GATE in Data Science and AI Study Materials
Linear Algebra
By Piyush Wairale

Instructions:

- Kindly go through the lectures/videos on our website www.piyushwairale.com
- Read this study material carefully and make your own handwritten short notes. (Short notes must not be more than 5-6 pages)
- Attempt the question available on portal.
- Revise this material at least 5 times and once you have prepared your short notes, then revise your short notes twice a week
- There are few topics like Projection matrix, partitioned matrix, SVD for these topics we don't have previous years questions. So focus on concept only.
- **If you are not able to understand any topic or required detailed explanation, please mention it in our discussion forum on webiste**
- **Let me know, if there are any typos or mistake in study materials. Mail me at piyushwairale100@gmail.com**

1 Vector:

An ordered 'n'-tuple X= $(x_1, x_2, ...x_n)$ is called an n-vector and $x_1, x_2, ...x_n$ are called components of 'X'.
A vector may be written as either a row matrix X= $[x_1 x_2 ....x_n]$ which is called row vector (or) a column matrix X= $\begin{bmatrix} x_1 \\ x_2 \\ ... \\ x_n \end{bmatrix}$ which is called column vector.
If A is a matrix of order mn then each row of A is an n-vector and each column of A is an m-vector (m-tuple vector).
In particular, if m=1 then A is a row vector and if n=1 then A is a column vector.

Equality of two vectors:
If X= $(x_1, x_2, ...x_n)$ and Y= $(y_1, y_2, ...y_n)$are said to be equal if $x_1 = y_1, x_2 = y_2, ...x_n = y_n$
For example
X= (1, 3, 5) and Y =(1, 3, 5) then X=Y.
If X= (1, 3, 5) and Y = (3, 5, 1) then X $\neq$ Y.)

Addition of two vectors:
If X= $(x_1, x_2, ...x_n)$ and Y= $(y_1, y_2, ...y_n)$ then X+ Y = $(x_1 + y_1, x_2 + y_2, ...x_n + y_n)$
For example,
X= (2, 4, 5), Y = (1,-3, 7)
Then X+ Y = (2+1, 4-3, 5+7) = (3, 1, 12)

Multiplication of a vector by a scalar:
Let 'k' be any number and X = $(x_1, x_2, ...x_n)$ then kX = = $(kx_1, kx_2, ...kx_n)$
For example X = (1, 3, 2) then 4X = (4, 12,8)

Linear combination of vectors:
If $X_1, X_2, ...X_r$, are 'r' vectors of order 'n' and $k_1, k_2, ...k_r$ are 'r' scalars then the expression of the form $k_1X_1 + k_2X_2 + ...k_rX_r$, is also a vector and it is called linear combination of the vectors $X_1, X_2....X_r$

Linearly dependent vectors:
The vectors $X_1, X_2, ...X_r$ of same order 'n' are said to be linearly dependent if there exist scalars (or numbers) $k_1, k_2, ...k_r$ not all zero such that $k_1X_1 + k_2X_2 + ...k_rX_r =$ O, where O denotes the zero vector of order n.

Linearly independent vectors:
The vectors $X_1, X_2, ...X_r$ of same order n are said to be linearly independent vectors if every relation of the type

$k_1X_1 + k_2X_2 + ...k_rX_r = O$
$\implies k_1 = k_2 = ... = .k_r = 0$

Note:

- If $X_1, X_2, ...X_r$ are linearly dependent vectors then at least one of the vectors can be expressed as a linear combination of other vectors.
- If A is a square matrix of order n and $|A| = 0$ then the rows and columns are linearly dependent.
- If A is a square matrix of order 'n' and $|A| \neq 0$ then the rows and columns are linearly independent.
- Any subset of a linearly independent set is itself linearly independent set.
- If a set of vectors includes a zero vector then the set of vectors is linearly dependent set.

Inner product:
The inner product of two vectors

X = $\begin{bmatrix} x_1 \\ x_2 \\ ... \\ x_n \end{bmatrix}$ and Y = $\begin{bmatrix} y_1 \\ y_2 \\ ... \\ y_n \end{bmatrix}$ is denoted by X.Y and defined as

$X.Y = X^TY = \begin{bmatrix} x_1 & x_2 & ... & x_n \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ ... \\ y_n \end{bmatrix}$ which is a scalar quantity.

Note:
1. $X^TY = Y^TX$ i.e Inner Product is symmetric
2. $X.Y = 0 \implies$ the vectors X and Y are perpendicular.
3. X.Y = $\pm 1 \implies$ the vectors X and Y are parallel.

Length or norm of a vector:

If X = $\begin{bmatrix} x_1 \\ x_2 \\ ... \\ x_n \end{bmatrix}$ is a vector of order n then the positive square root of inner product of X and X^T i.e., X^TX is called length of X and it is denoted by $||X||$.
$||X|| = \sqrt{X.X} = \sqrt{x_1^2 + x_2^2 + ... + x_n^2}$
Ex: If $||X|| = [123]$then $\sqrt{1^2 + 2^2 + 3^2} = 3$

Unit vector or normal vector:
A vector X is said to be a unit vector if $||X|| = 1$

A unit vector is a vector with a length of 1. To find a unit vector in the direction of $\mathbf{v}$, divide $\mathbf{v}$ by its norm:

$$\mathbf{u} = \frac{\mathbf{v}}{\|\mathbf{v}\|}$$

Orthogonal vectors:
The two column vectors X_1 and X_2 are said to be orthogonal if $X_1.X_2 = X_1^T X_2 = X_2^T X_1 = 0$

Orthogonal set :
The column vectors $X_1, X_2, ...X_n$ of same order are said to be orthogonal if $X_i.X_j = X_i^T X_j = 0$ for all $i \neq j$.

A set of vectors $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$ is an orthogonal set if every pair of distinct vectors in the set is orthogonal. An orthogonal set is linearly independent.

