Representing Relations

Matrix Representation

The relation with finite sets can be represented using the matrix (zero one matrix). Let A be a set (a_1, a_2, …a_n) and B be the set (b_1, b_2, …, b_m), where elements are listed in some arbitrary order we represent relation from A to B by matrix M_R = [m_{ij}], where

$$m_{ij} = \begin{cases} 1 \text{ if } (a_i, b_j) \in R. \\ 0 \text{ if } (a_i, b_j) \notin R. \end{cases}$$

Example:

Represent the relation {(1,1), (1,2), (1,3), (2,2), (2,3), (3,2), (3,3)} on the set {1,2,3) with matrix, where elements of the set is listed in increasing order.

Solution:

$$M_R = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix}$$

Identifying properties

Reflexive: If all the diagonal elements are 1 i.e. all m_{ij} = 1 whenever i = j, then the relation represented by the matrix is reflexive **(is above matrix reflexive? Yes)**.

Symmetric: If m_{ij} = 1 in the matrix then m_{ji} = 1 must be true and if m_{ij} = 0 then m_{ji} = 0 is also true. (it means that the relation represented by a matrix is symmetric if and only if the matrix is equal to its transpose). **(What about above matrix? No)**.

Antisymmetric: If m_{ij} = 1 and i ≠j, then m_{ji} = 0 or, in other words, either m_{ij} = 0 or m_{ji} = 0 when i ≠j. **(What about above matrix? No)**.

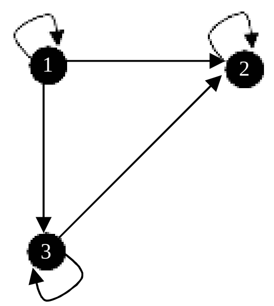
Directed Graph Representation

A directed graph, or digraph is a set of vertices V together with the set of edges. The vertex a is called initial vertex of the edge (a, b), and the vertex b is called the terminal vertex of this edge.

Example:

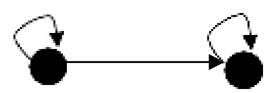
Draw the directed graph for the relation given in above example (example in matrix representation relation is {(1,1), (1,2), (1,3), (2,2), (2,3), (3,2), (3,3)}).

Solution:



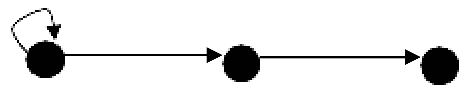
Identifying properties

Reflexive: If every vertex has edge from the vertex to itself.



Symmetric: If for every edge of one direction there is another edge in opposite direction joining same two vertices as of first edge.

Antisymmetric: If no two distinct vertices have an edge going in both directions.



Closures of Relations

Let R be a relation on a set A. R may or may not have some property P, like symmetry, reflexivity, etc. If there is a relation S with property P containing R such that S is a subset of every relation with property P containing R, then S is called the closure of R with respect to P.

Reflexive closure

For any relation R on A, reflexive closure of R is formed by adding to R all pairs of the form (a, a) with $a \in A$, not already in R i.e. $R \cup D$, where $D = \{(a, a) \mid a \in A\}$ is the diagonal relation on A.

Example:

Let R be the relation on the set {1, 2, 3, 4} containing the ordered pairs (1,2), (1,3), (2,2), (2, 4), (3,1), (3,2), (3, 4), and (4, 4) find reflexive closure.

Solution:

We have R = {(1,2), (1,3), (2,2), (2, 4), (3,1), (3,2), (3, 4), (4, 4)} and

D = {(1,1), (2,2), (3,3), (4,4)}

So, $R \cup D$ = {(1,1), (1,2), (1,3), (2,2), (2, 4), (3,1), (3,2), (3,3), (3, 4), (4, 4)} is the reflexive closure of R.

Symmetric closure

If all ordered pairs of the form (b, a) is added to the relation R on A, where (a, b) is in the relation, that are not already in R, then the newly formed set after addition is symmetric closure of R i.e. for the relation R on A $R \cup R^{-1}$ is symmetric closure of relation R, where $R^{-1} = \{(b, a) \mid (a, b) \in A\}$.

Example:

Find symmetric closure for the relation given in above example.

Solution:

We have R = {(1,2), (1,3), (2,2), (2, 4), (3,1), (3,2), (3, 4), (4, 4)} and

R^{-1} = {(2,1), (3,1), (4,2), (1,3), (2,3), (4,3)}

So, $R \cup R^{-1}$ = {(1,2), (1,3), (2,1), (2,2), (2,3), (2,4), (3,1), (3,2), (3,4), (4,2), (4,3), (4,4)} is the symmetric closure of R.

Equivalence Relations

Equivalence Relations

A relation R on set A is called equivalence relation if it satisfies the three properties namely, reflexive, symmetric, and transitive. The two elements related by equivalence relations are called equivalent.

