Counting

Introduction

Combinatorics is the study of arrangements of objects. **Enumeration**, the counting of objects with certain properties, is an important part of combinatorics. We must count objects to solve many different types of problems. For example counting is used to determine the complexity of algorithms, to determine whether there are enough telephone numbers or Internet protocol addresses to meet demand etc.

Basics of Counting

There are two basic counting principles that can be used to solve the counting problems. These two basic counting principles are discussed below:

Sum Rule

If a task can be done in n_1 ways and a second task in n_2 ways, and if these tasks cannot be done at the same time, then there are $n_1 + n_2$ ways to do one of these tasks.

The sum rule can be phrased in terms of sets: If $A_1, A_2, \ldots, A_m$ are disjoint sets, then the number of elements in the union of these sets is the sum of the numbers of elements in them, that is,

$|A_1 \cup A_2 \cup \ldots \cup A_m| = |A_1| + |A_2| + \ldots + |A_m|$

Remember: This equality applies only when the sets are disjoint. For overlapping set we use different principle called inclusion exclusion principle.

Example 1: In how many ways we can draw a heart or a diamond from an ordinary deck of playing cards?
Solution: There are total 13 cards of heart and 13 card of diamond. So, by sum rule total number of ways of picking heart or diamond is 13 + 13 = 26.

Example 2: How many ways we can get a sum of 4 or of 8 when two distinguishable dice (say one die is red and the other is white) are rolled?
Solution: Since dice are distinguishable outcome (1, 3) is different form (3, 1) so to get 4 as sum we have the pairs (1, 3), (3, 1), (2, 2), so total of 3 ways. And similarly getting 8 can be from pairs (2, 6), (6, 2), (3, 5), (5, 3), (4, 4), so total 5 ways. Hence getting sum of 4 or 8 is 3 + 5 = 8.

Exercise 1: Suppose that either a member of the computer faculty or a student who is a computer major is chosen as a representative to a university committee. How many different choices are there for this representative if there are 37 members of the computer faculty and 83 computer majors?

Exercise 2: A student can choose a computer project from one of three lists. The three lists contain 23, 15, and 19 possible projects, respectively. How many possible projects are there to choose from?

Product Rule

If a task can be done in n_1 ways and a second task in n_2 ways after the first task has been done, then there are n_1n_2 ways to do the work that consists both the task.

The product rule is often phrased in terms of sets in this way: If $A_1, A_2, \ldots, A_m$ are finite sets, then the number of elements in the Cartesian product of these sets is the product of the numbers of elements in each set, that is,

$|A_1 \times A_2 \times \ldots \times A_m| = |A_1|.|A_2| \ldots\ldots |A_m|$

Example 1: An office building contains 27 floors and has 37 offices on each floor. How many offices are there are in the building?
Solution: By the product rule there are 27.37 = 999 offices in the building.

Example 2: How many different three-letter initials with none of the letters can be repeated can people have?
Solution: Here the first letter can be chosen in 26 ways, since the first letter is assigned we can choose second letter in 25 ways and in the same manner we can choose third letter in 24 ways. So by product rule number of different three-letter initials are 26.25.24 = 15600.

Example 3: How many strings are there of four lowercase letters that have the letter x in them?
Solution: There are total 26.26.26.26 strings of four lowercase letters, by product rule. In the same way we can say that there are 25.25.25.25 strings of four lowercase letters without x, since without x there will be a set of 25 characters only. So there are total of 26.26.26.26 - 25.25.25.25 = 66351 four lowercase letter strings with x in them. This is true because we are decrementing total numbers of strings with the number of strings that do not contain x in them so at least one x will be in the strings.

Exercise 1: The chairs of an auditorium are to be labeled with a letter and a positive integer not exceeding 100. What is the largest number of chairs that can be numbered?

Exercise 2: How many different license plates are available if each plate contains sequence of three letters followed by three digits?

The Inclusion-Exclusion Principle

When tow tasks can be done at the same time, we cannot use the sum rule to count the number of ways to do one of the two tasks. Adding the number of ways to do each task leads to an over count, since the ways to do both tasks are counted twice. To correctly count the number of ways to do one of the two tasks, we add the number of ways to do each of the two tasks and then subtract the number of ways to do both tasks. This technique is called **principle of inclusion-exclusion**.

