Graph (Representation of Network)
[Graphs makes us easy for depiction]

Def^n -> A graph is a collection of vertices and edges. Mathematically $G = (V, E)$ where,

V is vertex set (point)

E is Edge set (connection line)

a) To scale down real time system or entities
b) To study networks & networking devices
c) To study the chemical compound & their bonds
d) To study & understand flow of a system.
e) Graphs are used to find out all the connecting flights between any two cities
f) Graphs are used to model electric circuits, project plannings & genetics etc.

Note:
nodes - Data points
Data points relation - Edge

Types of graph

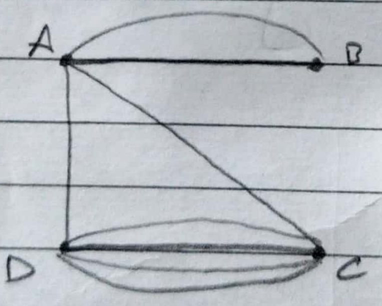
i) Simple graph
ii) Multi graph
iii) Pseudo graph

Explaination

1. Multigraph -> यस्तो Graph जुनमा कुनै पनि दुईवटा vertices हरुको बिचमा Multiple Edge हुन्छ।

(Same source & same destination with no intermediate vertex.)

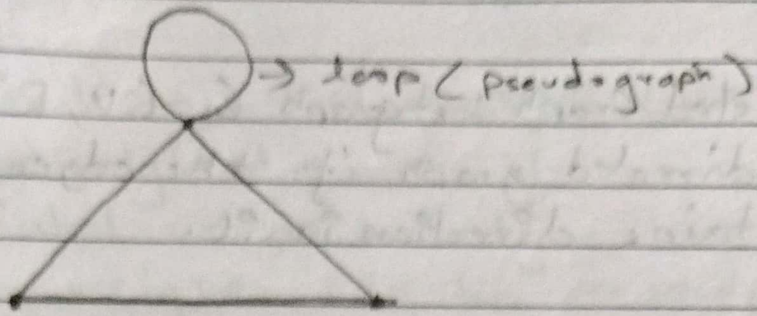
Example:-



2. A graph $G = (V, E)$ is said to be a multi graph if there exist two or more edges between any two pair of vertices, such edge are called as parallel edges.

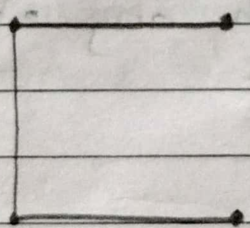
3. A Pseudo graph
↳ A graph $G = (V, E)$ is said to be a pseudo graph if any of its vertices contain loop in it.

Example:-



4. Simple graph
↳ A graph $G = (V, E)$ is said to be a simple graph if it doesn't contains any parallel edges & loop in it.

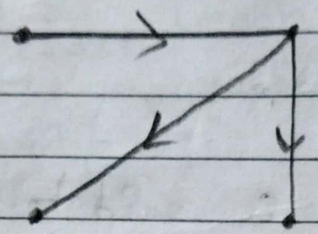
Example:-



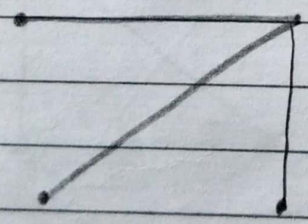
Types of graph based on direction

1. Directed Graph
2. Undirected Graph

Directed Graph : A graph $G=(V,E)$ is said to be directed graph if the edges of graph contains direction in it.



Undirected Graph :- A graph $G=(V,E)$ is said to be an undirected graph if there is no direction associated with any of its edges.

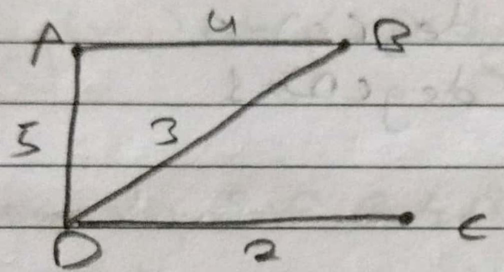


Types of graph based on weight

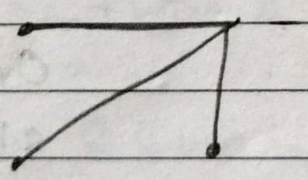
1. Weighted Graph
2. Unweighted Graph

~~1st~~ Weighted Graph : A graph $G=(V,E)$ is said ~~to~~ be weighted graph if some weights are associated with its edges.

↳ These weights can be resource utilized, time consumed, cost or anything



Unweighted graph :- A graph $G=(V,E)$ is said to unweighted graph if there is no any weights ~~ass~~ assigned to its edges.



