CS 229 - PS 1: Problem 1

1. **[25 points] Logistic regression**

   (a) [10 points] Consider the average empirical loss (the risk) for logistic regression:

$$J(\theta) = \frac{1}{m}\sum_{i=1}^{m}\log(1+e^{-y^{(i)}\theta^T x^{(i)}}) = -\frac{1}{m}\sum_{i=1}^{m}\log(h_\theta(y^{(i)}x^{(i)}))$$

where $y^{(i)} \in \{-1, 1\}$, $h_\theta(x) = g(\theta^T x)$ and $g(z) = 1/(1+e^{-z})$. Find the Hessian H of this function, and show that for any vector z, it holds true that

$$z^T H z \geq 0.$$

Hint: Be careful that the range for label values, $\{-1, 1\}$, is different than the range used in lecture notes, which is $\{0, 1\}$. Please read the supplementary notes if you are having trouble. You might want to start by showing the fact that $\sum_i \sum_j z_i x_i x_j z_j = (x^T z)^2 \geq 0$.
Remark: This is one of the standard ways of showing that the matrix H is positive semi-definite, written "$H \succeq 0$." This implies that J is convex, and has no local minima other than the global one.[1] If you have some other way of showing $H \succeq 0$, you're also welcome to use your method instead of the one above.

   (b) [10 points] We have provided two data files:

   - `http://cs229.stanford.edu/ps/ps1/logistic_x.txt`
   - `http://cs229.stanford.edu/ps/ps1/logistic_y.txt`

These files contain the inputs ($x^{(i)} \in \mathbb{R}^2$) and outputs ($y^{(i)} \in \{-1, 1\}$), respectively for a binary classification problem, with one training example per row. Implement[2] Newton's method for optimizing $J(\theta)$, and apply it to fit a logistic regression model to the data. Initialize Newton's method with $\theta = \vec{0}$ (the vector of all zeros). What are the coefficients θ resulting from your fit? (Remember to include the intercept term.)

In [9]:

```python
#imports
import pandas as pd
import numpy as np
import math
```

The solution to part a was

$$H_{ij} = \frac{1}{m}\sum_{k=1}^{m}(y^{(k)})^2 x_i^{(k)} x_j^{(k)} g(y^{(k)}\theta^T x^{(k)})(1 - g(y^{(k)}\theta^T x^{(k)}))$$

where $g(z) = \frac{1}{1+e^{-z}}$

Newton Method update rule

$$\theta := \theta - H^{-1}\nabla_\theta \ell(\theta).$$

Steps

1. Calculate H for current value of θ
2. Use this to find H^{-1}
3. Calculate δ
4. Update θ by the update rule described previously

In [24]:

```
#main
theta = np.zeros((3))
loss = [J(X_data,Y,theta,n)]
for i in range(10):
    theta,H,delta,cur_loss = update(theta,H,delta,X_data,Y,n)
    loss.append(cur_loss)

print(theta)
print(loss)
```

```
[-2.6205116   0.76037154  1.17194674]
[-0.6931471805599458, -0.3747247075760391, -0.33425013576955137, -0.
32928119947970075, -0.32914756442755344, -0.32914743129585516, -0.32
9147431295712, -0.32914743129571195, -0.329147431295712, -0.32914743
1295712, -0.329147431295712]
```

Results

The **parameters** we found are as follows:

- $\theta_0 = -2.62$
- $\theta_1 = 0.76$
- $\theta_2 = 1.17$

Read on for how we got these results and the helper functions.

Loading the Data

In [11]:

```
x = pd.read_csv('logistic_x.txt', sep="  ", header = None)
x
```

```
/Users/anjayfriedman1/opt/anaconda3/lib/python3.7/site-packages/ipyk
ernel_launcher.py:1: ParserWarning: Falling back to the 'python' eng
ine because the 'c' engine does not support regex separators (separa
tors > 1 char and different from '\s+' are interpreted as regex); yo
u can avoid this warning by specifying engine='python'.
  """Entry point for launching an IPython kernel.
```

Out[11]:

	0	1
0	1.343250	-1.331148
1	1.820553	-0.634668
2	0.986321	-1.888576
3	1.944373	-1.635452
4	0.976734	-1.353315
...	...	...
94	4.774854	0.099415
95	5.827485	-0.690058
96	2.289474	1.970760
97	2.494152	1.415205
98	2.084795	1.356725

99 rows × 2 columns

In [12]:

```
y = pd.read_csv('logistic_y.txt', header = None)
y
```

Out[12]:

	0
0	-1.0
1	-1.0
2	-1.0
3	-1.0
4	-1.0
...	...
94	1.0
95	1.0
96	1.0
97	1.0
98	1.0

99 rows × 1 columns

In [13]:

```
X = x.to_numpy()
Y = y.to_numpy()
```

We now have our x and y data. Since we are doing logistic regression, we must add a column of 1's to our X data since one of our θ's will be an intercept. We also must initialize our θ, H and δ

In [14]:

```
n = 99
i = np.ones((n,1))
X_data = np.concatenate((i,X),1)

H = np.zeros((3,3))
theta = np.zeros((3)) #dimension of x + 1
delta =  np.zeros((3))
```