1.1 Examples

Let's consider a few examples to illustrate these concepts:

Inner Product Example
Given two vectors $\mathbf{u} = [2, 3, -1]$ and $\mathbf{v} = [4, -1, 2]$, calculate their inner product:

$$\mathbf{u^T}.\mathbf{v} = 2 \cdot 4 + 3 \cdot (-1) + (-1) \cdot 2 = 8 - 3 - 2 = 3$$

Length or Norm Example
Find the norm of the vector $\mathbf{v} = [3, 4]$:

$$\|\mathbf{v}\| = \sqrt{\mathbf{v^T}.\mathbf{v}} = \sqrt{3^2 + 4^2} = 5$$

Unit Vector Example
Given a vector $\mathbf{v} = [6, 8]$, find the unit vector in the direction of $\mathbf{v}$:

$$\|\mathbf{v}\| = \sqrt{\mathbf{v^T}.\mathbf{v}} = \sqrt{6^2 + 8^2} = 10$$

$$\mathbf{u} = \frac{\mathbf{v}}{\|\mathbf{v}\|} = \frac{[6, 8]}{10} = [0.6, 0.8]$$

Orthogonal Vectors Example
Check if the vectors $\mathbf{u} = [1, 2]$ and $\mathbf{v} = [-4, 2]$ are orthogonal:

$$\mathbf{u^T}.\mathbf{v} = 1 \cdot (-4) + 2 \cdot 2 = -4 + 4 = 0$$

Therefore, $\mathbf{u}$ and $\mathbf{v}$ are orthogonal.

Orthogonal Set Example
Consider the set of vectors: $\{\mathbf{v}_1 = [1, 0], \mathbf{v}_2 = [0, -1]\}$. Check if this set is orthogonal:

$$\langle \mathbf{v}_1, \mathbf{v}_2 \rangle = 1 \cdot 0 + 0 \cdot (-1) = 0$$

The set $\{\mathbf{v}_1, \mathbf{v}_2\}$ is orthogonal.

2 Vector Space

A vector space is a set of vectors that satisfies the following properties:

1. Closure under addition: For any vectors u and v in the vector space, $u + v$ is also in the vector space.
2. Closure under scalar multiplication: For any vector u in the vector space and any scalar α, αu is in the vector space.
3. Associativity of addition: $(u + v) + w = u + (v + w)$ for all vectors u, v, and w in the vector space.
4. Commutativity of addition: $u + v = v + u$ for all vectors u and v in the vector space.
5. Identity element with respect to addition: There exists a zero vector (denoted as 0) such that $u + 0 = u$ for all vectors u in the vector space.
6. Inverse elements with respect to addition: For every vector u, there exists a vector $-u$ such that $u + (-u) = 0$.
7. Compatibility of scalar multiplication with field multiplication: For any scalars α and β and any vector u in the vector space, $\alpha(\beta u) = (\alpha\beta)u$.
8. Distributivity of scalar multiplication with respect to vector addition: For any scalar α and vectors u and v in the vector space, $\alpha(u + v) = \alpha u + \alpha v$.
9. Distributivity of scalar multiplication with respect to field addition: For any scalars α and β and any vector u in the vector space, $(\alpha + \beta)u = \alpha u + \beta u$.
10. Scalar multiplication identity: For any vector u in the vector space, $1u = u$, where 1 is the multiplicative identity of the scalar field.

2.1 Subspaces

Let 'V' be a vector space defined under vector addition and scalar multiplication. A non-empty subset 'W' of 'v' such that 'W' is also a vector space under the same two operations of vector addition and scalar multiplication is called a subspace of 'V'.

A subspace is a subset of a vector space that is itself a vector space. It inherits all the vector space properties from the parent vector space. To be a subspace, a set S must satisfy:

1. Contains the zero vector: The zero vector (denoted as 0) must be in S.
2. Closed under addition: For any vectors u and v in S, $u + v$ is in S.
3. Closed under scalar multiplication: For any vector u in S and any scalar α, αu is in S.

2.2 Dimension and Basis

Let 'V' be a vector space. If for some positive integer 'n', there exists a set of 'n' linearly independent vectors of 'V' and if every set of (n+1) or more vectors in 'V' is linearly dependent then 'V' is said to have dimension 'n' and is written as dim(V) = n i.e., the maximum number of linearly independent vectors of 'V' is the dimension of 'V'. The set 'S' of 'n' linearly independent vectors is called the basis of 'V'.

2.3 Linear Dependence and Independence

Vectors in a vector space can be linearly dependent or independent. A set of vectors $\{v_1, v_2, \ldots, v_n\}$ is:

- Linearly dependent if there exist scalars $\alpha_1, \alpha_2, \ldots, \alpha_n$, not all zero, such that $\alpha_1 v_1 + \alpha_2 v_2 + \ldots + \alpha_n v_n = 0$.
- Linearly independent if the only solution to $\alpha_1 v_1 + \alpha_2 v_2 + \ldots + \alpha_n v_n = 0$ is $\alpha_1 = \alpha_2 = \ldots = \alpha_n = 0$.

A set of vectors is linearly independent if and only if no vector in the set can be expressed as a linear combination of the others.

3 Intro to Matrices:

A set of mn numbers (real or complex) or functions arranged in the form of a rectangular array of m horizontal lines and n vertical lines is known as a matrix of order or type $m \times n$

An $m \times n$ matrix is usually written as
$$\begin{bmatrix} a11 & a12 & ... & ... & a1n \\ a21 & a22 & ... & ... & a2n \\ ... & .. & ... & ... & a3n \\ am1 & a12 & ... & ... & amn \end{bmatrix}$$

The above matrix in a compact form is represented by A $= (a_{ij})_{mxn}$ Or $A = [aj]_{mxn}$ where $i = 1, 2, ...m$ and $j = 1, 2, ...n$.

Types of Matrices:

Real matrix:

If all the elements of a matrix A are real numbers then the matrix A is called a real matrix.

For example: $\begin{bmatrix} 3 & 0 & 3 \\ 2 & 2 & 0 \\ -3 & 2 & 1 \end{bmatrix}$

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Complex matrix:

If at least one of the elements of a matrix A is purely imaginary (or) complex then the matrix A is called complex matrix.

