

Bayesian Hierarchical Modeling of COVID-19 Cases and Government Response in the United States

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Introduction

Methods

Model

Algorithm

We use a component-wise Metropolis Hastings (MH) algorithm that updates the p parameters at the k th iteration one at a time. In contrast to the Gibbs sampler, the MH algorithm uses the full joint distribution to generate potential values, allowing us to avoid computing the full conditional posteriors for each of the parameters.

For our algorithm, we first define some notation:

- N_{iter} is the number of iterations that the algorithm was run for.
- $\theta_i^{(k)}$ and $\theta_i^{(k+1)}$ are the current and updated values, respectively, for the i th parameter at the k th iteration.
- $\boldsymbol{\theta}^{(k)} = (\theta_1^{(k)}, \theta_2^{(k)}, \dots, \theta_p^{(k)})$ is the vector of parameters $\boldsymbol{\theta}$ at the start of the k th iteration.
- $\boldsymbol{\theta}_{-i}^{(k)} = (\theta_1^{(k+1)}, \theta_2^{(k+1)}, \dots, \theta_{i-1}^{(k+1)}, \theta_{i+1}^{(k)}, \dots, \theta_p^{(k)})$ is the current vector of parameters $\boldsymbol{\theta}$ at the k th iteration, excluding the i th parameter.
- θ_i^* is the proposal value for the i th parameter.
- $Q_i(\theta_i^* | \theta_i^{(k)})$ is the proposal distribution for the i th parameter. More generally, the proposal distribution can also depend on $\boldsymbol{\theta}_{-i}^{(k)}$, but for our algorithm, our choice of proposal distribution depends only on $\theta_i^{(k)}$.

The algorithm is given as follows:

1. Choose starting values $\boldsymbol{\theta}^{(0)}$.
2. For the k th iteration where $k = 1, \dots, N_{iter}$:
 1. For the i th parameter where $i = 1, \dots, p$:
 1. Generate θ_i^* from $Q_i(\theta_i^* | \theta_i^{(k)})$.

2. Calculate the MH ratio given by:

$$\begin{aligned}
r_i^{(k)} &= \frac{\pi(\theta_i^*, \boldsymbol{\theta}_{-i}^{(k)} | \mathbf{X})}{\pi(\theta_i^{(k)}, \boldsymbol{\theta}_{-i}^{(k)} | \mathbf{X})} \frac{Q_i(\theta_i^{(k)} | \theta_i^*)}{Q_i(\theta_i^* | \theta_i^{(k)})} \\
&= \frac{f(\theta_i^*, \boldsymbol{\theta}_{-i}^{(k)}, \mathbf{X})}{f(\theta_i^{(k)}, \boldsymbol{\theta}_{-i}^{(k)}, \mathbf{X})} \frac{Q_i(\theta_i^{(k)} | \theta_i^*)}{Q_i(\theta_i^* | \theta_i^{(k)})} \\
&= \frac{f(\theta_i^*, \boldsymbol{\theta}_{-i}^{(k)}, \mathbf{X})}{f(\theta_i^{(k)}, \boldsymbol{\theta}_{-i}^{(k)}, \mathbf{X})}
\end{aligned} \tag{1}$$

3. Calculate acceptance probability $\alpha_i(\theta_i^{(k)}, \theta_i^*) = \min(1, r_i^{(k)})$.

4. Draw $U \sim \text{Unif}(0, 1)$. If $U < \alpha_i(\theta_i^{(k)}, \theta_i^*)$, then set $\theta_i^{(k+1)} = \theta_i^*$. Otherwise, set $\theta_i^{(k+1)} = \theta_i^{(k)}$.

For each $Q_i(\theta_i^* | \theta_i^{(k)})$, we use a uniform distribution. More specifically, θ_i^* is generated as $\theta_i^* = \theta_i^{(k)} + \text{Unif}(-a_i, a_i)$, where a_i represents the chosen window length for the i th parameter. We tune a_i for each of the parameters by trying different a_i until we find ones that allow us to accept about 30-60% of N_{iter} iterations for a parameter. We keep track of the number of acceptances by computing the number of unique $\theta_i^{(k)}$ across all N_{iter} iterations. Tuning a_i is important because: 1) a_i being too large means the proposed moves will be too large and unlikely to be accepted, taking the chain a long time to sample the entire parameter space (i.e. the chain is sampling a lot of values outside of the support of the posterior distribution); 2) a_i being too small means that the proposed moves will be too small and accepted too often, taking the chain a long time to move around the parameter space.