This counting principle can be phrased in terms of sets. Let A_1 and A_2 be sets and let T_1 be the task of choosing an element from A_1 and T_2 the task of choosing an element from A_2. There are $|A_1|$ ways to do T_1 and $|A_2|$ ways to do T_2. The number of ways to either T_1 or T_2 is the sum of the number of ways to do T_1 and the number of ways to do T_2, minus the number of ways to do both T_1 and T_2. Since, there are $|A_1 \cup A_2|$ ways to do either T_1 or T_2 and $|A_1 \cap A_2|$ ways to do both T_1 and T_2, we have $|A_1 \cup A_2| = |A_1| + |A_2| - |A_1 \cap A_2|$.

Example 1: How many bit strings of length eight either start with a 1 bit or end with the two bits 00?
Solution: The first task, constructing a bit string of length eight beginning with a 1 bit, can be done in $2^7 = 128$ ways (using product rule).

The second task, constructing a bit string of length eight ending with the two bits 00, can be done in $2^6 = 64$ ways (using product rule).

Both tasks, constructing a bit string of length eight that begins with a 1 and ends with 00 can be done in $2^5 = 32$ ways (using product rule).

Hence, the number of bit strings of length eight that either start with a 1 bit or end with the two bits 00 is 128 + 64 – 32 = 160.

Pigeonhole Principle

The pigeonhole principle states that if there are more pigeons than pigeonholes, then there must be at least one pigeonhole with at least two pigeons. The concept of pigeons can be extended to any objects.

Theorem 1: The pigeonhole principle: If k + 1 or more objects are placed into k boxes, then there is at least one box containing two or more of the objects.
Proof: Here, We use proof by contradiction. Suppose that k+1 or more boxes are placed into k boxes and no boxes contain more than one object in it. If there are k boxes then there must be k objects such that there are no two objects in a box. This contradicts our assumption. So there is at least one box containing two or more of the objects.

Example 1: Show that if there are 30 students in a class, then at least two have last names that begin with the same letter.
Proof: There are 30 students in the class and we have 26 letters in English alphabet that can be used in beginning of the last name. Since there are only 26 letters and 30 students, by pigeonhole principle at least two students have the last name that begins with the same letter.

Example 2: How many students must be in a class to guarantee that at least two students receive the same score on the final exam, if the exam is graded on a scale from 0 to 100 points?
Proof: There are 101 possible scores on the final. The pigeonhole principle shows that among any 102 students there must be at least 2 students with the same score.

Theorem 2: The generalized pigeonhole principle: If N objects are placed into k boxes, then there is at least one box containing at least $\lceil N/k \rceil$ objects.

Proof: Suppose N objects are placed into k boxes and there is no box containing more than $\lceil N/k \rceil$-1 objects. So the total number of objects is at most k($\lceil N/k \rceil$-1) < k((N/k + 1) –1) = N. This is the contradiction that N objects are placed into k boxes (since we showed that there is total number of objects less than N).

Example 1: Find the minimum number of people among 100 people who were born in the same month.
Solution: Among 100 people there are at least $\lceil 100/12 \rceil$ = 9 who were born in the same month.

Example 2: If a class has 24 students, what is the maximum number of possible grading that must be done to ensure that there at least two students with the same grade.
Solution: There are total 24 students and the class and at least two students must have same grade. If the number of possible grades is k then by pigeonhole principle we have $\lceil 24/k \rceil$ = 2. Here the largest value that k can have is 23 since 24 = 23.1 + 1. So the maximum number of possible grading to ensure that at least two of the students have same grading is 23.

Example 3: What is the minimum number of students required in a discrete mathematics class to be sure that at least six will receive the same grade, if there are five possible grades A, B, C, D, and F?
Solution: The minimum number of students needed to ensure that at least six students receive the same grade is the smallest integer N such that $\lceil N/5 \rceil$ = 6. The smallest such integer is N = 5.5 + 1 = 26. If you have only 25 students, it is possible for there to be five who have received each grade so that no six students have received the same grade.

Thus, 26 is the minimum number of students needed to ensure that at least six students will receive the same grade.

Permutations and Combinations

Here, we will study the techniques for counting the unordered selections of objects and the ordered arrangement of objects of a finite set.

Permutations

A permutation of a set of distinct objects is an ordered arrangement of these objects. We are also interested in ordered arrangement of some of the elements of a set. An ordered arrangement of **r** elements of a set is called an **r-permutation**. The number of *r-permutations* of a set with n elements is denoted by $P(n, r)$. We can find $P(n, r)$ using the product rule.