For example: $\begin{bmatrix} 3 & 0 & 0 \\ 2 & 2i & 0 \\ -3i & 2 & 1 \end{bmatrix}$

. **Row matrix:**

If a matrix A has only one row and any number of columns then the matrix A is called a row matrix (or) row vector.

For example: $\begin{bmatrix} 4 & -1 & 1 \end{bmatrix}$

Column matrix:

If a matrix A has only one column and any number of rows then the matrix A is called a column matrix (or) column vector.

For example: $\begin{bmatrix} 4 \\ 2 \\ 3 \end{bmatrix}$

Null matrix or zero matrix:

If every element of a matrix is zero then the matrix is called a null matrix and the null matrix of order $m \times n$ is denoted by O_{mn}.

For example: $\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

Rectangular matrix:

If the number of rows is not equal to the number of columns in a matrix then the matrix is called a rectangular matrix.

For examples: $\begin{bmatrix} 4 & -1 & 1 \\ 2 & -3 & -3 \end{bmatrix}$

$\begin{bmatrix} 4 & -1 \\ 2 & -3 \\ 3 & 1 \end{bmatrix}$

Square matrix:

If the number of rows is equal to the number of columns in a matrix then the matrix is called square matrix. A square matrix of order $n \times n$ is sometimes called as n-rowed matrix A or simply a Square matrix of order 'n'.

For example: $\begin{bmatrix} 2 & 3 & 0 \\ 2 & 0 & 1 \\ 0 & 3 & 0 \end{bmatrix}$

Diagonal or principal diagonal elements:

If A = $(a_{ij})_{m \times n}$ then the elements a_{ij} of a square matrix for which $i = j$ i.e., the elements $a_{11}, a_{22}, ...a_{nn}$ are called diagonal elements (or) leading diagonal elements.

For example: $\begin{bmatrix} 4 & -1 & 1 \\ 2 & -3 & -3 \\ 3 & 4 & 3 \end{bmatrix}$, the elements 4, -3 and 3 are called as diagonal element

Principal diagonal:

The line along which the diagonal elements lie is called principal diagonal of the matrix.

Trace of a matrix:

The sum of the diagonal elements of a square matrix A is called trace of A and it is denoted by trace (A) or tr(A).

For example: A= $\begin{bmatrix} 4 & -1 & 1 \\ 2 & -3 & -3 \\ 3 & 4 & 3 \end{bmatrix}$, the tr(A) = 4 + (-3) + 3 = 4

Properties: If A and B are square matrices of order n, then

- $tr(A + B) = tr(A) + tr(B)$
- $tr(A - B) = tr(A) - tr(B)$
- $tr(AB) \neq tr(A)tr(B)$
- $tr(BA) \neq tr(B)tr(A)$
- tr(AB) = tr(BA)
- tr(kA) = k tr(A) where k is a scalar.

Diagonal matrix:
If all the non-diagonal elements in a square matrix are zero, then the matrix is called a diagonal matrix.

For example: $\begin{bmatrix} 3 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & 2 \end{bmatrix}$

Note: A diagonal matrix of order $n \times n$ having di,d2,. ...d, as diagonal elements is denoted by diag[di, d2 , ...dn] or diag [d1, d2 ...dn].

If A and B are diagonal matrices then

- A+B is a diagonal matrix.
- A-B is a diagonal matrix.
- A^2, B^2 are diagonal matrices.
- $A^n \pm B^n$ (n $\in$ N) are diagonal matrices.
- A^T is a diagonal matrix.
- $A^T \pm B^T$ are diagonal matrices.
- $adj(A), adj(B)$ are diagonal matrices.
- A^{-1}, B^{-1} are diagonal matrices.

Scalar matrix:
If all the diagonal elements of a diagonal matrix are same or equal then the matrix is called a scalar matrix.

For example: $\begin{bmatrix} 3 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{bmatrix}$

Unit matrix or identity matrix:
If all the diagonal elements of a diagonal matrix are one then the matrix is called an identity matrix.

For example: $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

Upper triangular matrix:
If all the elements below the principal diagonal are zero in a square matrix then the matrix is called an upper triangular matrix.

For example: $\begin{bmatrix} 3 & 2 & 1 \\ 0 & 2 & 3 \\ 0 & 0 & -2 \end{bmatrix}$
.

Lower triangular matrix: If all the elements above the principal diagonal are zero in a square matrix then the matrix is called a lower triangular matrix.

For example: $\begin{bmatrix} 3 & 0 & 0 \\ 2 & 2 & 0 \\ -3 & 2 & 1 \end{bmatrix}$

Thus, A $= [aj]_{nxn}$ is a lower triangular matrix if $a_{ij} = 0$ for all $i < j$. For example, the matrix is a lower triangular matrix.

Note: A diagonal matrix is both upper and lower triangular matrix.

Triangular matrix: A matrix that is either upper triangular or lower triangular is called a triangular matrix.

Transpose of Matrix If a matrix $B_{n\times m}$ is obtained from a matrix $A_{m\times n}$ by changing its rows into columns and its columns into rows then the matrix $B_{n\times m}$ is called transpose of A and is denoted by A^T

For example: A $= \begin{bmatrix} 4 & -1 \\ 2 & -3 \\ 3 & 4 \end{bmatrix}_{2\times 3}$ then, $A^T = \begin{bmatrix} 4 & 2 & 3 \\ -1 & -3 & 4 \end{bmatrix}_{2\times 3}$

Properties of Transpose:

If $A\&B$ are two matrices and A^T and B^T are transpose of $A\&B$ respectively then

- $(A^T)^T =$ A
- $(A+B)^T = A^T + B^T$
- $(kA)^T = kA^T$ where k is a scalar (real or complex)
- $(AB)^T = B^T A^T$
- $(A^2)^T = (A^T)^2$

Symmetric matrix:

If $a_{ij} = a_{ji}$ for all i,j; then A is called a symmetric matrix.

For example, the matrix $\begin{bmatrix} 1 & 2i & -3 \\ 2i & 4 & 6 \\ -3 & 6 & 0 \end{bmatrix}$

is a symmetric matrix of order 3.

Note: The necessary and sufficient condition for a square matrix A to be symmetric is that $A^T =$ A

Skew-Symmetric matrix:

If $a_{ij} = -a_{ji}$ for all i,j; then A is called a skew-symmetric matrix.

