x^2} + \frac{x}{1+x^2}\,dx = \frac{-1}{1+y^2} - \frac{y}{1+y^2}\,dy$

Integrating both sides,

$$\int \frac{1}{1+x^2} + \frac{x}{1+x^2}\,dx = \int \frac{-1}{1+y^2} - \frac{y}{1+y^2}\,dy$$

or, $\int \frac{1}{1+x^2}\,dx + \int \frac{x}{1+x^2}\,dx = -\int \frac{1}{1+y^2}\,dy - \int \frac{y}{1+y^2}\,dy$

or, $\tan^{-1}x + \frac{1}{2}\int \frac{2x}{1+x^2}\,dx = -\tan^{-1}y - \frac{1}{2}\int \frac{2y}{1+y^2}\,dy$

or, $\tan^{-1}x + \frac{1}{2}\log(1+x^2) = -\tan^{-1}y - \frac{1}{2}\log(1+y^2) + c$

or, $\tan^{-1}x + \tan^{-1}y + \frac{1}{2}\left[\log(1+x^2) + \log(1+y^2)\right] = c$ //

Homogeneous differential Equation

2) $\frac{dy}{dx} = \frac{x^2 y}{x^3 + y^3}$

Soln.

Let $y = vx$, we get

$$\frac{d(vx)}{dx} = \frac{\cancel{d}\ (x^2 vx)}{x^3 + (vx)^3}$$

or, $v + x \cdot \frac{dv}{dx} = \frac{vx^3}{x^3 + v^3 x^3}$

or, $v + x \frac{dv}{dx} = \frac{v\cancel{x^3}}{\cancel{x^3}(1+v^3)}$

or, $v + x \frac{dv}{dx} = \frac{v}{1+v^3}$

or, $x \frac{dv}{dx} = \frac{v}{1+v^3} - v$

or, $x \frac{dv}{dx} = \frac{\cancel{v} - \cancel{v} - v^4}{1+v^3}$

or, $-\frac{v^4}{1+v^3} = x \frac{dv}{dx}$

or, $-\frac{1}{x} dx = \frac{1+v^3}{v^4} dv$

or, $-\frac{1}{x} dx = \left(\frac{1}{v^4} + \frac{1}{v} \right) dv$

~~or,~~ $\cancel{\frac{1}{x} dx =}$ Integrating on both sides,

$$-\int \frac{1}{x} dx = \int \left(\frac{1}{v^4} \right) dv + \int \frac{1}{v} dv$$

$$\frac{dy}{dx} = \frac{xy}{(x^2+y^2)}$$

Let $y = vx$

$$\frac{d(vx)}{dx} = \frac{xvx}{(x^2+v^2x^2)}$$

or, $v + x\frac{dv}{dx} = \frac{v\cancel{x^2}}{\cancel{x^2}(1+v^2)}$

or $v + x\cdot\frac{dv}{dx} = \frac{v}{1+v^2}$

or, $x\frac{dv}{dx} = \frac{v}{1+v^2} - v$

or, $x\frac{dv}{dx} = \frac{v - v(1+v^2)}{1+v^2}$

or, $x\frac{dv}{dx} = \frac{\cancel{v} - \cancel{v} - v^3}{1+v^2}$

or, $x\frac{dv}{dx} = -\frac{v^3}{1+v^2}$

or, $\frac{1+v^2}{v^3}\,dv = -\frac{1}{x}dx$

or, Integrating on both sides

$$\int \frac{1+v^2}{v^3}\,dv = -\int \frac{1}{x}dx$$

or, $\int \frac{1}{v^3}\,dv + \int \frac{1}{v}\,dv = -\int \frac{1}{x}dx$

or, $-\frac{1}{2v^2} + \log v = -\log x + \log c$

or, $-\frac{1}{2v^2} + \log v + \log x = \log c$

4) $y\frac{d^2y}{dx^2} + \left(\frac{dy}{dx}\right) = 5$

order - 2

degree - 1

It is second order & first degree diff eqn.

5) $\frac{d^2y}{dx^2} = \left[1 + \left(\frac{dy}{dx}\right)^2\right]^{3/2}$.

order - 2

degree - 2

Solve:

Squaring

$$\left(\frac{d^2y}{dx^2}\right)^2 = \left(\left[1 + \left(\frac{dy}{dx}\right)^2\right]^{3/2}\right)^2$$

$$= \left(1 + \left(\frac{dy}{dx}\right)^2\right)^3$$

$$\left(\frac{d^2y}{dx^2}\right)^2 = = \left(1 + \left(\frac{dy}{dx}\right)^6\right)$$

∴ It is second order & second degree diff eqn.

6) $\sqrt{\frac{d^3y}{dx^2}} = \frac{dy}{dx}$

$\frac{d^3y}{dx^2} = \left(\frac{dy}{dx}\right)^2$

order - 3

degree - 1

∴ It is third order & first degree diff eqn.

$(xy^2+x)dx + (yx^2+y)dy = 0$

Solⁿ.

$x(y^2+1)\,dx = -y(x^2-1)\,dy$

or, $\frac{x}{x^2-1}\,dx = \frac{-y}{y^2+1}\,dy$

Integrating both sides, we get.

$\int \frac{x}{x^2-1}\,dx = -\int \frac{y}{y^2+1}\,dy$

or, Dividing & Multiplying by 2 on both sides,

$\frac{1}{2}\int \frac{2x}{x^2-1}\,dx = -\frac{1}{2}\int \frac{2y}{y^2+1}\,dy$

or, $\frac{1}{2}\cdot\log(x^2-1) = -\frac{1}{2}\log(y^2+1) + \frac{c}{2}$

or, $\log(x^2-1) + \log(y^2+1) = c$ #