In equation (1), the second equality holds since $\pi(\boldsymbol{\theta} | \mathbf{X}) = \frac{f(\boldsymbol{\theta}, \mathbf{X})}{m(\mathbf{X})}$, so the $m(\mathbf{X})$'s cancel out. Additionally, the third equality holds since θ_i^* is drawn from $\text{Unif}(\theta_i^{(k)} - a_i, \theta_i^{(k)} + a_i)$. Since the uniform distribution is a symmetric distribution, the Q_i 's cancel out.

To allow the results to converge more quickly and save computation time, we choose the $\boldsymbol{\theta}^{(0)}$ at the beginning of the algorithm based on results from model fitting using the frequentist approach (under the assumption that the Bayesian and frequentist approaches will yield similar results). More specifically, we fit a linear mixed effect model (LMM) with random intercepts on $\log(\text{infection rate})$ for the i th state during the j th week (see Table 1 for results). The LMM is given as follows:

$$\log(\lambda_{ij}) = \alpha + \boldsymbol{\beta} \mathbf{x}_{ij}^T + \gamma P_{ij} + \delta E_{ij} + u_i + e_{ij}$$

where α is the fixed intercept, $\boldsymbol{\beta}$ is the vector of fixed-effect coefficients associated with the vector of covariates \mathbf{x}_{ij}^T , γ is the fixed-effect coefficient associated with population density P_{ij} , δ is the fixed-effect coefficient associated with the percentage of elderly population E_{ij} , $u_i \sim N(0, \sigma_u^2)$ is the state-specific random intercept, and $e_{ij} \sim N(0, \sigma_e^2)$ is the residual error term.

Results

Convergence

Posterior Summaries

Table 2 shows the posterior means and 95% credible intervals of the parameters for mobility changes, government response, population density, and percentage of elderly population. The posterior mean is calculated by taking the mean of the MCMC chain after burn-in (i.e., last 50,000 states of the chain), and the 95% credible interval is the 2.5% and 97.5% percentiles of the chain after burn-in. Because the Government Response Index measures the overall government response to the pandemic – including containment measures, economic support, and stringency of public health measures – we only present the posterior results of the

parameter for Government Response Index (β_4) here. Based on the posterior summaries, changes in mobility trends (compared to baseline) for retail and recreation were associated with a slight decrease in the expected log infection rate, whereas changes in mobility trends for parks and transit stations were associated with an increase in the expected log infection rate. A 1-unit increase in the weekly average Government Response Index was associated with a 4.95 unit decrease in the expected log infection rate, adjusting for the other covariates. A 1% increase in percentage of elderly population was associated with a 12.74 unit decrease in the expected log infection rate, adjusting for the other covariates. The credible interval with respect to population density suggests that population density had no significant impact on the infection rate.

Conclusion

Our posterior results suggest that the overall government response to the pandemic may be crucial for slowing down the spread of COVID-19. Larger mobility changes in retail and recreation may also slow down the spread of COVID-19. However, the temporal order of changes in mobility trends and infection rate is unclear given the available data. A next step for future studies is thus to evaluate the lagged relationship between mobility trends and infection rate. In terms of demographics, higher percentage of elderly population appeared to be associated with lower infection rate, which may be due to other factors (e.g., less interactions among older adults). We therefore suggest further investigation on the causal mechanisms underlying the spread of COVID-19 among the younger population. Finally, population density did not appear to have a significant impact on the infection rate. There have been mixed findings in the literature regarding the importance of population density on the spread of COVID-19. Although a higher population density indicates a higher chance of being in contact with a positive case, larger cities also tend to have more resources to combat the pandemic than rural areas. More in-depth research on these relationships is thus needed to determine the important predictors for the spread of COVID-19.

References

Tables and Figures

Table 1: Summary of LMM fit on log(infection rate)

Coefficient	Estimate
(Intercept)	-0.601
Retail/rec. % change	-0.024
Parks % change	0.010
Transit % change	0.001
Gov. response index	-4.956
Containment index	4.456
Economic support index	0.633
Stringency index	-0.027
Week	0.155
Population density	-0.001
% over 65	-17.024
sigma_u	0.821
sigma_e	0.783

Table 2: Posterior summaries of the parameters

Parameter	Posterior Mean	95% Credible Interval
beta1	-0.035	(-0.041 -0.028)
beta2	0.006	(0.005 0.007)
beta3	0.024	(0.02 0.029)
beta4	-4.95	(-4.971 -4.927)
delta	-12.741	(-15.066 -10.614)
gamma	0.00018	(-2e-05 0.00038)

beta1: mobility changes for retail and recreation

beta2: mobility changes for parks

beta3: mobility changes for transit stations

beta4: government response index

delta: percentage of elderly population

gamma: population density