For example, the matrix $\begin{bmatrix} 0 & 2i & -3 \\ -2i & 0 & 6 \\ 3 & -6 & 0 \end{bmatrix}$
is a skew-symmetric matrix of order 3.

Note:

(a). The diagonal elements of a skew-symmetric matrix are all zero.

(b). The necessary and sufficient condition for a square matrix A to be skew-symmetric matrix is that $A^T = -A$.

Properties of symmetric and Skew- symmetric matrices:

If A and B are symmetric matrices then

- A+B pr A-B are symmetric
- AB and BA need not be symmetric
- AB+BA is symmetric
- AB-BA is skew-symmetric
- $A^2, B^2, A^2 + B^2$ are symmetric
- A^3, A^4, B^3, B^4 are symmetric (In general A^k, B^k are symmetric when $k \in N$)
- kA is symmetric when k is a scalar.

If A and A^T are any two square matrices then

- $A + A^T$ is symmetric
- $A - AT, AT - A$ are skew-symmetric.
- AA^T, A^TA are symmetric.

If A and B are skew-symmetric then

- A + B, A -B are skew-symmetric.
- AB, BA are not skew-symmetric.
- $A^2, B^2$are symmetric.
- $A^2 + B^2$ or $A^2 - B^2$ are symmetric.
- A^2, A^4, A^6 are symmetric

- A^3, A^5A^7 are skew-symmetric V
- kA is skew-symmetric.

1. If A and B are square symmetric matrices then AB is symmetric means AB = BA.
2. If A and B are skew-symmetric matrices then AB is symmetric means A and B are commute.
3. The matrix B^TAB is symmetric (or) skew-symmetric according as A is symmetric (or) skew-symmetric.
4. O_{nxn} is symmetric as well as skew-symmetric. $I_n$is symmetric.

Idempotent matrix:

If A^2 =A for a square matrix A of order n then $An \times n$ is called an idempotent matrix.
For example, the matrices $\begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$ are idempotent matrices.

Note:

- If AB = A and BA = B then A and B are idempotent.
- If A is an idempotent matrix then I-A is also an idempotent matrix.
- If AB = BA = O, then the sum of two idempotent matrices A and B is also an idempotent matrix.

Involutary matrix:

If A^2 = In for a square matrix A of order n, then matrix A is called an involutary matrix.
For example, $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ is involuntary matrices

Note: A is involuntary matrix (I+A)(I-A) = 0

Nilpotent matrix:

If there exists a positive integer m for a square matrix A of order n such that A^m = O then the matrix A is called a nilpotent matrix.

If m is a least positive integer for which $A^m = 0$ then 'm' is called the index of the nilpotent matrix.
For example, $\begin{bmatrix} 1 & -1 \\ 1 & -1 \end{bmatrix}$ is a nilpotent matrix of index 2,

Check if A = $\begin{bmatrix} 1 & -3 & 4 \\ -1 & 3 & 4 \\ 1 & -3 & -4 \end{bmatrix}$ is a nilpotent matrix of index 2.

Orthogonal Matrix:

A square matrix $A_{n\times n}$ is said to be an orthogonal matrix if $AA^T = A^TA$ = I or $A^{-1} = A^T$

For examples:

$1/3\begin{bmatrix}1 & 2 & 2\\ 2 & 1 & -2\\ 2 & -2 & 1\end{bmatrix}$ **Refer to the videos/slides for better clarification**

Note: If A and B are Orthogonal matrices then AB and BA are also orthogonal matrices.

4 Minor, Cofactor of an element and Adjoint Matrix

Minor of an element
If A = a_{ij} is a square matrix of order n then the minor of an element a_{ij}in A is the determinant of a square matrix that remains after deleting corresponding the ith row and jth column of A. It is denoted by M_{ij}.

Thus, if A = $\begin{bmatrix} a11 & a12 & a13 \\ a21 & a22 & a23 \\ a31 & a32 & a33 \end{bmatrix}$ then

the minor of a11 is M11 = $\begin{vmatrix} a22 & a23 \\ a32 & a33 \end{vmatrix}$

= a22 * a33 - a32*a23

Similarly, we can calculate for all elements.

Cofactor of an element:
If A = (a_{ij}) is a square matrix of order 'n' then the cofactor of an element aij is denoted by Aij and defined as $(-1)^{i+j}$ M_{ij} where M_{ij} is a minor of a_{ij}.

Note:

1. If A = (a_{ij}) is a square matrix of order 'n' then the sum of the products of the elements of any row (column) with their cofactors is always equal to A or det (A).

2. If A = (a_{ij}) is a square matrix of order 'n' then the sum of the products of elements of any row (or column) with the cofactors of the corresponding elements of some other row (or column) is zero.

Adjoint matrix:
If B is a cofactor matrix of matrix A then the adjoint matrix of A is denoted by adj(A) and is defined as B^T.
$adj(A) = B^T$

Properties of adjoint matrix:

1. IF A is a square matrix of order 'n' then A adj(A) = adj(A) A = $|A|I_n$.
2. If O is a zero matrix of order 'n' then adj(O) = O
3. If I_n is a unit matrix of order 'n° then adj(I_n)= I_n
4. If D is a diagonal matrix of order 'n' then Adj(D) is also a diagonal matrix.
5. If A is a square matrix of order 'n' then $adj(A^T) = (adj(A))^T$.

6. If A and B are non-singular matrices then adj(AB) = adj(B).adj(A).
(If A or B is a singular matrix then adj(AB) = adj(B).adj(A)).

7. $|A| = 0$ then $|adj(A)| = 0$.

8. 1f A is a square matrix of order 'n' then $|adj(A)| = |A|^{n-1}$

9. If A is a non-singular matrix of order 'n' then adi(adj(A)) = $|A|^{n-2}A$

10. If A is a square matrix of order 'n' then $|adj(adj(A)) = |A|^{(n-2)^2}$

11. If A is a square matrix then $adj(A^\theta) = (adj(A))^\theta$

12. If A is a Hermitian matrix then adj(A) is also hermitian.

13. If A is a symmetric matrix then adi (A) is also symmetric.

Singular matrix:

A square matrix A of order 'n' is said to be a singular matrix if $|A_{n\times n}| = 0$.