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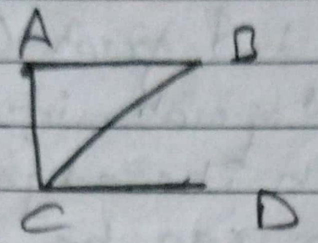
Basic Terminologies of Graph

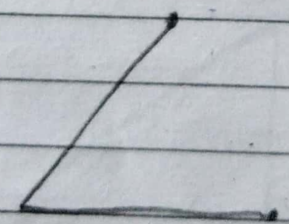
1. Degree of a node : The no of edges incident on a node is said to be degree of a node



deg(A) = 2
deg(A) = 2
deg(B) = 2
deg(C) = 3
deg(D) = 1

Note :- Loop contributes 2 in degree count.

2. Order and size of the graph :- The number of vertices in a graph is said to be order of the graph. The number of edges in a graph is said to be size of the graph



Order 3
size - 2

3. Theorem : The Handshaking theorem :- The sum of degree of all vertices is twice the number of edges

$$\sum_{i=1}^{n} \deg(v_i) = 2|E|$$

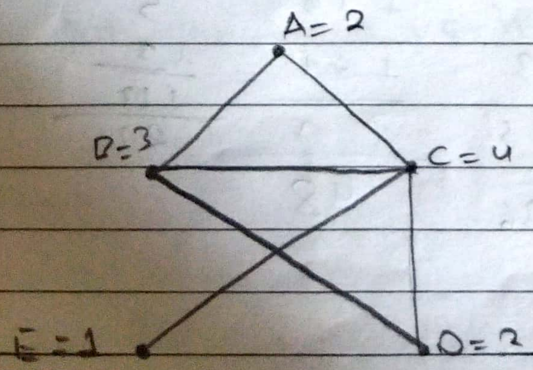
$$v_1 + v_2 + v_3 + \dots + v_n = 2|E|$$

or,

$$\deg(v_1) + \deg(v_2) + \deg(v_3) + \dots + \deg(v_n) = 2|E|$$

Consider a graph $G = (V, E)$ with n vertices say $v_1, v_2, v_3, \dots, v_n$ and $|E|$ be the number of edges. Since each ~~eac~~ edge & contributes a degree of 2. Therefore when we are counting degree an edge is repeated (twice counted) which leads to the total no of degrees to be twice as no of edge.

4. Odd vertices & Even vertices :- The vertex with odd degree is called odd vertex and with ~~the~~ even degree is called even vertex



Odd vertices = B, E
Even vertices = A, C, D

Theorem: The number of odd vertices in a graph is always even.

In hand shaking theorem

$$\sum_{i=1}^{n} deg(v_i) = 2|E|$$

$$deg(v_1) + deg(v_2) + deg(v_3) + \cdots + deg(v_n) = 2|E|$$

$$\Rightarrow deg(O_1) + deg(O_2) + deg(O_3) + \cdots + deg(O_m) +$$

$$deg(E_1) + deg(E_2) + deg(E_3) + \cdots + deg(E_n) = 2|E|$$

$$\Rightarrow deg(O_1) + deg(O_2) + deg(O_3) + \cdots + deg(O_m) + even = even$$

$$\Rightarrow deg(O_1) + deg(O_2) + deg(O_3) + \cdots + deg(O_m) = even$$

Example

1 +5 = 6

1 +3 +5 = 9

1 +7 +13 +15 = 36

1 +19 +21 +7 = 48

5 +3 +13 = 21

Proof :

Let G be a graph. If G contains no odd vertices, then the result follows immediately. Suppose that G contains k number of odd vertices which are $v_1, v_2 \dots v_k$ and n number of even vertices which are $u_1, u_2 \dots u_3$. Clearly $d(u_1) + d(u_2) + \dots + d(u_n) = \text{even}$

By handshaking theorem

$$[d(v_1) + d(v_2) + \dots + d(v_k)] + [d(u_1) + d(u_2) + \dots + d(u_n) + \dots + d(u_n) = 2|E|.$$

where $|E|$ is the total number of edges in G.

Then,

$$d(v_1) + d(v_2) + \dots (v_k) = 2|E| - [d(u_1) + d(u_2) + \dots + d(u_n)]$$
$$= 2|E| - \text{even}$$
$$= \text{even}$$

Here, sum of the degree of odd vertices is even, to hold this condition k must be even i.e, G has an even number of odd vertices. If G has no even vertices then we have

$$d(v_1) + d(v_2) + \dots + d(v_k) = 2|E| \text{ is}$$
even

From ~~whe~~ which we again conclude that k is even. This completes the proof of the theorem.