Theorem: The number of *r-permutations* of a set with n distinct elements is

$$P(n, r) = n(n - 1)(n - 2)\dots(n - r + 1)$$

Proof: The first element of the permutation can be chosen in n ways, since there are n elements in the set. There are $n - 1$ ways to choose the second element of the permutation, since there are $n - 1$ elements left in the set after using the element picked for the first position. Similarly, there are $n - 2$ ways to choose the third element, and so on, until there are exactly $n - (r - 1) = n - r + 1$ ways to choose the r^{th} element.

Hence, by the product rule, there are $(n - 1)(n - 2)\dots(n - r + 1)$ *r-permutations* of the set. Therefore, $P(n, r) = n(n - 1)(n - 2)\dots(n - r + 1) = n! / (n-r)!$
In particular, note that, $P(n, n) = n!$

Example 1: How many ways are there to select a first-prizewinner, a second prize winner, and a third prize winner from 100 different people who have entered a contest?
Solution: The number of ways to pick the three prize winners is the number of ordered selections of three elements from a set of 100 elements, that is, the number of 3 – permutations of a set of 100 elements. Consequently, the answer is
P(100, 3) = 100.99.98 = 970,200.

Example 2: Suppose that there are eight runners in a race. The winner receives a gold medal, the second-place finisher receives a silver medal, and the third-place finisher receives a bronze medal. How many different ways are there to award these medals, if all possible outcomes of the race can occur and there are no ties?
Solution: The number of different ways to award the medals is the number of 3-permutations of the set with eight elements. Hence, there are P(8, 3) = 8.7.6 = 336 possible ways to award the medals.

Example 3: Suppose that a saleswoman has to visit eight different cities. She must begin her trip in a specified city, but she can visit other seven cities in any order she wishes. How many possible orders can the saleswoman use when visiting these cities?
Solution: The number of possible paths between the cities is the number of permutations of seven elements, since the first city is determined, but the remaining seven can be ordered arbitrarily. Hence, there are 7! = 7.6.5.4.3.2.1 = 5040 ways for the saleswoman to choose her tour.

Example 4: How many permutations of the letters ABCDEFGH contain the string ABC?
Solution: Because the letters ABC must occur as a block, we can find the answer by finding the number of permutations of six objects, namely, the block ABC and the individual letters D, E, F, G, and H. because these six objects can occur in any order,

there are 6! = 720 permutations of the letters ABCDEFGH in which ABC occurs as a block.

Combinations

An **r-combination** of elements of a set is an unordered selection of *r* elements from the set. Thus, an *r-combination* is simply a subset of the set with r elements.

The number of *r-combination* of a set with *n* distinct elements is denoted by C(n, r). C(n, r) is also denoted by $\binom{n}{r}$ and is called **binomial coefficient**. We can determine the number of *r-combinations* of a set with *n* elements using the formula for the number of *r-permutations* of a set.

Example: What is the value of *2-combinations* of the set {a, b, c, d}?
Solution: Since there are six subsets *{a, b}, {a, c}, {a, d}, {b, c}, {b, d},* and *{c, d},* the value of 2-combinations *C(4, 2) = 6*.

Theorem: The number of *r-combinations* of a set with *n* elements, where n is a nonnegative integer and r is an integer with $0 \leq r \leq n$, equals

$$C(n, r) = n! / r!(n - r)!$$

Proof: The *r-permutations* of the set can be obtained by forming *r-combinations, C(n, r)* of the set, and then ordering the elements in each r-combination, which can be done in *P(r, r)* ways. Hence,

$P(n, r) = C(n, r).P(r, r)$

$C(n, r) = P(n, r) / P(r, r) = n! / r!(n - r)!$

Corollary: Let n and r be nonnegative integers with $r \leq n$. Then, C(n, r) = C(n, n – r).
Proof: Form the above theorem, it follows that

$C(n, r) = n! / r!(n - r)!$

and

$C(n, n - r) = n! / (n - r)!\ [n - (n - r)]! = n! / (n - r)!\ r!$

Hence, *C(n, r) = C(n, n – r).*

Example 1: How many ways to select five players from a 10-member tennis team to make a trip to a match at another school?
Solution: The number of ways to select five players from a 10-member tennis team is C(10, 5) = 10! / 5!(10 – 5)! = 10! / 5!5! = 252.