Non-singular matrix:
A square matrix A of order 'n' is said to be a non-singular matrix if $|A_{n\times n}| \neq 0$

Note:
If A and B are non-singular matrices of the same order then AB is the non-singular matrix of the same order.

5 Inverse of Matrix

Inverse (or) reciprocal of a square matrix:
If for a non-singular matrix A of order 'n' there exists another non-singular matrix B of order 'n' such that AB = BA = I_n then B is called the inverse of A.
It is denoted by A^{-1}.

Therefore, B= A^{-1} and $A^{-1}A = AA^{-1} = I_n$.

Note:
1. If the inverse of a square matrix A exists then the matrix is called an invertible matrix.

2. If A is a non-singular matrix of order n then $A^{-1} = \frac{1}{|A|}adj(A)$

Properties of the inverse of a square matrix:

1. The necessary and sufficient condition for a square matrix A to possess (have) the inverse is that $|A| \neq 0$ (i.e., nonsingular matrix).
2. If the inverse of a square matrix A exists then it is unique.
3. If A, B and C are non-singular matrices then (i) $(AB)^{-1} = B^{-1}A^{-1}$ (reversal law) (ii) $(ABC)^{-1} = C^{-1}B^{-1}A^{-1}$
4. If A is non-singular matrix of order 'n' then $(A^T)^{-1} = (A^{-1})^T$ and $(A^\theta)^{-1} = (A^{-1})^\theta$
5. If A is non-singular and symmetric matrix then A^{-1} is also symmetric.
6. If AB = BA for two non-singular matrices then $A^{-1}B^{-1} = B^{-1}A^{-1}$
7. Cancellation law: If A is a non-singular matrix of order 'n' and B, C are square matrices of same order as A then
   (i) $AB = AC => B = C$(left cancellation law).
   (ii)$BA - CA => B = C$ (right cancellation law).
8. If the product of two non-zero square matrices is a zero matrix then both A and B must be singular matrices.
9. If A and B are 'n' rowed square matrices such that AB = O and B is non-singular matrix then A = O.
10. If A is an idempotent matrix and $A \neq I$ then A is singular.
11. If A is an orthogonal matrix then A^T and A^{-1} are also orthogonal.

12. If A $= \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ and k is scalar then $(kA)^{-1} = \frac{1}{k}A^{-1}$

13. The determinant of skew-symmetric matrix of odd order is zero.

14. If A $= \begin{bmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{bmatrix}$ such that $|A| \neq 0$ then,
$$A^{-1} = \begin{bmatrix} 1/a & 0 & 0 \\ 0 & 1/b & 0 \\ 0 & 0 & 1/c \end{bmatrix}$$

15. $I_n^{-1} = I_n$

6 Rank of Matrix

Definition:
A non-negative integer 'r' is said to be the rank of matrix A, if
(i) there exists at least one non-zero minor of order 'r'
(ii) all minors of order (r + 1) if they exist, are zeros.
then we write Rank of a = $\rho(A)$ = r
Note:

1. Rank of a matrix A = The order of any largest non-vanishing minor of A.
2. Rank of a matrix A = The number of linearly independent rows of A = The number of linearly independent columns of A.
3. A necessary and sufficient condition that the vectors $X_1 = [X_{11}, X_{12}...X_{1m}]$, $X_2 = [X_{21}, X_{22}....X_{2m}]$, $...X_n = [X_{n1}, X_{n2}...X_{2m}]$ of order m are

   (i) Linearly independent is that $\rho(A) = n$
   (ii) linearly dependent is that $\rho(A) \neq n$
   Where A = $\begin{bmatrix} X_{11} & X_{12} & ..... & X_{1m} \\ X_{21} & X_{22} & ..... & X_{2m} \\ .... & ... & ..... & .... \\ X_{n1} & X_{n2} & ..... & X_{nm} \end{bmatrix}$
   and n = number of given vectors.

4. If A is a matrix of order $m \times n$ and $\rho(A)$ = r then the maximum number of linearly independent rows or columns of A is 'r'.

Note:

- Rank of a matrix is zero if A is a null matrix.
- If $A \neq O$ then $\rho(A) \geq 1$.
- If $A_{m\times n} \neq O$ then $\rho(A_{m\times n}) < min(m, n)$
- If $|A_{n\times n}| \neq 0$ then $\rho(A_{n\times n}) = n$
- If $|A_{n\times n}| = 0$ then $\rho(A_{n\times n}) < n$
- $\rho(I_n) = n$
- $\rho(A) = \rho(A^T)$.
- $\rho(AB) \leq min(\rho(A), \rho(B))$

- $\rho(A+B) \leq (\rho(A)+\rho(B))$
- $\rho(A-B) \geq \rho(A)-\rho(B)$
- The rank of a diagonal matrix is equal to the number of non-zero diagonal elements.

7 Echelon form:

A matrix A of order mxn is said to be in row echelon form if
(i) zero rows (if any occur) then they must be below the non-zero rows
(ii) the number of zeros before the first non-zero element in each row is less than the number of such zeros in the next non zero row.
Note: If a matrix A of order mxn is in row echelon form then $\rho(A_{m\times n})$ = number of non-zero rows in A.

Example: A = $\begin{bmatrix} 0 & 2 & 0 & 3 \\ 0 & 0 & 5 & 0 \\ 0 & 0 & 0 & 6 \end{bmatrix}$ It is in row echelon form, so $\rho(A_{4x3}) = 3$

Example: A = $\begin{bmatrix} 0 & 2 & 5 \\ 0 & 0 & -1 \\ 0 & 0 & -4 \end{bmatrix}$ is not in row echelon form.

Because the number of zeros in 2nd row is not less than the number of zeros in the 3rd row.
But, we can reduce A into its echelon form by applying some elementary row operations on it.