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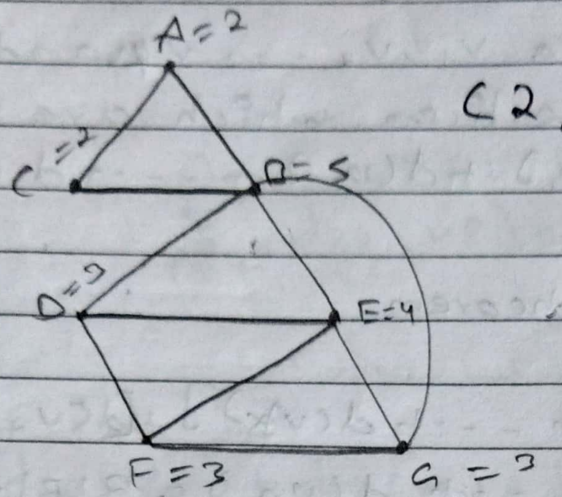
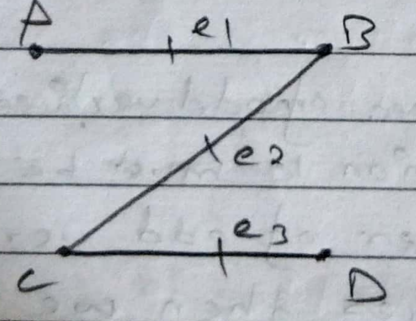
(*) Degree Sequence of a graph :- when we write all the degree of the respective vertices of the graph in ascending order then it is called Degree sequence of the graph



(2, 2, 3, 3, 3, 4, 5)

Adjacent nodes & Adjacent edges

Adjacent nodes :- when two nodes say u and v are separated by single edge or they are the end points for an edge, such node u and v are called as adjacent nods.



$e_1 = \{A, B\}$] → adjacent edge
$e_2 = \{B, C\}$] → adjacent edge
$e_3 = \{C, D\}$

Adjacent edges : If the ~~vertices~~

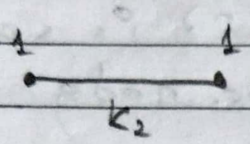
Adjacent edges: If one of the vertices in the two edges are same then it is said to be adjacent edge.

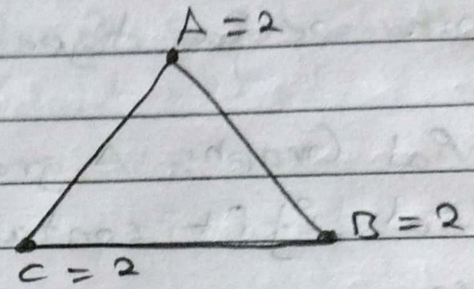
Some special type of graph

Trivial Graph:- A graph $G=(V,E)$ is said to be

Regular graph :- A graph $G=(V,E)$ is said to be a regulor graph if all the vertices of the graph have same degree. say if all the vertices of the graph have r degree then the graph is called as r - regulor graph.

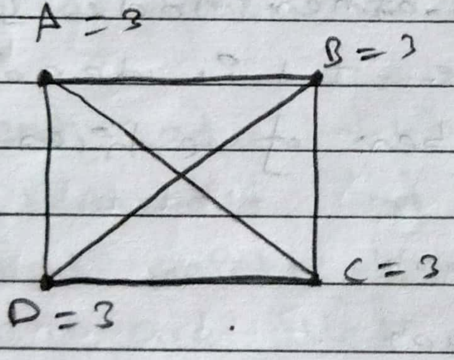
Example





1 - regulor

2 - regulor

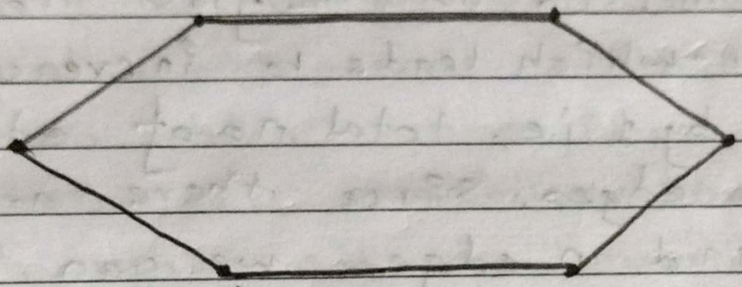


3 - regulor

(*) A complete graph k_n is $n-1$ regulor

Cycle graph :-

A graph $G=(V,E)$ is said to be a cycle graph if every vertices in a graph is connected to only two vertices on its either side & the last vertex is connected to the first vertex. The no of vertices in a cycle graph should be three or more. It is denoted by C_n



