Discrete Structure

Unit 3.1 Logic

Logic

- Logic is the basis of all mathematical reasoning.
- The rules of logic give precise meaning to mathematical statements so that we can apply these rules for mathematical reasoning
- Types
  – Propositional Logic
  – Predicate Logic

Proposition

- A **proposition** is a declarative sentence that is either true or false, but not both
- **<u>Example</u>**
  - Kathmandu is the capital of Nepal. ***TRUE***
  - $2 + 2 = 5$. ***FALSE***
  - $1 + 1 = 2$. ***TRUE***
  - $x > 5$. ***NOT A PROPOSITION***
  - What time is it? ***NOT A PROPOSITION***
  - Read this carefully. ***NOT A PROPOSITION***
  - $x + y = z$ ***NOT A PROPOSITION***
- Propositions are denoted conventionally by using small letters like ***p, q, r, s ….***
- The truth value of proposition is denoted by ***T*** for true proposition and ***F*** for false proposition

Logical Operators/Connectives

- Logical operators are used to construct mathematical statements having one or more propositions and the combined proposition is called **compound Proposition**
- **Truth table** is used to show the relationship between truth values of propositions
- Types of operators
  - Negation (NOT) :- ***denoted as*** $\neg$
    - Example :- let p = He passed the exam. Then $\neg$p=???
  - Conjunction (AND) :- ***denoted as*** $\wedge$
  - Disjunction (OR) :- ***denoted as*** $\vee$
  - Exclusive or (XOR) :- ***denoted as*** $\oplus$
  - Implication :- :- ***denoted as*** $\rightarrow$

p	Q	$p \vee q$
T	**T**	**T**
T	**F**	**T**
F	**T**	**T**
F	**F**	**F**

Implication

- Let ***p*** and ***q*** be propositions. The conditional statement ***p* → *q*** is the proposition "if p, then q." The conditional statement p → q is false when p is true and q is false, and true otherwise
- In the conditional statement p → q, p is called the hypothesis (or antecedent or premise) and q is called the conclusion (or consequence)
- also called an **implication**

p	q	p → q
T	**T**	**T**
T	**F**	**F**
F	**T**	**T**
F	**F**	**T**

Implication…

p	q	$p \rightarrow q$
T	**T**	**T**
T	**F**	**F**
F	**T**	**T**
F	**F**	**T**

- Resemble Real Life
  – Example:-
    - If you get 90 marks, I will give you A+.
    - If you gave me the vote, I will lower the tax
- Find the truth value for following implications
  – If 2+2=8 Then K. P. Oli is the president of USA.
  – If elephant can fly then snake can crawl.
  – If 3*3 = 9 then pink is black

Implication…

- ***Implication : p → q***
  - Example : ***If it is raining, then the home team wins***
- ***Converse : q → p***
  - Example : ***If the home team wins, then it is raining***
- ***Inverse : ¬p → ¬q***
  - Example : ***If it is not raining, then the home team does not win***
- ***Contra positive : ¬q → ¬p***
  - Example : ***If the home team does not win, then it is not rainin***g

Biconditional

- Let ***p*** and ***q*** be propositions. The biconditional statement ***p*** $\leftrightarrow$ ***q*** is the proposition "p if and only if q." The biconditional statement p $\leftrightarrow$ q is true when ***p*** and ***q*** have the same truth values, and is false otherwise

p	Q	p $\leftrightarrow$ q
T	**T**	**T**
T	**F**	**F**
F	**T**	**F**
F	**F**	**T**

- <u>**Example**</u>
  - You can take the flight if and only if you buy a ticket

Translating English Sentences to Propositional Logic

- **Steps**
  - Restate the given sentence into building block sentences
  - Give the symbol to each sentence and substitute the symbols using connectives
- **Example**
  - "if it is snowing then I will go to the beach"
  - Let ***p = it is snowing***, ***q = I will go to the beach***
  - Then the above sentence can be represented as $p \rightarrow q$