By applying R3 $\rightarrow$ R3 - 4R2 on A, we obtain A = $\begin{bmatrix} 0 & 2 & 5 \\ 0 & 0 & -1 \\ 0 & 0 & 0 \end{bmatrix}$ which is in an echelon form.
So, $\rho(A) = 2 =$ the number of non-zero rows.
Elementary operations do not change the order or rank of a matrix.

Note:
1. Rank of a matrix will not be affected by applying the elementary transformations.
2. To find rank of a matrix, we have to reduce the matrix A into its row Echelon form, using only elementary row transformations.
Then, rank of A = number of non-zero rows in the Echelon form of A.

Kindly refer to tutorials for examples and problem-solving

8 Rank and Nullity Theorem for Matrix

- The nullity of a matrix is determined by the difference between the order and rank of the matrix.
- The rank of a matrix is the number of linearly independent row or column vectors of a matrix.
- If n is the order of the square matrix A, then the nullity of A is given by n – r. Thus, the rank of a matrix is the number of linearly independent or non-zero vectors of a matrix, whereas nullity is the number of zero vectors of a matrix.
- The rank of matrix A is denoted as ρ(A), and the nullity is denoted as N(A). Evidently, if the rank of the matrix is equal to the order of the matrix, then the nullity of the matrix is zero

Nullspace
Let A be a real matrix of order m $\times$ n, the set of the solutions associated with the system of homogeneous equation AX = 0 is said to be the null space of A.
Nullspace of $A = \{x \in Rn|Ax = 0\}$. Then the nullity of A will be the dimension of the Nullspace of A.
If the rank of A is r, there are r leading variables in row-reduced echelon form of A and (n – r) free variables, which are solutions of the homogeneous system of equation AX = 0.
Thus, (n – r) is the dimension of the null space of A. This fact motivates the rank and nullity theorem for matrices.

If A is a matrix of order m $\times$ n, then Rank of A + Nullity of A = Number of columns in A = n

Important Facts on Rank and Nullity

- The rank of an invertible matrix is equal to the order of the matrix, and its nullity is equal to zero.
- Rank is the number of leading column or non-zero row vectors of row-reduced echelon form of the given matrix, and the number of zero columns is the nullity.
- The nullity of a matrix is the dimension of the null space of A, also called the kernel of A.
- If A is an invertible matrix, then null space (A) = 0.
- The rank of a matrix is the number of non-zero eigenvalues of the matrix, and the number of zero eigenvalues determines the nullity of the matrix.

9 System of Linear Equations

Non-homogenous system of linear equations

If the system of 'm' non-homogeneous linear equation in 'n' variables $x_1, x_2, ...x_n$ is given by

$a_{11}x_1 + a_{12}x_2 + ... + a_{1n}x_n = b_1$
$a_{21}x_1 + a_{22}x_2 + ... + a_{2n}x_n = b_2$
....
$a_{m1}x_1 + a_{m2}x_2 + ... + a_{mn}x_n = b_m$
then the set of these equations can be written in matrix form as
AX = B.
where A is a coefficient matrix, X is the column matrix of the variables and B is the column matrix of constants b1, b2, ... bn.
Note: The system of linear equations AX = B has a solution (consistent) if and only if Rank of$A$ = Rank of$[A|B]$

Theorem

The system AX = B has (i) a unique solution if and only if Rank (A) = Rank (A—B) = number of variables
(ii) infinitely many solutions if and only if $\rho(A) = \rho(A|B)$ <number of variables, and
(iii) no solution (inconsistent) if $\rho(A) \neq \rho(A|B)$
$\rho(A) < \rho(A|B)$

Homogenous Equations: System of Linear

If the system of 'm' homogeneous linear equation in 'n' variables $x_1, x_2, ...x_n$ is given by
$a_{11}x_1 + a_{12}x_2 + ... + a_{1n}x_n = b_1$
$a_{21}x_1 + a_{22}x_2 + ... + a_{2n}x_n = b_2$
....
$a_{m1}x_1 + a_{m2}x_2 + ... + a_{mn}x_n = b_m$
then the set of these equations can be written in matrix form as
AX = O.
where A is a coefficient matrix, X is the column matrix of the variables

Theorem

The system AX = O has
(i) Unique solution (zero solution or trivial solution), if $\rho(A)$ = number of variables (n) and
(ii) Infinitely many non-zero (or non-trivial) solutions, if $\rho(A) < n$.

Note:

1. In the system of homogenous linear equations AX = O, if A is a square matrix, then
(i) the system possesses only trivial solution (i.e. zero solution or unique solution), if $|A| \neq 0$.
(ii) the system possesses infinitely many non-trivial solution. (i.e. non-zero solutions), if $|A| = 0$.

2. If $\rho(A) = r$, and number of variables = n then, the number of linearly independent solutions of $AX = O$ is (n - r).

3. In a system of homogeneous linear equations, if the number of unknowns (or variables) exceeds the number of equations then the system necessarily possesses a nonzero solution.

Kindly refer to tutorials for examples and problem-solving

10 Eigen Values and Eigen vectors

Let A be a square matrix of order 'n' and $'\lambda'$ be a scalar.

$|A - \lambda 1| = 0$is called the characteristic equation of A.
The roots of the characteristic equation are called eigenvalues (characteristic roots / latent roots) of A.
Corresponding to each eigen value λ, there exists a non-zero vector X such that $AX = \lambda X$
Or $(A - \lambda I)X = O$
Here, X is called an eigen vector (characteristic vector or latent vector) of A.
Properties of eigenvalues:

- For lower triangular matrix (upper triangular matrix or diagonal matrix), the eigen values are same as diagonal elements of the matrix.
- If $\lambda_1, \lambda_2, ...\lambda_n$are eigenvalues of matrix A of order n, then
  1) $\lambda_1 + \lambda_2 + ... + \lambda_n$ = trace of A
  (2) $\lambda_1 \times \lambda_2 \times ...... \times \lambda_n = |A|$
- O is an eigenvalue of matrix A if and only if A is singular.
- If all the eigenvalues of A are non-zero then A is non-singular.
- If is an eigenvalue of a matrix A and k is a scalar then
  (i) λ^m is eigen value of matrix A^m $(m \in N)$
  (ii) kλ is an eigen value of matrix kA.
  (iii) λ+k is an eigen value of matrix $A + kI$
  (iv) λ-k is an eigen value of matrix $A - kI$
  (v) $a_0 + a_1\lambda + a_2\lambda^2$ is an eigen value of matrix $a_0I + a_1A + a_2A^2$
- If λ is an eigenvalue of a non-singular matrix A then
  (1) $1/\lambda$ is an eigen value of A^{-1} and
  (2) $|A|/\lambda$ is eigenvalue of $adj(A)$.
- The eigenvalues of A and A^T are same
- If λ is an eigen value of an orthogonal matrix then $1/\lambda$ is also another eigenvalue of same matrix A.
- If $a + \sqrt{b}$ is an eigen value of a real matrix A then $a - \sqrt{b}$ is also other eigen value of matrix A.
- If a+ib is an eigenvalue of a real matrix A then a-ib is also other eigen value of A.

- The eigen values of a real symmetric (or Hermitian) matrix are always real and the eigen values of a real skew-symmetric (or Skew-Hermitian) matrix are either zero or purely imaginary.
- The eigen values of an orthogonal (or unitary) matrix are of unit modulus.

Properties of eigen vectors:

1. λ is an eigenvalue of a matrix A $\implies$ There exists a non-zero vector X such that $AX = \lambda X$
2. For each eigen value of a matrix there are infinitely many eigenvectors. If X is an eigenvector of matrix A corresponding to the eigenvalue λ then kX (for every non-zero scalar k), is also an eigenvector of A corresponding to the same eigen value λ
3. If X is an eigen vector of a matrix A then X cannot correspond to more than one eigen value of A. i.e., We cannot get same eigen vector for two different eigen values of matrix A.
4. If some eigen values of matrix A are repeated then eigenvectors of A may or may not be linearly independent.
5. If $\lambda_1, \lambda_2, ...\lambda_n$ are distinct eigen values of A square matrix A of order 'n' then the corresponding eigenvectors $X_1, X_2, .., X_n$ of matrix A are linearly independent.
6. The eigenvectors of A and A^m are same
7. The eigen vectors of A and A^{-1} are same
8. The eigen vectors of A and $P(A) = a_0I + a_1A + a_2A^2$ are same
9. The eigen vectors of A and kA are same
10. The eigen vectors of A and A^T are not same.
11. Eigen vectors of symmetric matrix are orthogonal.

Cayley-Hamilton Theorem:
Statement: Every square matrix satisfies its own characteristic equation. For example, If $\lambda^3 - 12\lambda^2 + 36\lambda - 32 = 0$ is a characteristic equations of matrix
A $= \begin{bmatrix} 6 & -2 & 2 \\ -2 & 3 & -1 \\ 2 & -1 & 3 \end{bmatrix}$ then by
Cayley-Harmilton theorem, we have $A^3 - 12A^2 + 36A - 32I = 0$

Applications of Cayley-Hamilton theorem:
The important applications of Cayley-Hamilton theorem are
(i) To find higher powers of matrix A
(ii) To find the inverse of matrix A.

11 Gaussian Elimination Method

Gaussian elimination is a method for solving systems of linear equations by transforming the augmented matrix of the system into row-echelon form and then back-substituting to find the values of the variables.

1. Augmented Matrix: Start with the augmented matrix of the system of linear equations, where the coefficients of the variables and the constants are combined.

For example, consider the system of equations:

$$\begin{aligned} 2x + 3y - z &= 1 \\ 4x + 7y + 2z &= 3 \\ -2x + 5y + 2z &= 7 \end{aligned}$$

The augmented matrix for this system is:

$$\begin{bmatrix} 2 & 3 & -1 & | & 1 \\ 4 & 7 & 2 & | & 3 \\ -2 & 5 & 2 & | & 7 \end{bmatrix}$$

2. Row Reduction: Perform row operations to transform the matrix into row-echelon form (upper triangular form). The goal is to create zeros below the leading entries (the diagonal elements).

3. Example of Row Reduction:

a. Start with the first row (the leading entry) and use it to eliminate the leading entries below it. In this case, divide the first row by 2:

$$\begin{bmatrix} 1 & \frac{3}{2} & -\frac{1}{2} & | & \frac{1}{2} \\ 4 & 7 & 2 & | & 3 \\ -2 & 5 & 2 & | & 7 \end{bmatrix}$$

b. Use the first row to eliminate the leading entries below it:

$$\begin{array}{l} \text{Row } 2 - 4 \times \text{Row } 1: \\ \text{Row } 3 + 2 \times \text{Row } 1: \end{array} \begin{bmatrix} 1 & \frac{3}{2} & -\frac{1}{2} & | & \frac{1}{2} \\ 0 & 1 & 4 & | & 1 \\ 0 & 8 & 1 & | & 8 \end{bmatrix}$$

c. Now, work on the second row to create a zero below the leading entry. In this case, subtract 4 times the second row from the third row:

$$\text{Row } 3 - 8 \times \text{Row } 2: \begin{bmatrix} 1 & \frac{3}{2} & -\frac{1}{2} & | & \frac{1}{2} \\ 0 & 1 & 4 & | & 1 \\ 0 & 0 & -31 & | & 0 \end{bmatrix}$$

4. Back-Substitution: Once you have the matrix in row-echelon form, you can back-substitute to find the values of the variables.

a. Solve the last equation for the last variable:

$$-31z = 0 \implies z = 0$$

b. Substitute the value of z into the second equation to find y

$$y + 4z = 1 \implies y + 4.0 = 1 \implies y = 1$$

$Therefore,$
x = 1

5. Solution: The solution to the system of equations is (x, y, z) found from the back-substitution step.

12 LU Decomposition

LU decomposition of a matrix is the factorization of a given square matrix into two triangular matrices, one upper triangular matrix and one lower triangular matrix, such that the product of these two matrices gives the original matrix.
A square matrix A can be decomposed into two square matrices L and U such that A = L U where U is an upper triangular matrix formed as a result of applying the Gauss Elimination Method on A, and L is a lower triangular matrix with diagonal elements being equal to 1.
Consider the system of equations in three variables:

$$a_{11}x_1 + a_{12}x_2 + a_{13}x_3 = b_1$$
$$a_{21}x_1 + a_{32}x_2 + a_{23}x_3 = b_2$$
$$a_{31}x_1 + a_{32}x_2 + a_{33}x_3 = b_3$$