Example 2: How many ways are there to select a committee to develop a discrete mathematics course at a school if the committee is to consist of three faculty members from the mathematics department and four from the computer science department, if there are 9 faculty members of the mathematics department and 11 of the computer science department?
Solution: Here we can use product rule. Hence, the number of ways to select the committee is
$C(9, 3).C(11, 4) = 9! / 3!6!\ .\ 11! / 4!7! = 84.330 = 27{,}720.$

Binomial Coefficients

The number of *r-combinations* from a set with *n* elements is often denoted by$\binom{n}{r}$. This number is also called a *binomial coefficient* because these numbers occur as coefficients in the expansion of powers of *binomial expressions* such as $(a + b)^n$.

The Binomial Theorem

The binomial theorem gives the coefficients of the expansion of the expansion of powers of binomial expression. A binomial expression is simply the sum of two terms, such as $x + y$.

Theorem: Let *x* and *y* be variables, and let *n* be a nonnegative integer. Then,

$$(x+y)^n = \sum_{j=0}^{n} \binom{n}{j} x^{n-j} y^j$$

$$= \binom{n}{0} x^n + \binom{n}{1} x^{n-1} y + \binom{n}{2} x^{n-2} y^2 + \cdots + \binom{n}{n-1} x y^{n-1} + \binom{n}{n} y^n$$

Example 1: What is the expansion of $(x + y)^4$?

Solution: From the binomial theorem, it follows that

$$(x+y)^4 = \sum_{j=0}^{4} \binom{4}{j} x^{4-j} y^j$$

$$= \binom{4}{0} x^4 + \binom{4}{1} x^3 y + \binom{4}{2} x^2 y^2 + \binom{4}{3} x y^3 + \binom{4}{4} y^4$$

$$= x^4 + 4x^3y + 6x^2y^2 + 4xy^3 + y^4$$

Example 2: What is the coefficient of $x^{12}y^{13}$ in the expansion of $(x + y)^{25}$?

Solution: From the binomial theorem, it follows that this coefficient is

$$\binom{25}{13} = \frac{25!}{13!\,12!} = 5{,}200{,}300$$

Example 3: What is the coefficient of $x^{12}y^{13}$ in the expansion of $(2x - 3y)^{25}$?

Solution: From the binomial theorem, it follows that this coefficient is

$$\binom{25}{13} 2^{12} (-3)^{13} = -\frac{25!}{13!\,12!} 2^{12} 3^{13}$$

Corollary 1: Let n be a nonnegative integer. Then

$$\sum_{k=0}^{n} \binom{n}{k} = 2^n$$

Proof: Using the binomial theorem with x = 1 and y = 1, we see that

$$2^n = (1+1)^n = \sum_{k=0}^{n} \binom{n}{k} 1^k 1^{n-k} = \sum_{k=0}^{n} \binom{n}{k}$$

Corollary 2: Let n be a nonnegative integer. Then

$$\sum_{k=0}^{n} (-1)^k \binom{n}{k} = 0$$

Proof: By binomial theorem it follows that

$$0 = 0^n = ((-1)+1)^n = \sum_{k=0}^{n} \binom{n}{k} (-1)^k 1^{n-k} = \sum_{k=0}^{n} \binom{n}{k} (-1)^k$$

Corollary 3: Let n be a nonnegative integer. Then

$$\sum_{k=0}^{n}(2)^k\binom{n}{k} = 3^n$$

Proof: By binomial theorem it follows that

$$(1+2)^n = \sum_{k=0}^{n}\binom{n}{k}1^{n-k}2^k = \sum_{k=0}^{n}\binom{n}{k}2^k$$

Advanced Counting Techniques

Recurrence Relations

Some counting problems cannot be solved using the methods we have learnt before. One of the ways of solving counting problems is by finding relationships, called **recurrence relation**, between the terms of a sequence. When we represent some problem using recursive definition, we specify some **initial condition** and the **recursive condition**. We use such definition to solve the relation called recurrence relation.

A **recurrence relation** for the sequence $\{a_n\}$ is an equation that expresses a_n in terms of one or more of the previous terms of the sequence, namely, $a_0, a_1, \ldots, a_{n-1}$, for all integers n with $n \geq n_0$, where n_0 is a nonnegative integer. A sequence is called a solution of a recurrence relation if its term satisfies the recurrence relation.