Equivalence Classes

Given R, an equivalence relation on set A, the set of all elements that are related to an element a of A is called the equivalence class of a. The equivalence class of a with respect to R is denoted by $[a]_R$ or [a] when only one relation is in consideration. In notational term we can write $[a]_R = \{b \mid (a,b) \in R\}$. If $c \in [a]_R$, then c is called a representative of equivalence class $[a]_R$.

Equivalence Classes and Partitions

Given a set A, a partition of A is a collection P of disjoint subsets whose union is A i.e.

$\forall$ B $\in$ P, B $\subseteq$ A;

$\forall$ B,C $\in$ P, B $\cap$ C = ϕ, or B = C; and

$\forall$ x $\in$ A $\exists$ B $\in$ P such that x $\in$ B;

Example:

Define the relation R on the set A of positive integers by (a,b) $\in$ R iff a/b can be expressed in the form 2^m, where m is an arbitrary integer.

a) Show that R is an equivalence relation.
b) Determine the equivalence classes under R.

Solution:

a)

For all a$\in$ A, a/a = 1 =2^0, where m = 0 hence R is reflexive. If (a,b) $\in$ R then we have a/b = 2^m such that b/a = 2^{-m}. So we have (b,a) $\in$ R whenever (a,b) $\in$ R hence R is symmetric. Take (a,b) $\in$ R and (b, c) $\in$ R then we can write a/b = 2^m and b/c = 2^n so we have a/c = 2^{m+n} , where m+n is an arbitrary integer so that (a,c) $\in$ R hence R is transitive. For the facts above we showed that R is an equivalence relation.

b)

Some equivalence classes may be

[1] = {1,2,4,8,… …}

[3] = {3,6,12,24,48, … …}

[5] = {5,10,20,40, … …}

[7] = {7,14,28,56, … …}

… ……

Partial Orderings

A relation R on a set A is called **partial order** or partial ordering if it is reflexive, antisymmetric, and transitive. A set A together with the partial order R is called a partially ordered set, or a poset, denoted by (A, R).

Example:

Show that a relation ≤ "less than or equal" is partial order on the set of integers.

Solution:

We know $\forall a \in Z$, $a \leq a$, hence ≤ is reflexive. If $a \leq b$ and $b \leq a$, then $a = b$, hence ≤ is antisymmetric, and if $a \leq b$ and $b \leq c$, then $a \leq c$, hence ≤ is transitive. It follows that ≤ is a partial ordering on the set of integers Z and $(Z, \leq)$ is a poset.

In a poset (S, R), the elements a and b of a poset are comparable if either aRb or bRa. When neither aRb nor bRa, then a and b are incomparable. If every two elements of a set S are comparable, then S is called a **totally ordered** or **linearly ordered set** or a **chain**, and R is called **total order** or a linear order.

(S,R) is a **well ordered set** if it is a poset such that R is a total order and such that every nonempty subset of S has a least element.

Lexicographic order

Given two posets (A_1,L_1) and (A_2,L_2). The lexicographic ordering L on $A_1 \times A_2$ is defined by specifying that one pair is less than a second pair i.e. $(a_1, a_2)L(b_1, b_2)$ if $a_1L_1b_1$ or both $a_1 = b_1$ and $a_2L_2b_2$. Similarly we can extend this definition to Cartesian product or more than two sets.

Lexicographic Ordering of Strings

Suppose $a_1a_2 \ldots a_m$ and $b_1b_2 \ldots b_n$ are strings on a partially ordered set S and suppose two strings are not equal. If t is the minimum of m and n, then we define lexicographic ordering that the string $a_1a_2 \dots a_m$ is less than $b_1b_2 \dots b_n$ if and only if

$(a_1a_2 \dots a_t) < (b_1b_2 \dots b_t)$, or

$(a_1a_2 \dots a_t) = (b_1b_2 \dots b_t)$ and $m < n$.

Example #1:

Find the lexicographic ordering of the 3 tuples (1,1,2), (1,2,1)

Solution:

(1,1,2) < (1,2,1) because first elements of both the 3 tuples are same so if we look forward then the second element of first 3 tuple is less than the second element of second 3 tuple.

Example #2:

Find the lexicographic ordering of the strings *quack, quick, quicksilver, quicksand, quacking*.

Solution:

quack < *quick* [strings differ in third position and a < i]

quick < *quicksilver* [letter of the first string agrees but second string is longer]

quicksand < *quicksilver* [strings differ in seventh position and a < i]

quacking < *quicksand* [strings differ in third position and a < i]

Similarly others are

quack < *quacking, …..*

So the ordering is

quack < *quacking* < *quick* < *quicksand* < *quicksilver*

Hasse Diagrams

A partial ordering on a finite set can be represented using the pictorial notation as follows:

Construct the directed graph of a relation.

Remove all the loops (since it is clear that partial order is reflexive so every vertex has a loop)

Remove all the edges that is requires due to the transitivity

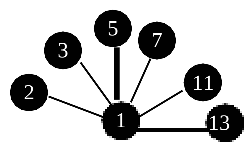
Arrange each edge so that its initial vertex is below its terminal vertex and remove all arrows from the edges since edges point in upward direction only.