Some Terminologies

- **Tautology**
  - A compound proposition that is always ***TRUE***
  - *Example :* $p \vee \neg p$
- **Contradiction**
  - A compound proposition that is always ***FALSE***
  - *Example :* $p \wedge \neg p$
- **Contingency**
  - A compound proposition that is neither a tautology nor a contradiction
  - *Example :* $p \vee q$

Logical Equivalences

$p \wedge \mathbf{T} \Leftrightarrow p$ $p \vee \mathbf{F} \Leftrightarrow p$	Identity laws
$p \wedge \mathbf{F} \Leftrightarrow \mathbf{F}$ $p \vee \mathbf{T} \Leftrightarrow \mathbf{T}$	Domination laws
$p \wedge p \Leftrightarrow p$ $p \vee p \Leftrightarrow p$	Idempotent laws
$\neg(\neg p) \Leftrightarrow p$	Double negation law
$p \wedge q \Leftrightarrow q \wedge p$ $p \vee q \Leftrightarrow q \vee p$	Commutative laws
$(p \wedge q) \wedge r \Leftrightarrow p \wedge (q \wedge r)$ $(p \vee q) \vee r \Leftrightarrow p \vee (q \vee r)$	Associative laws
$p \wedge (q \vee r) \Leftrightarrow (p \wedge q) \vee (p \wedge r)$ $p \vee (q \wedge r) \Leftrightarrow (p \vee q) \wedge (p \vee r)$	Distributive laws
$\neg(p \wedge q) \Leftrightarrow \neg p \vee \neg q$ $\neg(p \vee q) \Leftrightarrow \neg p \wedge \neg q$	De Morgan's laws
$p \wedge \neg p \Leftrightarrow \mathbf{F}$ $p \vee \neg p \Leftrightarrow \mathbf{T}$	Negation laws
$p \wedge (p \vee q) \Leftrightarrow p$ $p \vee (p \wedge q) \Leftrightarrow p$	Absorption laws
$p \rightarrow q \Leftrightarrow \neg p \vee q$	Implication Law
$p \rightarrow q \Leftrightarrow \neg q \rightarrow \neg p$	Contra positive Law

p	q	p∨q	p∧(p∨q)	p∧(p∨q)↔p
T	T	T	T	T
T	F	T	T	T
F	T	T	F	T
F	F	F	F	T

Show that $(p \wedge q) \rightarrow (p \vee q)$ is a tautology

Rules of Inferences

Rule of Inference	*Tautology*	*Name*
p $p \to q$ $\therefore q$	$(p \wedge (p \to q)) \to q$	Modus ponens
$\neg q$ $p \to q$ $\therefore \neg p$	$(\neg q \wedge (p \to q)) \to \neg p$	Modus tollens
$p \to q$ $q \to r$ $\therefore p \to r$	$((p \to q) \wedge (q \to r)) \to (p \to r)$	Hypothetical syllogism
$p \vee q$ $\neg p$ $\therefore q$	$((p \vee q) \wedge \neg p) \to q$	Disjunctive syllogism
p $\therefore p \vee q$	$p \to (p \vee q)$	Addition
$p \wedge q$ $\therefore p$	$(p \wedge q) \to p$	Simplification
p q $\therefore p \wedge q$	$((p) \wedge (q)) \to (p \wedge q)$	Conjunction
$p \vee q$ $\neg p \vee r$ $\therefore q \vee r$	$((p \vee q) \wedge (\neg p \vee r)) \to (q \vee r)$	Resolution

Example: If You study hard then you will pass the exam.
You study hard.
Therefore you will pass the exam.

