# Sandpile model: Study the power laws

Saroj Anjesh, Instructor: Prof Mahendra Verma

anjesh@iitk.ac.in, mkv@iitk.ac.in

**Indian Institute of Technology Kanpur, India**

## 1 Introduction

In most systems which exhibit critical phenomena, there exist adjustable parameters which have to be finely tuned in order for the system to reach the critical point. For example, in the Ising model, one needs to adjust both the temperature and the external magnetic field to close to their critical values. In the language of the Renormalization Group, these are the relevant directions of the critical fixed point that we are interested in. If such a system is initially prepared such that the temperature and the external magnetic field are far from their critical values, the system will not show critical behavior even if other parameters are changed, or if the system is perturbed. Since most systems with phase transitions exhibit the above behavior, it would seem reasonable to guess that critical phenomena can be triggered only if all relevant parameters have been fine tuned.

However, in 1987, Bak, Tang, and Wiesenfeld introduced the sandpile model, which displayed spatial and temporal power laws and scale invariance, without controlling the external parameters. The evolution of the system was such that it spontaneously moved towards the critical point. Because of this, the critical behavior exhibited by this model was termed as self-organized criticality.

Self-organized criticality was gradually understood to be a feature of out-of-equilibrium systems with a slow driving force. These models of SOC provide a mechanism which can be used to explain the emergence of complexity in many natural phenomena. The behavior of such systems is unlikely to be governed by the fine-tuning of parameters, and the complexity must arise from the evolution of the system itself. In their original paper, Bak et al. claimed that the ubiquitous 1/f can be explained in terms of SOC.

A wide range of other natural phenomena, such as naturally-occurring fractals, earthquakes, rainfall patterns, have all been investigated in terms of these models of self organizing behavior spontaneous critical phenomena. In fact, the model has also been used to analyze systems which have no connection to physics, such as stock markets and sociology.

In this report we will analyze the power law distribution in the sandpile model i.e, the size of an event versus its frequency of occurring. In section 2 we define the model and the rules for the same. In section 3 we discuss the parameters of the model. In section 4 we explain the approach to find the distribution. Section 5 talks about the plot and analysis and in the final conclusion of the report

## 2 Sandpile model in 2 dimensions

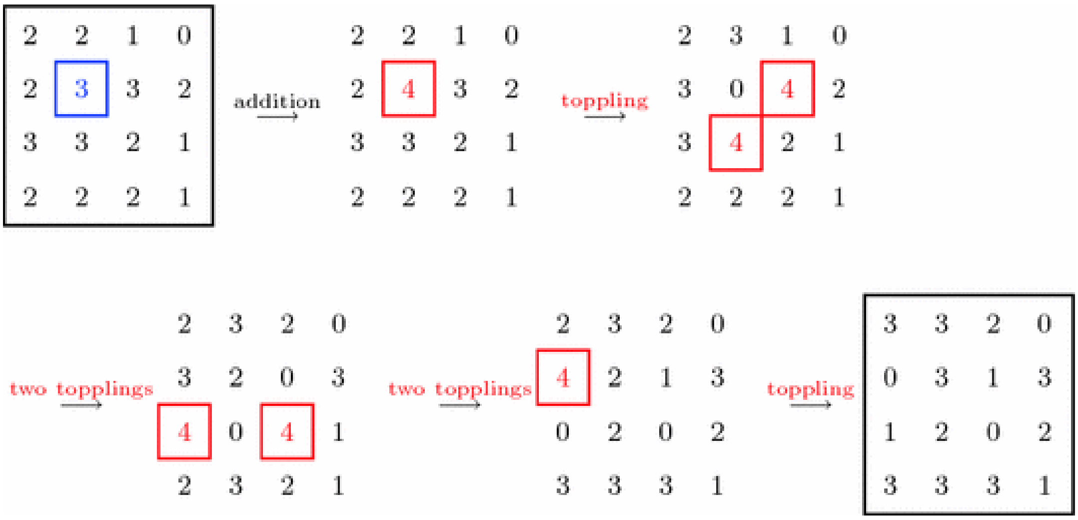
Most studies of the sandpile model were done on a 2-dimension model, we have an N ×N grid, we drop the sand particles randomly on the grid. At each site, we can place grains of sand, one on top of each other, and if any site(grid point) height becomes equal to critical value (i.e $z_c$= 4) then toppling happens as follows.