Q. The order and size of cycle graph is equal.

Soln

To connect any two nodes we need only one edge. In a cycle graph every nodes are connected to other in a linear fashion. So for a graph of n nodes $n-1$ edges are required to connect them in a linear fashion. To make the graph as cycle graph, the last node is connected to the first node by single edge which leads to increase in edge count by 1 i.e, total no of edges $= n-1+1 = n$ edges. Since there are n vertices and n edges we can conclude the order & size of cycle graph G is equal

Q. A complete graph (k_n) is n-1 regular

soln

In a complete graph every node is connected to every other nodes by single edge. let the complete graph contains n nodes (say k_n). when we look for the first vertex it is connected to every other vertex by individual edges ie. it is connected to (n-1) vertices which show (n-1) edges are incident on vertex 1. Thus, vertex 1 has (n-1) degree. similarly vertex 2 is also connected with other (n-1) vertices which makes its degree (n-1) and this case continues for all n vertices which makes all the n vertices having (n-1) degree. Thus we can say the graph k_n is (n-1) regular since all the vertices have (n-1) degree.

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(*) The total number of edges in a complete graph K_n is $\frac{n(n-1)}{2}$

⇒ In a completegraph every node is connected to every other nodes by single edge. let the complete graph contains n nodes (say K_n). When we look for the first vertex it is connected to every other vertex by individual edges i.e. it is connected to (n-1) vertices which shows ~~(n-1) edges~~ leads that all the n vertices have (n-1) degree.

By handshaking theorem,

$$\sum_{i=1}^{n} d(v_i) = 2|E|$$

where, E = no of edges with n vertices in the graph.

⇒ $deg(v_1) + deg(v_2) + \cdots + d(v_n) = 2|E|$ —(i)

Since we know that,

the ~~de~~ maximum no of degree of each vertex in the graph can be (n-1).

Then eqn (i) becomes

$(n-1)+(n-1)+ \ldots + n\text{'terms} = 2|E|$

$\Rightarrow n(n-1) = 2|E|$

$\Rightarrow E = \frac{n(n-1)}{2}$

i. The total number of edges in a complete graph K_n is $\frac{n(n-1)}{2}$

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* The no of vertices in a R regular graph is even if R is odd.

=> ~~In a~~

=> A graph $G=(V,E)$ is said to be a regular graph if all the vertices of the graph have same degree.

By handshaking theorem

$$\sum_{i=1}^{n} deg(v_i) = 2|E|$$

In the above theorem, the sum of degree of the vertices is even because it is two times the number of edges in graph

Thus if r is odd, every r-regular graph must have even number of vertices.
To proove the fact we have

$$\sum_{i=1}^{n} deg(v_i) = 2|E|$$

$\Rightarrow deg(v_1) + deg(v_2) + deg(v_3) + \cdots + deg\, v_n$

$\Rightarrow deg(O_1) + deg(O_2) + \cdots deg(O_n) + deg(\varepsilon_1) + deg(\varepsilon_2) + \cdots + deg(\varepsilon_n) = 2|E|$

$\Rightarrow deg(O_1) + deg(O_2) + \cdots + deg(O_n) = 2|E| - \{deg(\varepsilon_1) + deg(\varepsilon_2) + \cdots + deg(\varepsilon_n)\}$

So, The end result is even number of vertices even if R is odd proved.

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Q. The total number of edges in a complete graph kn is $\frac{n(n-1)}{2}$.

=) A Graph $G=(V,E)$ is said to be complete graph if each node of the graph is connected to every other nodes by a single dedicated edge. It is denoted, kn where n is number of vertices in a graph

The no of ~~graph i.~~ edges in a graph is the no of ways in which two vertices can be selected from n vertices i.e. $C(n,2)$ because only two vertices is required to form an edge.

$$\therefore C(n,2) = \frac{n!}{(n-2)! \times 2!}$$

$$= \frac{n \times (n-1) \times (n-2)!}{(n-2)! \times 2}$$

$$= \frac{n(n-1)}{2}$$

Ans #

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Q. The no of vertices in a R regular graph is even if R is odd.

⇒ Let $G=(V,E)$ be a R-regular graph of N vertices & T be the sum of degree of R-regular graph with N vertices then
$T = R \times N$

∵ The sum of degree of vertices is even from handshaking theorem

So, T must be even and we are provided with R as odd.

∴ N must be even hence the no of vertices in a R-regular graph is even when R is odd.