Using Rules of Inference to Build Arguments

- **Example 1**
- For the set of premises "If I play hockey, then I am sore the next day." "I use the whirlpool if I am sore." "I did not use the whirlpool". Show that conclusion "I don't play hockey" can be drawn.
- ***Solution***
- Let p = "I play hockey", q = "I am sore", r = "I use the whirlpool"
- Then the above premises are
  - 1. $p \rightarrow q$ [hypothesis]
  - 2. $q \rightarrow r$ [hypothesis]
  - 3. $\neg r$ [hypothesis]
  - 4. $p \rightarrow r$ [from 1 and 2 using hypothetical syllogism]
  - 5. $\neg p$ [from 3 and 4 using modus tollens]
  - hence proved

Using Rules of Inference to Build Arguments…

- **Example 2**
- Construct an argument using rules of inference to show that the hypotheses “If it does not rain or if it is not foggy, then the sailing race will be held and the life saving demonstration will go on,” “If the sailing race is held, then the trophy will be awarded,’ and “The trophy was not awarded” imply the conclusion “It rained”
- **Solution**
  - Let p = “It rains”, q = “It is foggy”, r = “the sailing race is held”, s = “Life saving demonstration is done”, and t = “Trophy is awarded”
  - [1] $(r \rightarrow t)$ [Hypothesis]
  - [2] $\neg t$ [Hypothesis]
  - [3] $\neg r$ [Modus Tollens using steps 1 and 2]
  - [4] $((\neg p \vee \neg q) \rightarrow (r \wedge s))$ [Hypothesis]
  - [5] $\neg(\neg p \vee \neg q) \vee (r \wedge s)$ [Implication of Step 4]
  - [6] $(p \wedge q) \vee (r \wedge s)$ [De Morgan’s Law in Step 5]
  - [7] $p \vee (r \wedge s)$ [Simplification using step 6]
  - [8] $p \vee r$ [Simplification using step 7]
  - [9] $r \vee p$ [Commutative law in step 8]
  - [10] $\neg r \vee p$ [Addition using step 3]
  - [11] $p \vee p$ [Resolution using steps 9 and 10]
  - [12] p [Idempotent law]

Using Rules of Inference to Build Arguments…

- **Exercises**
  - Show that the premises “It is not sunny this afternoon and it is colder than yesterday,” “We will go swimming only if it is sunny,” “If we do not go swimming, then we will take a canoe trip,” and “If we take a canoe trip, then we will be home by sunset” lead to the conclusion “We will be home by sunset.”
  - Show that the premises “If you send me an e-mail message, then I will finish writing the program,” “If you do not send me an e-mail message, then I will go to sleep early,” and “If I go to sleep early, then I will wake up feeling refreshed” lead to the conclusion “If I do not finish writing the program, then I will wake up feeling refreshed.”
  - Use rules of inference to show that the hypotheses “Randy works hard,” “If Randy works hard, then he is a dull boy,” and “If Randy is a dull boy, then he will not get the job” imply the conclusion “Randy will not get the job.”

Using Rules of Inference to Build Arguments…

- **Exercise**
  - If I passed the exam or win the lottery then I will be happy. If I am lucky then I will win the lottery. I am lucky but I didn’t pass the exam. Am I happy?
  - If i pass the exam and win the medal then my mother will give me a chocolate. My mother won't give me the chocolate. I pass the exam. Prove that i don't win the medal.

Predicate

- Statements involving variables are neither true nor false when the values of the variables are not specified, for example, "***x > 3***"
- We can denote this statement by ***P(x)***, where ***P*** denotes the **predicate** and ***x*** is variable or subject.
- Once value is assigned to the propositional function then we can tell whether it is true or false
- **Example** ***P(4) is TRUE*** and ***P(2) is FALSE***

Quantifiers

- Quantifiers are the tools that are used to make a propositional function a proposition
- two types of quantifiers:
  - **Universal Quantifier** (denoted by $\forall$)
    - *The universal quantification of P(x), denoted by "$\forall$x P(x), is a proposition "P(x) is true* ***for all*** *the values of x in the universe of discourse"*
    - *Eg:- Pass(x) is a predicate in which universe of discourse is CSIT students of PMC*
    - *Then $\forall$x Pass(x) means all CSIT students of PMC have passed the exam*
  - **Existential Quantifier** (denoted by $\exists$)
    - *The existential quantification of P(x), denoted by $\exists$x P(x), is a proposition "P(x) is true* ***for some*** *values of x in the universe of discourse"*
    - *Eg:- Pass(x) is a predicate in which universe of discourse is CSIT students of PMC*
    - *Then $\exists$x Pass(x) means some CSIT students of PMC have passed the exam*