The diagram formed using above steps contains sufficient information to find the partial ordering. This diagram is called **Hasse diagram**.

Example #1:

Draw the Hasse diagram for divisibility on the set {1,2,3,5,7,11,13}

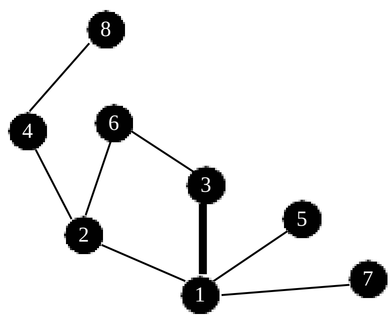
Solution:



Example #2:

Draw the Hasse diagram for divisibility on the set {1,2,3,4,5,6,7,8}

Solution:



Maximal and Minimal Elements

Some Definitions:

Maximal Elements: An element of a poset is maximal if it is not less than any elements of the poset i.e. a is maximal in the poset $(S,\leq)$ if there is no $b \in S$ such that $a < b$. In Hasse diagram maximal elements are top elements.

Minimal Elements: An element of a poset is minimal if it is not greater than any elements of the poset i.e. a is minimal in the poset $(S,\leq)$ if there is no $b \in S$ such that $b <$
a. In Hasse diagram minimal elements are bottom elements.

Greatest Element: An element in a poset that is greater than every other element i.e. a is the greatest element of the poset $(S,\leq)$ if $b \leq a$ for all $b \in S$. The greatest element may exist or may not exist in a poset.

Least Element: An element in a poset that is less than every other element i.e. a is the least element of the poset $(S,\leq)$ if $a \leq b$ for all $b \in S$. The least element may exist or may not exist in a poset.

Upper Bound: If u is an element of S such that $a \leq u$ for all elements $a \in A$, then u is called an upper bound of A, where $A \subseteq S$.

Lower Bound: If l is an element of S such that $l \leq a$ for all elements $a \in A$, then l is called a lower bound of A, where $A \subseteq S$.

Least Upper Bound: l_u is the least upper bound of the subset A if $a \leq l_u$ whenever $a \in A$ and $l_u \leq u$ whenever u is an upper bound of A.

Greatest Lower Bound: g_l is the greatest lower bound of the subset A if g_l is a lower bound and $l \leq g_l$ whenever l is a lower bound of A.

Example:

Answer these questions for the poset $(\{3,5,9,15,24,45\},|)$

a) Find the maximal elements
b) Find the minimal elements
c) Is there a greatest element?
d) Is there a least element?
e) Find all upper bounds of {3,5}
f) Find the least upper bound of {3,5} if exists
g) Find all lower bounds of {15,45}
h) Find the greatest lower bound of {15,45}, if exists.

Solution:

a) 24 and 45 are maximal elements.
b) 3 and 5 are minimal elements
c) No greatest element since there is no element in a set is divisible by all the elements in a set.
d) No least element since no element in a set divides all the elements in a set.
e) Since 15 and 45 are divisible by both 3 and 5 and they are follow the relation we have 15 and 45 as upper bounds of {3,5}

f) Since among the upper bounds 15 is the least, 15 is least upper bound.

g) The elements 3, 5 , and 15 divides the elements 15 and 45. Hence 3, 5, and 15 are lower bounds

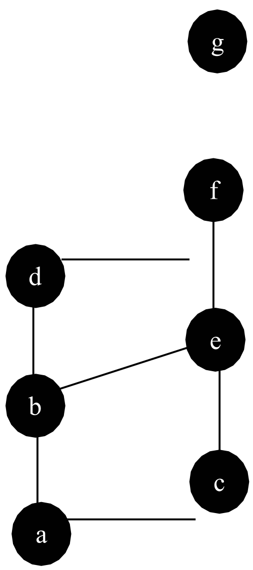
h) Amongst the lower bounds 15 is the greatest lower bound.

Lattices

A partially ordered set in which every pair of elements has both the least upper bound and the greatest lower bound is called **lattice**.

Example #1:

Identify whether the poset given by following Hasse diagram is lattice or not?



Solution:

Every pair of elements for e.g. (a,b), (a,c), (a,e), (b,e) …. Has least upper bound and greatest lower bound so the poset given by above Hasse diagram is a lattice.

Example #2:

Determine whether the poset $(Z, \geq)$ is a lattice

Solution:

Suppose x and y are two integers. If we have a relation $x \geq y$ for all $x,y \in Z$, then we can say that x is the least upper bound and y is the greatest lower bound. Conversely $y \geq x$ for all $x, y \in Z$, then we can say that y is the least upper bound and x is the greatest lower bound. These conditions hold for all elements in Z, hence poset $(Z, \geq)$ is a lattice.