Example: Let $\{a_n\}$ be a sequence that satisfies the recurrence relation $a_n = a_n + 1$ for $n = 2, 3, \ldots$, and suppose that $a_1 = 1$. What is the sequence?
Solution: We have $a_1 = 1$, $a_2 = a_1 + 1 = 1 + 1 = 2$, $a_3 = a_2 + 1 = 2 + 1 = 3$, and so on. Hence, the sequence is 1, 2, 3, 4, 5,…….

Example: For $a_n = -2a_{n-1}$ and $a_0 = -1$ find a_1, a_2, a_3, a_4, and a_5.
Solution:
We have $a_0 = -1$
$a_1 = -2a_0 = -2.-1 = 2.$
$a_2 = -2a_1 = -2.2 = -4.$
$a_3 = -2a_2 = -2.-4 = 8.$
$a_4 = -2a_3 = -2.8 = -16.$
$a_5 = -2a_4 = -2.-16 = 32.$

Solving Recurrence Relations

We encounter different types of recurrence relations. There is no specific technique to solve all the recurrence relation. However, we solve recurrence relation with some particular forms by using the systematic methods. In this section we are going to see few of them.

Linear Homogeneous Recurrence Relation of Degree k with Constant Coefficients

A linear homogeneous recurrence relation of degree k with constant coefficients is a recurrence relation of the form $a_n = c_1a_{n-1} + c_2a_{n-2} + \ldots + c_ka_{n-k}$, where $c_1, c_2, \ldots, c_k$ are real numbers, and $c_k \neq 0$.

The above relation is **linear** since right hand side is the sum of the multiples of previous terms of the sequence. It is **homogeneous** because no term occurs without being multiple of some a_js. All the coefficients of the terms are **constants** because they do not depend on n. And, the **degree** of the relation is k because a_n is expressed in terms of previous k terms of the sequence.

Examples: The recurrence relation $P_n = (1.11)P_{n-1}$ is a linear homogeneous recurrence relation of degree one. The recurrence relation $f_n = f_{n-1} + f_{n-2}$ is a linear homogeneous recurrence relation of degree two. The recurrence relation $a_n = a_{n-5}$ is a linear homogeneous recurrence relation of degree 5.

The recurrence relation $a_n = a_{n-1} + a^2_{n-2}$ is not linear. The recurrence relation $H_n = 2H_{n-1} + 1$ is not homogeneous. The recurrence relation $B_n = nB_{n-1}$ does not have constant coefficient.

Solving Linear Homogeneous Recurrence Relation of Degree k with Constant Coefficients

In solving the recurrence relation of this type, the approach is to look for the solution of the form $a_n = r^n$, where r is a constant. $a_n = r^n$ is a solution of a recurrence relation $a_n = c_1a_{n-1} + c_2a_{n-2} + \ldots + c_ka_{n-k}$ if and only if $r^n = c_1r^{n-1} + c_2r^{n-2} + \ldots + c_kr^{n-k}$. When we divide both sides by r^{n-k} and transpose the right hand side we have $r^k - c_1r^{k-1} - c_2r^{k-2} - \ldots - c_k = 0$. Here we can say $a_n = r^n$ is a solution if and only if r is the solution if the equation $r^k - c_1r^{k-1} - c_2r^{k-2} - \ldots - c_k = 0$ (**characteristic equation** of the recurrence relation) and solutions to this equations are called **characteristic roots** of the recurrence relation.

Theorem 1: (without proof)
Let c_1 and c_2 be real numbers. Suppose that $r^2 - c_1r - c_2 = 0$ has two distinct roots r_1 and r_2. Then the sequence $\{a_n\}$ is a solution of the recurrence relation $a_n = c_1a_{n-1} + c_2a_{n-2}$ if and only if $a_n = \alpha_1r_1^n + \alpha_2r_2^n$ for n = 0, 1, 2, …, where α_1 and α_2 are constants.

Example: Solve the recurrence relation $a_n = a_{n-1} + 6a_{n-2}$ for $n \geq 2$, $a_0 = 3$, $a_1 = 6$.
Solution: Characteristic equation of the given relation is $r^2 - r - 6 = 0$. Its roots are r = 3 and r = -2 since (r - 3)(r + 2) = 0. Hence, the sequence $\{a_n\}$ is a solution to the recurrence relation if and only if $a_n = \alpha_1 3^n + \alpha_2(-2)^n$, for some constants α_1 and α_2. From the initial conditions we have $a_0 = 3 = \alpha_1 + \alpha_2$, $a_1 = 6 = 3\alpha_1 + (-2)\alpha_2$. Solving these two equations we have $\alpha_1 = 12/5$ and $\alpha_1 = 3/5$. Hence, the solution is the sequence $\{a_n\}$ with $a_n = (12.3^n + 3(-2)^n)/5$.