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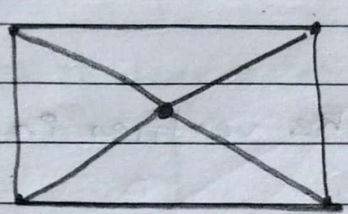
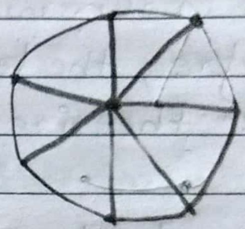
Wheel graph :- A graph G=(V,E) is said to be a wheel graph if it is a cycle graph with one additional vertex where all other vertices of the cycle graph get connected to by an edge. It is denoted as W_n

Example



W_4



Properties of wheel graph and cycle

i) The size of W_n is twice the size of C_n

ii) The sum of degrees of the vertices W_n is 4 times the size of C_n.

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HW

i) The size of W_n is twice the size of C_n

=> let W_n be the wheel graph which is obtained from the cycle graph by adding a vertex inside and then joining to every vertex in C_n. Thus, by joining the newly introduced vertex to each vertex in C_n we get n number of new edges. Therefore, the total number of edges in W_n is twice the size of C_n.

ii) The sum of degrees of the vertices in W_n is four times the size of C_n.

=> Let C_n be a cycle graph and W_n be a wheel graph obtained from C_n. By handshaking theorem, Sum of degrees of vertices in graph is twice the no of edges. so for wheel graph W_n

Sum of degree of verties in W_n = 2x no of edges in W_n

But the size of W_n is twice the size of C_n

$\therefore$ The sum of the degree of verties in
W_n = 2 x 2 x number of edges in C_n
= 4 x number of edges in C_n
= 4 x size of C_n.

Thus, the sum of degree of verties in W_n is four times the size of C_n.

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Alternative

i) The size of W_n is twice the size of C_n.

⇒ W_n is C_n with one additional vertex. where all the n vertices of C_n is connected to that additional vertex which gives W_n with n more edges

∴ Total no of edges in $W_n = n + n = 2n$

ii) The sum of degree of the vertices in W_n is four times the size of C_n.

⇒ Acc to Handshaking theorem :-

$$\sum_{i=1}^{n} \deg(v_i) = 2|E|$$

In wheel graph,

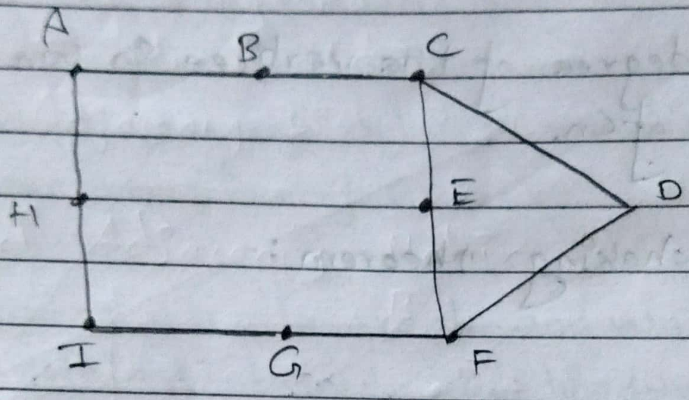
$$\sum_{i=1}^{n} \deg(v_i) = 2|E_{W_n}|$$

$$= 2 \times 2 \, |E_{C_n}|$$

$$= 4 \, |E_{C_n}|$$

6. Bipartite Graph:-

A graph $G = (V, E)$ is said to be a bipartite graph if the vertices of graph G can be divided into two sets say V_1 & V_2 such as $V_1 \cup V_2 = V$ and $V_1 \cap V_2 = \phi$, such that for every edge in Edge set E, one point lies in V_1 & other endpoint lies in V_2.

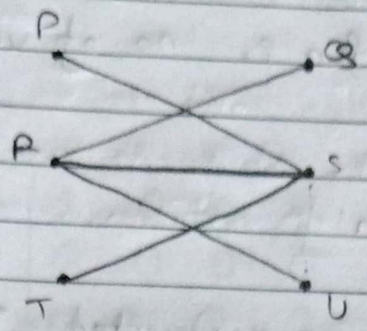


$G = \{V, E\}$

$V = \{A, B, C, D, E, F, G, H, I\}$

$E = \{AB, BC, CD, DF, CE, EF, FG, GI, IH, AH\}$

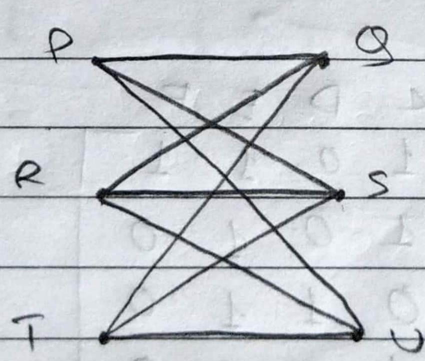
$V_1 = \{A, C, F, I\}$
$V_2 = \{B, D, E, G, H\}$