If

$$z_{x,y} = z_c$$

Then

$$\begin{aligned} z_{x,y} &\longrightarrow z_{x,y} - 4 \\ z_{x\pm1,y} &\longrightarrow z_{x\pm1,y} + 1 \\ z_{x,y\pm1} &\longrightarrow z_{x,y\pm1} + 1 \end{aligned}$$




We work with free boundary conditions on all sides. Any grain crossing the boundary is lost. If we add a grain of sand to a site which is at the critical value of z(x, y), we trigger off an avalanche onto the adjoining sites. If these sites themselves are at the critical value, then the avalanche propagates, else it stops. A very important rule which one must obey while adding subsequent grains of sand, is to allow all possible avalanches to occur within the system before one adds another grain. In physical systems, this would correspond to the situation where the frequency of the driving force is small compared to the relaxation time-scale, that is, when the system is driven weakly away from equilibrium.

The configuration where all $z_{x,y} = z_c$, is not stable to perturbations, nor is it the final configuration starting from some random configuration. Instead, all that can be said about the system is that, given sufficient time, it will reach a steady state where on the average, the number of grains added to the grid is the same as that lost at the boundaries.

## 3 Parameters of the model

As we saw in the last section, once the slope at a point on the grid exceeds the critical value $z_c$, it sets of on avalanche, with grains toppling onto the adjacent sites. These avalanches can be parameterized by three variables:

- number of topplings s
- area affected by the avalanche a
- duration of the avalanche T

Though at first glance, it seems that the number of topplings, ‘s’ and the area ‘a’ both measure the number of affected sites, one must remember that a single site may topple more than once in a single avalanche, and hence the two are truly different variables. The duration of the avalanche is defined as the number of updates one must perform on the system before all the sites become stable after the addition of one grain of sand.

## 4 Approach

We wrote the python code declaring a NxN matrix and adding the grain one by one on the site by the above mentioned rules. And then finally analysing the parameters by plotting the scatter plots.

Description about the code:

SIZE, TOTAL_GRAINS, RANDOM_DROPPING, CREATE_VIDEO are adjustable variables of the code depending on the type of output required. ASM is a class in the code which adds the grain one by one on the site and records the parameters of the model. In the array name ‘aval_time’, ‘aval_size’. And ‘crtvideo’ creates the video animation for the model.

## 5 Plots and Analysis

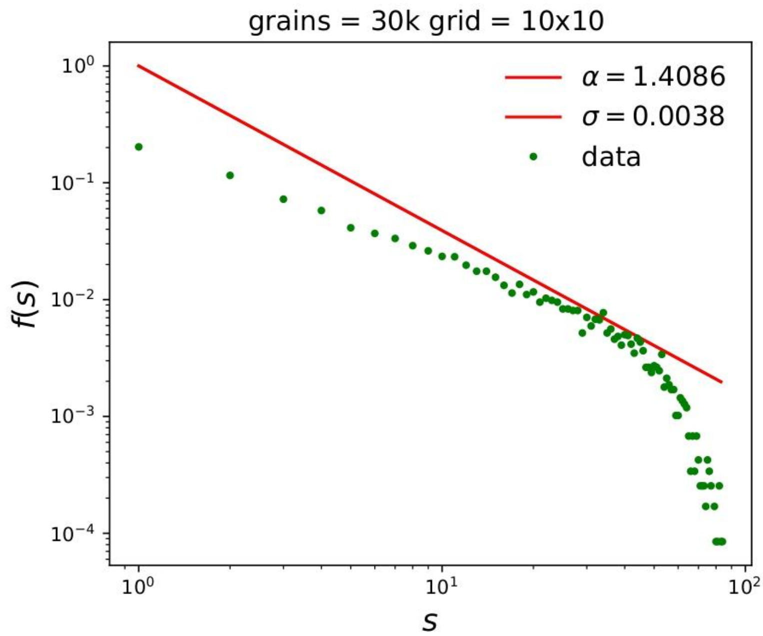
We analyzed the avalanche size and time for 3 different grid sizes, (10x10, 50x50, and 100x100). Using total grains from 10 thousand to 2 lakhs. It is quite impossible to display all the 21 plots on one page. Hence the table given below shows the slope of the line for all the 21 graph plots. For visualization of how slope changes with the increase in numbers of grain.