Exercise: What is the solution of the recurrence relation $a_n = a_{n-1} + 2a_{n-2}$ with $a_0 = 2$ and $a_1 = 7$.

Exercise: Find the explicit formula for the Fibonacci numbers. [Use $f_n = f_{n-1}+f_{n-2}$ as recursive condition and $f_0 = 0$ and $f_1 = 1$ as initial condition]

Theorem 2: (without proof)
Let c_1 and c_2 be real numbers with $c_2 \neq 0$. Suppose that $r^2 - c_1r - c_2 = 0$ has only one root r_0. Then the sequence $\{a_n\}$ is a solution of the recurrence relation $a_n = c_1a_{n-1} + c_2a_{n-2}$ if and only if $a_n = \alpha_1r_0^n + \alpha_2nr_0^n$ for n = 0, 1, 2,…, where α_1 and α_2 are constants.

Example: Solve the recurrence relation $a_n = 2a_{n-1} - a_{n-2}$ for $n \geq 2$, $a_0 = 3$, $a_1 = 6$.
Solution: Characteristic equation of the given relation is $r^2 - 2r + 1 = 0$. Its only root is r = 1. Hence, the sequence $\{a_n\}$ is a solution to the recurrence relation if and only if $a_n = \alpha_1 1^n + \alpha_2 n1^n$, for some constants α_1 and α_2. From the initial conditions we have $a_0 = 3 = \alpha_1$ and $a_1 = 6 = \alpha_1 + \alpha_2$. Solving these two equations we have $\alpha_1 = 3$ and $\alpha_2 = 3$. Hence, the solution is the sequence $\{a_n\}$ with $a_n = 3(1^n + n1^n) = 3(1 + n)$.

Exercise: What is the solution of the recurrence relation $a_n = 6a_{n-1} – 9a_{n-2}$ with initial conditions $a_0 = 1$ and $a_1 = 6$?

Theorem 3: (without proof)
Let $c_1, c_2, \ldots, c_k$ be real numbers. Suppose that $r^k - c_1r^{k-1} - \ldots - c_k = 0$ has k distinct roots $r_1, r_2, \ldots, r_k$. Then the sequence $\{a_n\}$ is a solution of the recurrence relation $a_n = c_1a_{n-1} + c_2a_{n-2} + \ldots + c_ka_{n-k}$ if and only if $a_n = \alpha_1r_1^n + \alpha_2r_2^n + \ldots + \alpha_kr_k^n$ for n = 0, 1, 2, …, where $\alpha_1, \alpha_2, \ldots, \alpha_k$ are constants.

Example: Solve the recurrence relation $a_n = 2a_{n-1} + a_{n-2} - 2a_{n-3}$ for $n \geq 3$, $a_0 = 3$, $a_1 = 6$ and $a_2 = 9$.
Solution: Characteristic equation of the given relation is $r^3 - 2r^2 - r + 2 = 0$. Its roots are $r = 1$, $r = -1$, and $r = 2$. Hence, the sequence $\{a_n\}$ is a solution to the recurrence relation if and only if $a_n = \alpha_1 1^n + \alpha_2(-1)^n + \alpha_3 2^n$, for some constants α_1, α_2, and α_3. From the initial conditions we have $a_0 = 3 = \alpha_1 + \alpha_2 + \alpha_3$, $a_1 = 6 = \alpha_1 - \alpha_2 + 2\alpha_3$, and $a_2 = 9 = \alpha_1 + \alpha_2 + 4\alpha_3$. Solving these equations we have $\alpha_1 = 3/2$, $\alpha_2 = -1/2$, and $\alpha_3 = 2$. Hence, the solution is the sequence $\{a_n\}$ with $a_n = (3/2)1^n – (1/2)(-1)^n + 2.2^n$.

Exercise: Find the solution to the recurrence relation $a_n = 6a_{n-1} – 11a_{n-2} + 6a_{n-3}$ with the initial conditions $a_0 = 2$, $a_1 = 5$, and $a_2 = 15$.