$V = \{P, Q, R, S, T, U\}$
$E = \{PS, RS, QR, UR, ST\}$

$V_1 = \{P, R, T\}$
$V_2 = \{S, Q, U\}$

7. Complete Bipartite Graph:- A bipartite graph is a complete bipartite graph if each vertex in set V_1 is connected to every vertex in set V_2 by an edge



$V_1 = \{P, R, T\}$
$V_2 = \{Q, S, U\}$

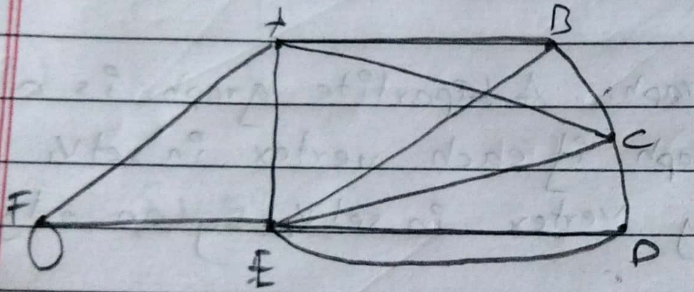
This graph is denoted as $K_{m,n}$ where m is no of vertices in V_1 & n is no of vertices in V_2.

Representation of Graph :-

Graphs in a system are represented by :

1. Adjacency Matrix
2. Incidence Matrix
3. Adjacency List

Construct Adjacency Matrix



$$A(G) = \begin{array}{c} \\ A \\ B \\ C \\ D \\ E \\ F \end{array} \begin{array}{c} \begin{array}{cccccc} A & B & C & D & E & F \end{array} \\ \begin{bmatrix} 0 & 1 & 1 & 0 & 1 & 1 \\ 1 & 0 & 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 & 2 & 0 \\ 1 & 1 & 1 & 2 & 0 & 1 \\ 1 & 0 & 0 & 0 & 1 & 1 \end{bmatrix} \end{array}$$

Construct Incidence matrix for the provided graph above.

$$I(G) = \begin{array}{c} \\ A \\ B \\ C \\ D \\ E \\ F \end{array} \begin{array}{c} \begin{array}{cccccccccccc} e_1 & e_2 & e_3 & e_4 & e_5 & e_6 & e_7 & e_8 & e_9 & e_{10} & e_{12} & e_{12} \end{array} \\ \left[\begin{array}{cccccccccccc} 1 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 & 0 & 0 & 1 & 1 & 0 & 0 & 1 & 1 \\ 1 & 1 & 2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{array} \right] \end{array}$$

Adjacency list

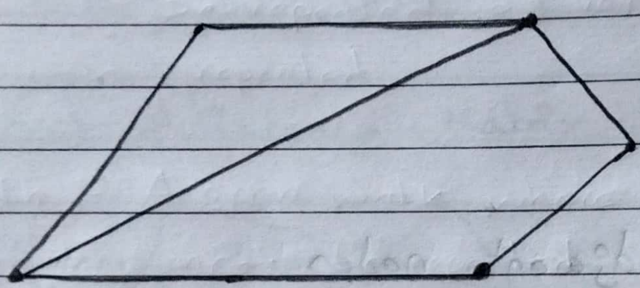
Vertices	Adjacent nodes
A	B, C, E, F
B	A, E, C
C	A, B, D, E
D	C, E
E	A, B, C, D, F
F	A, E, F

Connectivity

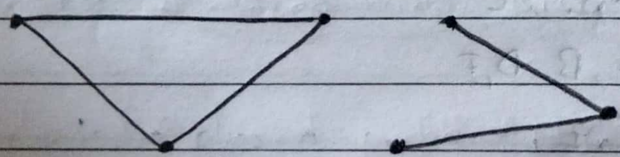
A graph $G=(V,E)$ is said to be connected graph if there exists at least one path between any pair of vertices.