Free and Bound Variables

- When the variable is assigned a value or it is quantified using some quantifier, it is called ***bound variable***. If the variable is not bounded then it is called ***free variable***
- **Example**
  - *P(x,y)→* *has two free variables x and y*
  - *P(2,y) →* *has one bound variable 2 and one free variable y*
  - *P(2,y) where y = 4 →* *is bounding the variable y also*
  - *∀x P(x) →* *has a bound variable x*
  - *∀x P(x,y) →* *has one bound variable x and one free variable y*

Negation of Quantifies Expression

- Let ***P(x)*** denotes ***x*** is beautiful and universe of discourse for ***x*** is ***girls in Kathmandu***
- $\forall x\, P(x) \rightarrow$ every girls in Kathmandu are beautiful
- $\neg\forall\, x\, P(x) \equiv ???$

At least one girl in kathmandu is not beautiful $\longrightarrow$ $\exists\, x \neg\, P(x)$

- $\exists x\, P(x) \rightarrow$ at least a girl in Kathmandu is beautiful
- $\neg\exists\, x\, P(x) \equiv ???$

All girls in Kathmandu are not beautiful $\longrightarrow$ $\forall\, x \neg\, P(x)$

Translating Sentences into Logical Expressions using Quantifiers

- All over smart person are stupid
- All animal who can bark are dog
- If a person is female and is a parent, then this person is someone’s mother
- Children of stupid person are naughty
- All American who sale weapon to hostile nations are criminal
- Every student loves some student
- Some lions do not drink coffee
- All living things are either animal or plant.

Rules of Inference for Quantified Statements

- Universal Instantiation

$$\frac{\forall x P(x)}{\therefore P(c),\ for\ all\ c}$$

- Universal Generalization

$$\frac{P(c),\ for\ all\ c}{\therefore \forall x P(x)}$$

- Existential Instantiation

$$\frac{\exists x P(x)}{\therefore P(c),\ for\ some\ c}$$

- Existential Generalization

$$\frac{P(c),\ for\ some\ c}{\therefore \exists x P(x)}$$

Inference with quantified statements

- All over smart person are stupid. Children of stupid person are naughty. Ram is children of Hari. Hari is over smart. Is Ram naughty?
- **<u>Solution</u>**
  - $O(x) \rightarrow$ x is oversmart
  - $S(x) \rightarrow$ x is stupid
  - $C(x,y) \rightarrow$ x is children of y
  - 1. $\forall x O(x) \rightarrow S(x)$ [Hypothesis]
  - 2. $\forall x \forall y\ C(x,y) \land S(y) \rightarrow N(x)$ [Hypothesis]
  - 3. C(Ram, Hari) [Hypothesis]
  - 4. O(Hari) [Hypothesis]
  - 5. O(Hari) $\rightarrow$ S(Hari) [U.I. over 1 and 4]
  - 6. S(Hari) [Modus ponens over 4 and 5]
  - 7. C(Ram, Hari) $\land$ S(Hari) [conjunction over 3 and 6]
  - 8. C(Ram, Hari) $\land$ S(Hari) $\rightarrow$ N(Ram) [U.I. over 2 and 7]
  - 9. N(Ram) [Modus ponens over 7 and 8]

Inference with quantified statements…

- Consider the statement: “The law says that it is a crime for an American to sell weapons to hostile nations. The country Nono, an enemy of America, has some missiles, and all of it missiles were sold to it by Colonel West, who is American. Convert the above statement into predicate logic and using resolution prove the “West is Criminal”.
- Every living thing is either animal or plant. All animals have heart. Hari’s dog is alive and it is not a plant. Prove that Hari’s dog has heart.
- If X is on the top of Y, Y supports X. If X is above Y and they are touching each other, then X is on the top of Y. A cup is above a book. A cup is touching a book. Show that supports(book, cup) is true.
- All Nepalese who love their country are hero. All politicians ignore heroes. Ram is Nepalese and loves his country. Hari is a politician. Prove that Hari ignores Ram.

End of Session 3.1