Theorem 4: (without proof)
Let $c_1, c_2, \ldots, c_k$ be real numbers. Suppose that $r^k - c_1r^{k-1} - \ldots - c_k = 0$ has t distinct roots $r_1, r_2, \ldots, r_t$ with multiplicity $m_1, m_2, \ldots, m_t$, respectively, so that $m_i \geq 1$ for $i = 1, 2, \ldots, t$ and $m_1 + m_2 + \ldots + m_t = k$. Then the sequence $\{a_n\}$ is a solution of the recurrence relation $a_n = c_1a_{n-1} + c_2a_{n-2} + \ldots + c_ka_{n-k}$ if and only if

$$a_n = (\alpha_{1,0} + \alpha_{1,1}n + \ldots + \alpha_{1,m1-1}n^{m1-1})\, r_1^n$$

$$+ (\alpha_{2,0} + \alpha_{2,1}n + \ldots + \alpha_{2,m2-1}n^{m2-1})\, r_2^n$$

$$+ \ldots + (\alpha_{t,0} + \alpha_{t,1}n + \ldots + \alpha_{t,mt-1}n^{mt-1})\, r_t^n$$

for $n = 0, 1, 2, \ldots$, where $\alpha_{i,j}$ are constants for $1 \leq i \leq t$ and $0 \leq j \leq m_i-1$.

Example: Solve the recurrence relation $a_n = 5a_{n-1} - 7a_{n-2} + 3a_{n-3}$ for $n \geq 3$, $a_0 = 1$, $a_1 = 9$ and $a_2 = 15$.
Solution: Characteristic equation of the given relation is $r^3 - 5r^2 + 7r - 3 = 0$. Its roots are $r = 1$, $r = 3$, and $r = 1$. i.e. $r_1 = 1$, $m_1 = 2$ and $r_2 = 3$, $m_2 = 1$ Hence, the sequence $\{a_n\}$ is a solution to the recurrence relation if and only if $a_n = (\alpha_{1,0} + \alpha_{1,1}n)\, 1^n + (\alpha_{2,0})\, 3^n$, for some constants $\alpha_{1,0}$, $\alpha_{1,1}$, and $\alpha_{2,0}$. From the initial conditions we have $a_0 = 5 = \alpha_{1,0} + \alpha_{2,0}$, $a_1 = 9 = \alpha_{1,0} + \alpha_{1,1} + 3\alpha_{2,0}$, and $a_2 = 15 = \alpha_{1,0} + 2\alpha_{1,1} + 9\alpha_{2,0}$. Solving these equations we have $\alpha_{1,0} = 3/2$, $\alpha_{1,1} = 9$, and $\alpha_3 = -1/2$. Hence, the solution is the sequence $\{a_n\}$ with $a_n = (3/2)1^n + 9n1^n - (1/2)3^n$.

Exercise: Find the solution to the recurrence relation an $= -3a_{n-1} – 3a_{n-2} – a_{n-3}$ with initial conditions $a_0 = 1$, $a_1 = -2$ and $a_2 = -1$.

Solving Linear Nonhomogeneous Recurrence Relation of Degree k with Constant Coefficients

The recurrence relation of the form $a_n = c_1a_{n-1} + c_2a_{n-2} + \ldots + c_ka_{n-k} + F(n)$, where $c_1, c_2, \ldots, c_k$ are real numbers and F(n) is a function depending upon n. The recurrence relation preceding F(n) is called **associated homogeneous recurrence relation**. For example, $a_n = 7a_{n-1} + 3a_{n-2} + 6n$ is a linear nonhomogeneous recurrence relation with constant coefficients.

Examples: Each of the recurrence relations $a_n = 3a_{n-1} + 2n$, $a_n = a_{n-1} + 2^n$, $a_n = a_{n-1} + a_{n-2} + n^2 + n + 1$, $a_n = 3a_{n-1} + n3^n$, and $a_n = a_{n-1} + a_{n-2} + a_{n-3} + n!$ is a linear nonhomogeneous recurrence relation with constant coefficients. The associated linear homogeneous recurrence relations are $a_n = 3a_{n-1}$, $a_n = a_{n-1}$, $a_n = a_{n-1} + a_{n-2}$, $a_n = 3a_{n-1}$ and $a_n = a_{n-1} + a_{n-2} + a_{n-3}$, respectively.