A graph $G=(V,E)$ is said to be disconnected if there doesn't exist path between any point of vertices.

a)

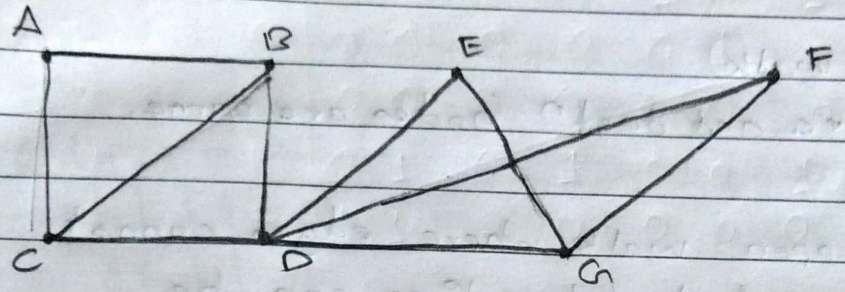


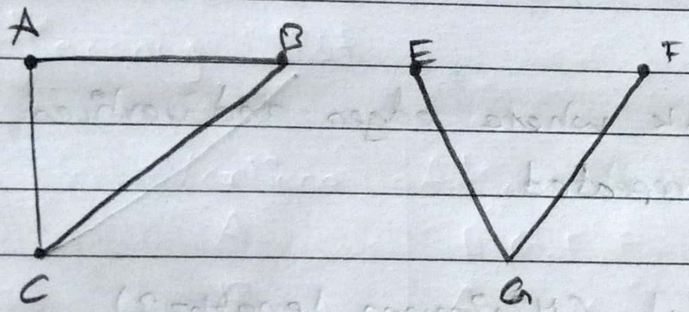
b)



Cut vertex and cut edge :-

A cut vertex is a single vertex in a connected graph where its removal from connected graph makes the graph disconnected.





∴ Here, removal of D makes the graph disconnected.

Cutedge :- (Bridge):- An edge is said to be bridge in a connected graph where its removal from the connected graph makes the graph disconnected.

Walk :- An alternating sequences of vertices & edges is said as a walk.

1) Open walk
   ↳ source & destⁿ nodes are different

↳ closed walk
   ↳ src and destⁿ nodes are same.

open walk

Trail :- An open walk where edges cannot be repeated but vertices can be repeated

Path :- A open walk where edges and vertices cannot be repeated.

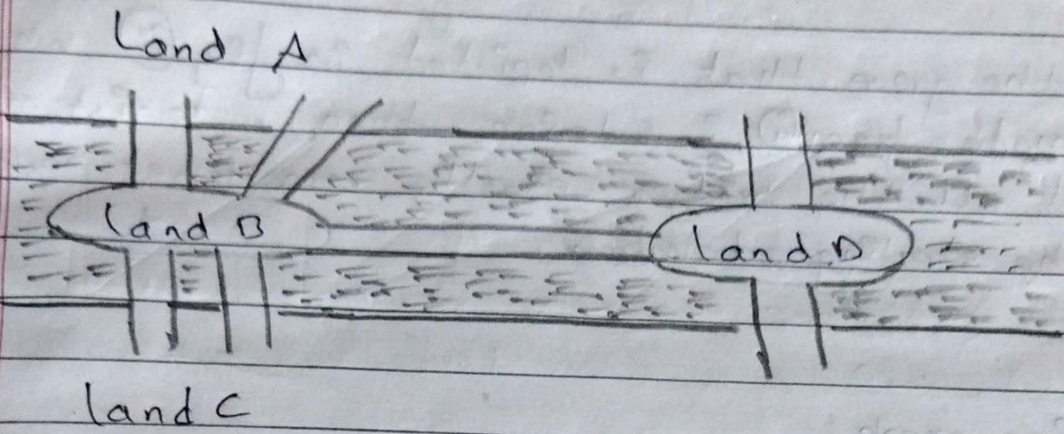
closed walk

Circuit :- A closed trial. (Minimum length=3)

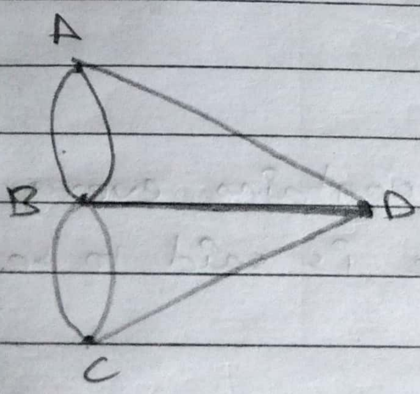
Cycle :- A closed path (Minimum length=3)

Euler Trail and Euler Circuit

The konisgberg bridge problem :-



i



ii

Q. Is it possible for a tourist to travel all the bridges exactly once & return to the initial print?

=> No

=> If the group that is depicted in fig (ii) which is made from (i) is Eulerian then a tourist can travel all the bridges exactly once & get back to the starting location