All graphs links :-
https://drive.google.com/drive/folders/1JdKJ6_yGJk37sDclh7FKRdSOA5n_KOEb?usp=sharing

## 5a. Avalanche size

**Grain number: 10,000 to 2 lakhs**

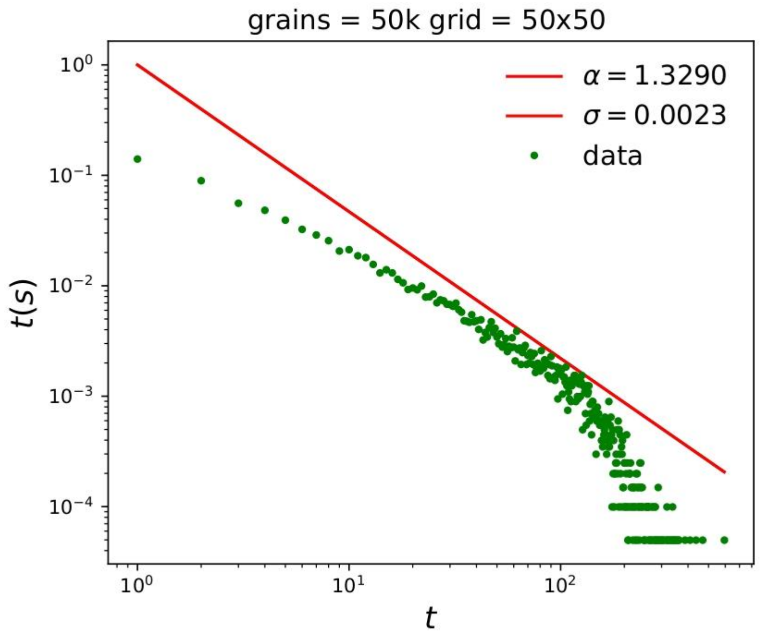
| | 10x10 | 50x50 | 100x100 |
|---|---|---|---|
| 10k | 1.4817 | 1.3661 | 2.2275 |
| 30k | 1.4760 | 1.3325 | 1.3714 |
| 50k | 1.4768 | 1.3290 | 1.3198 |
| 200k | 1.4066 | 1.2563 | 1.2296 |




## 5b. Avalanche time

**Grain number: 10,000 to 50,000**

| | 10x10 | 50x50 | 100x100 |
|---|---|---|---|
| 10k | 1.4817 | 1.3661 | 2.2275 |
| 30k | 1.4760 | 1.3325 | 1.3714 |
| 50k | 1.4768 | 1.3290 | 1.3198 |



## For grid size 50x50. 10 data points of alpha

| grain | alpha | grain | grain |
|---|---|---|---|
| 1.5k | 2.1763 | 6.0k | 1.4128 |
| 2.0k | 2.4427 | 7.0k | 1.3327 |
| 3.0k | 2.0965 | 8.0k | 1.3228 |
| 4.0k | 1.8870 | 9.0k | 1.3084 |
| 5.0k | 1.6088 | 10.0k | 1.3661 |

## 6 Result

### Avalanche size

- Value of alpha for avalanche size from BTW review letter: **$\alpha$ = 0.98 (in 50x50)**
- Value of alpha from my code: **$\alpha$ = 1.2682 (in 50x50)**

Power law: $Y = bx^{\alpha}$

frequency $\propto$ (toppling)$^{\alpha}$

### Avalanche time

- Value of alpha for avalanche time from BTW review letter: **$\alpha$ = 0.42 (in 50x50)**
- Value of alpha from my code: **$\alpha$ = 1.3425 (in 50x50)**

**We run the code for grid size 50x50. From 1500 grains to 10,000 grains, and it is found that the value of alpha decreases with the increase in the number of grains dropped.**

Video: https://youtu.be/8abR3l6Gj5w

Reference: https://journals.aps.org/prl/pdf/10.1103/PhysRevLett.59.381