Theorem 5: (without proof)
If $\{a_n^{(p)}\}$ is a particular solution of the nonhomogeneous linear recurrence relation with constant coefficients $a_n = c_1a_{n-1} + c_2a_{n-2} + \ldots + c_ka_{n-k} + F(n)$, then every solution of the form $\{a_n^{(p)} + a_n^{(h)}\}$, where $a_n^{(h)}$ is a solution of the associated homogeneous recurrence relation $a_n = c_1a_{n-1} + c_2a_{n-2} + \ldots + c_ka_{n-k}$.

Example: Find all the solutions of the recurrence relation $a_n = 4a_{n-1} + n^2$. Also find the solution of the relation with initial condition $a_1 = 1$.
Solution: We have associated linear homogeneous recurrence relation as $a_n = 4a_{n-1}$. The root is 4, so the solutions are $a_n^{(h)} = \alpha 4^n$, where α is a constant.

Since $F(n) = n^2$ is a polynomial of degree 2, a trial solution is a quadratic function in n, say, $p_n = an^2 + bn + c$, where a, b, and c are constants. To determine whether there are any solutions of this form, suppose that $p_n = an^2 + bn + c$ is such solution. Then the equation $a_n = 4a_{n-1} + n^2$ becomes

$$an^2 + bn + c = 4(a(n-1)^2 + b(n-1) + c) + n^2$$
$$= 4an^2 - 8an + 4a + 4bn - 4b + 4c + n^2$$
$$= (4a + 1)n^2 + (-8a + 4b)n + (4a - 4b + 4c)$$

Here $an^2 + bn + c$ is the solution if and only if $4a + 1 = a$ i.e. $a = -1/3$; $-8a + 4b = b$ i.e. $b = -8/9$; $4a - 4b + 4c = c$ i.e. $c = -20/27$. So $a_n^{(p)} = -(n^2 + 8n + 28)/3$ is a particular solution and all solutions are $a_n = \{a_n^{(p)} + a_n^{(h)}\} = -(n^2 + 8n/3 + 20/9)/3 + \alpha 4^n$,where α is a constant.

For solution with $a_1 = 1$, we have $a_1 = 1 = -(1 + 8/3 + 20/9)/3 + \alpha 4$ i.e. $\alpha = 20/27$. Then the solution is $a_n = - (n^2 + 8n/3 + 20/9)/3 + 20.4^n/27$.

Exercise: Find all solutions of the recurrence relation $a_n = 3a_{n-1} + 2n$. What is the solution with $a_1 = 3$.

Exercise: Find all solutions of the recurrence relation $a_n = 5a_{n-1} - 6a_{n-2} + 7^n$.

Theorem 6: (without proof)
Suppose that $\{a_n\}$ satisfies the linear nonhomogeneous recurrence relation $a_n = c_1a_{n-1} + c_2a_{n-2} + \ldots + c_ka_{n-k} + F(n)$, where $c_1, c_2, \ldots, c_k$ are real numbers and $F(n) = (b_tn^t + b_{t-1}n^{t-1} + \ldots + b_1n + b_0)s^n$, where $b_0, b_1, \ldots, b_t$ and s are real numbers.

When s is not a root of the characteristic equation of the associated linear homogeneous recurrence relation, there is a particular solution of the form $(p_tn^t + p_{t-1}n^{t-1} + \ldots + p_1n + p_0)s^n$.

When s is a root of the characteristic equation and its multiplicity is m, there is a particular solution of the form $n^m(p_tn^t + p_{t-1}n^{t-1} + \ldots + p_1n + p_0)s^n$.

Example: Find the solution of the recurrence relation $a_n = 2a_{n-1} + n.2^n$.
Solution: We have the associated linear homogeneous recurrence relation is $a_n = 2a_{n-1}$. The characteristic equation for this would be $r-2 = 0$, so the root is 2 and hence the solutions are $a_n^{(h)} = \alpha 2^n$, where α is a constant. We have $F(n) = n.2^n$. (Of the form $n.s^n$) where s is the root of the characteristic equation and the multiplicity of 2 is 1 so, the particular solution has the form $n.(p_1n + p_0)2^n = p_1n^22^n + p_0n2^n$. Hence, all solutions of the original recurrence relation are given by $a_n = \alpha 2^n + p_1n^22^n + p_0n2^n$.

Exercise: What form does a particular solution of the linear nonhomogenous recurrence relation $a_n = 6a_{n-1} - 9a_{n-2} + F(n)$ have when $F(n) = 3^n$, $F(n) = n3^n$, $F(n) = n^22^n$, and $F(n) = (n^2 + 1)3^n$?