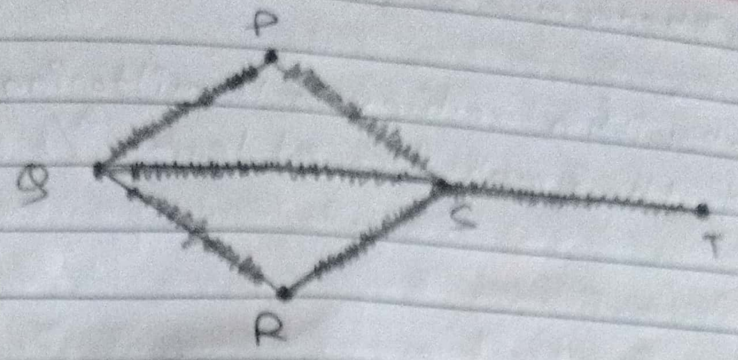
Eulerian graph :-

A graph $G = (V, E)$ is said to be Eulerian if it contains Euler circuit

Euler circuit

A closed walk which contains every edge of the graph exactly once is said to be Euler circuit

Example :-



q - p - s - q - r - s - t

Hamiltonian Graph :

A graph G = (V, E) is said to be Hamiltonian Graph if it contains hamiltonian cycle.

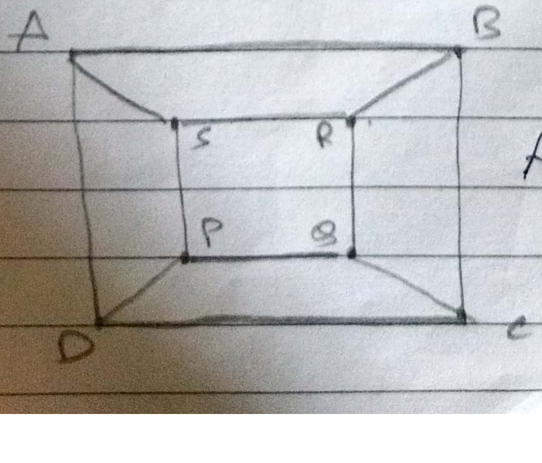
Hamiltonian cycle

↳ A closed walk with length greater than 3, which contains all the vertices of the graph exactly once except the first and the last

Hamiltonian path

↳ An open walk which contains all the vertices of the graph exactly once.

Example:



A - S - R - B - C - Q - P - D - A

Dirac's Theorem

An graph with n vertices in Hamiltonian if every vertex of the graph has at least $n/2$ degree.

Ore's theorem

A graph is hamiltonian if for any two non-adjacent vertices U & V, $deg(u) + deg(v) \geq n$

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Theorem 1: A graph is Eulerian if it contains even degree of vertices only

(A connected multigraph with at least two vertices has an euler circuit if and only if each of its vertices has an even degree).

→ It has ~~an~~ is that every time a circuit passes through a vertex, it adds twice of its degree. Since it is a circuit, it starts and ends at the same ~~to~~ vertex, which makes it contribute one degree when the circuit starts and one when it ends. In this way, every vertex has an even degree.

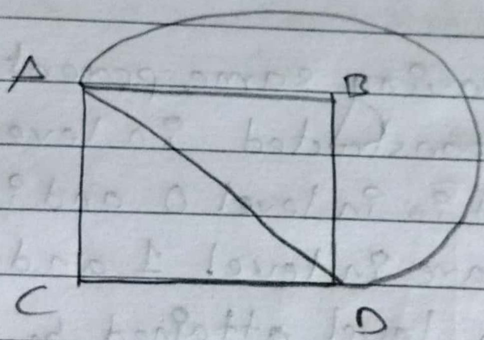
Since, the konisberg graph has vertices having ~~ad~~. odd degrees, a Euler circuit doesnot exist in the graph.

Theorem 2 : A graph contains Euler trail if only two its vertices are odd.

(A connected multigraph and simple graph) has an euler path but not an Euler circuit if only if it has exactly two vertices of odd degree)

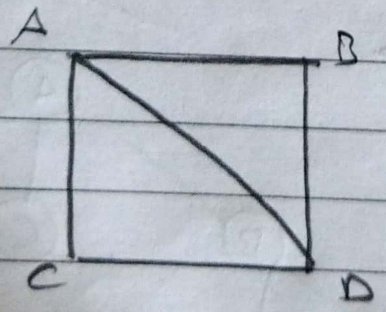
→ Since a path may start and end at different vertices, the vertices where the path starts and ends are allowed to have odd degrees

Example



G_1

G_1 has four vertices all of even degree, so it has euler circuit. The circuit is (a, d, b, a, c, d, a



G_2

G_2 has two vertices of odd degree a and d and the rest of them have even degree. So, this graph has an euler path but not an euler circuit.

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