In [15]:

```
def sigmoid(z):
    return 1/(1 + math.exp(-z))
```

In [16]:

```
def h(yi,xi,thet):
    if(yi.shape[0]==99):
        print("y")
    if(xi.shape[0]==99):
        print("x")
    a = yi*xi
    return sigmoid(np.dot(thet,a))
```

In [17]:

```
#loss function as defined in question
def J(x,y,thet,n):
    loss = 0
    for rows in range(x.shape[0]):
        loss+= math.log(h(y[rows],x[rows],thet))
    return loss/n
```

In [18]:

```
#returns ijth index of H
def H_ij(i,j,x,y,thet,n):
    cur = 0
    for rows in range(x.shape[0]):
        yr = y[rows]
        xr = x[rows]
        ht = h(yr,xr,thet)

        cur += (yr**2)*xr[i]*xr[j]*ht*(1-ht)

    return cur/n
```

In [19]:

```
#return the partial derivative of J wrt thet[i]
def del_i(i,x,y,thet,n):
    cur = 0
    for rows in range(x.shape[0]):
        yr = y[rows]
        xr = x[rows]
        ht = h(yr,xr,thet)

        cur += (1-ht)*xr[i]*yr

    return -1*cur/n
```

In [20]:

```
#Calculates the Hessian
def calc_H(x,y,thet,n):
    h = np.zeros((3,3))
    for i in range(3):
        for j in range(3):
            h[i][j] = H_ij(i,j,x,y,thet,n)

    return h
```

In [21]:

```
#Calculates delta
def calc_delta(x,y,thet,n):
    d = np.zeros((3))
    for i in range(3):
        d[i] = del_i(i,x,y,thet,n)

    return d
```

In [22]:

```
#Update theta
def update_theta(theta,H,delta):
    H_inv = np.linalg.inv(H) #FIXME
    theta = theta - np.matmul(H_inv,delta) #FIXME
    return theta
```

In [23]:

```
#update function
def update(theta,H,delta,x,y,n):
    H = calc_H(x,y,theta,n)
    delta = calc_delta(x,y,theta,n)
    theta = update_theta(theta,H,delta)
    loss = J(x,y,theta,n)

    return theta,H,delta,loss
```

Part (c)

(c) [5 points] Plot the training data (your axes should be x_1 and x_2, corresponding to the two coordinates of the inputs, and you should use a different symbol for each point plotted to indicate whether that example had label 1 or -1). Also plot on the same figure the decision boundary fit by logistic regression. (This should be a straight line showing the boundary separating the region where $h_\theta(x) > 0.5$ from where $h_\theta(x) \leq 0.5$.)

In [3]:

```
#imports for (c)
import matplotlib.pyplot as plt
```

The **parameters** we found previously are as follows:

- $\theta_0 = -2.62$
- $\theta_1 = 0.76$
- $\theta_2 = 1.17$

The **decision boundary** $h_\theta(x) = 0.5$ will be the line given by solving $(\theta_0, \theta_1, \theta_2)^T(1, x_1, x_2) = 0.5$

The **solution** with x_2 as a function of x_1 is:

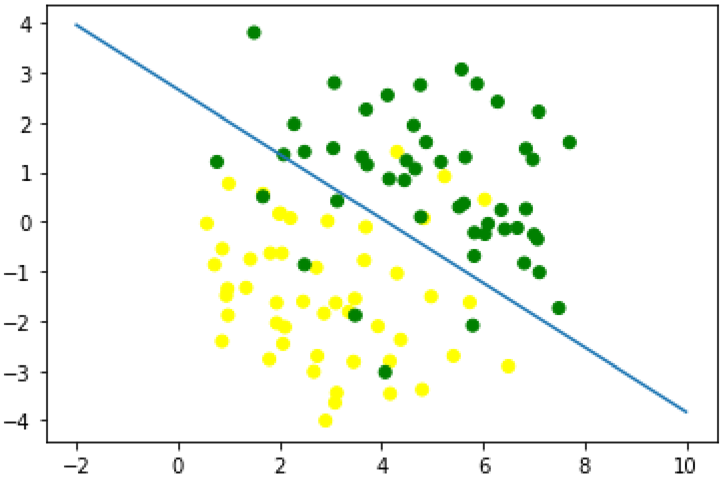
> $x_2 = \frac{0.5 - x_1\theta_1 - \theta_0}{\theta_2} = ax + b$ where $a = \frac{-\theta_1}{\theta_2}$ and $b = \frac{0.5 - \theta_0}{\theta_2}$

In [40]:

```
#label colors for each point
colors = []
for c in range(99):
    colors.append('green') if Y[c]==1 else colors.append('yellow')
```

In [41]:

```
#plot boundary
xvals = np.arange(-2,10,0.001)
yvals = xvals*(-0.76/1.17) + ((0.5+2.62)/1.17) #linear equation with parameters
 found
plt.scatter(x[0],x[1],c=colors)
plt.plot(xvals, yvals)
plt.show()
```



The above graph has the dots with a +1 classification as green and those with -1 as yellow. It demonstrates that the decision boundary generated by applying Newton's method to the cost function $J(\theta)$ with logistic regression is generally correct.

Thanks for reading :)