These can be written in the form of AX = B as:

$$\text{A} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}, \text{X} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \text{ and B} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$$

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$$

Which can be written in the form

$$AX = B$$

Let, $A = LU$

where
$$L = \begin{bmatrix} 1 & 0 & 0 \\ l_{21} & 1 & 0 \\ l_{31} & l_{32} & 1 \end{bmatrix}$$

and
$$U = \begin{bmatrix} u_{11} & u_{12} & u_{13} \\ 0 & u_{22} & u_{23} \\ 0 & 0 & u_{33} \end{bmatrix}$$

Now, the equation becomes. LUX = B
Let us assume UX = Y, then the equation becomes LY = B

Where $\text{Y} = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}$

Now, solve LY = B and find out the value of y_1, y_2 and y_3.
After this solve UX = Y and solve for x_1, x_2 and x_3.

Important for GATE

This method is based on the fact that a square matrix A can be factorised into the form LU where L is unit lower triangular and U is a upper triangular, if all the principal minors of A are non singular ie., it is a standard result of linear algebra that such a factorisation, when it exists, is unique.

Consider A = $\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$

If in the exam question asked for the matrix on which LU decomposition can be applied, then all the principal minors of A must be non-singular

First order minor $a_{11} \neq 0$

Second order minor $\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} \neq 0$

Third order minor $\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} \neq 0$

All above conditions must satisfied in order matrix A to be LU decomposition.

LU Decomposition Example (3x3 Matrix)

Let's perform LU decomposition for the following 3x3 matrix A:

$$A = \begin{bmatrix} 2 & 3 & -1 \\ 4 & 7 & 2 \\ -2 & 5 & 2 \end{bmatrix}$$

Refer to the video lectures to get the detailed solution

13 Introduction to SVD

(Only concept will be required for GATE DA exam.)

Singular Value Decomposition (SVD) is a fundamental matrix factorization technique used in linear algebra and various applications in data analysis, signal processing, and machine learning. It decomposes a matrix into three simpler matrices, revealing essential information about the data's structure and properties.

We decomposed the given matrix A into U, $\sum$ and V^T
A $= U \sum V^T$, where,

U= Matrix of a unit eigen vectors of $A.A^T$

$\sum$= diagonal matrix of singular eigen values of $A.A^T$ or $A^T.A$

V= Matrix of a unit eigen vectors of $A^T.A$
Components of SVD

SVD decomposes a matrix A into three matrices:

1. U: Left singular vector matrix.
2. Σ: Diagonal matrix of singular values.
3. V^T: Transpose of the right singular vector matrix.

SVD is applicable to any real or complex matrix, not just square ones, making it a versatile tool in data analysis.

Calculation of SVD
Given a matrix A of dimensions $m \times n$, SVD is calculated as follows:

1. Compute the matrices $A^T A$ and AA^T.
2. Find the eigenvalues and eigenvectors of $A^T A$ and AA^T.
3. The square root of the eigenvalues of $A^T A$ or AA^T forms the singular values in Σ.
4. The eigenvectors of $A^T A$ are the columns of U, and those of AA^T are the columns of V.

Singular Values and Rank
The singular values in Σ are ordered in descending order. The number of non-zero singular values is the rank of the matrix A. SVD allows you to determine the rank of A and to

identify the most significant features or dimensions in the data.

Dimensionality Reduction

SVD is used for dimensionality reduction. By keeping only the top k singular values and corresponding columns of U and V, you can approximate the original matrix A with reduced dimensions. This is particularly useful in reducing noise, compressing data, and identifying principal components in applications like image compression and recommendation systems.

Watch the video on SVD to understand the concept using example

Note:

If A is $m \times n$ matrix then,

U is $m \times m$ matrix

$\sum$ is $m \times n$ matrix

V^T is $n \times n$ matrix

i:e,

$A_{m\times n} = \mathrm{U}_{m\times m} \cdot \sum_{m\times n} . V^T_{n\times n}$

14 Introduction to Projection Matrices

Projection matrices are fundamental in linear algebra and various fields, providing a means to project vectors onto subspaces. They are extensively used in geometry, optimization, statistics, and computer graphics, among others. In this set of notes, we will explore the key concepts and properties of projection matrices, along with practical examples.

Projection onto a Subspace

The primary purpose of a projection matrix is to project a vector onto a subspace. Consider a vector v and a subspace S. The projection of v onto S, denoted as $\text{proj}_S(v)$, is the vector in S that is closest to v. Projection matrices enable us to compute this projection efficiently.

Properties of Projection Matrices

Projection matrices possess the following properties:

Idempotent Property

A projection matrix P is idempotent, meaning that applying the projection twice results in the same vector:

$$P^2 = P$$

Symmetry

If the subspace S is orthogonal, the projection matrix P is symmetric:

$$P = P^T$$

Rank and Trace

The rank of a projection matrix P is equal to the dimension of the subspace S. Furthermore, the trace of P is equal to the rank of P

Projection Matrix Calculation

The projection of a vector v onto a subspace with an orthogonal basis $\{u_1, u_2, \ldots, u_k\}$ can be computed using the projection matrix P:

$$P = \sum_{i=1}^{k} \frac{u_i \cdot v}{\|u_i\|^2} u_i$$

Here, $u_i \cdot v$ represents the dot product of u_i and v, and $\|u_i\|$ is the norm of u_i.

Example 1: Projection onto a Line

Consider a line S in 2D space spanned by the basis vector $\mathbf{u} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$. Given a vector $\mathbf{v} = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$, let's calculate the projection of $\mathbf{v}$ onto S:

$$P = \frac{\mathbf{u} \cdot \mathbf{v}}{\|\mathbf{u}\|^2} \mathbf{u} = \frac{5}{2} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} \frac{5}{2} \\ \frac{5}{2} \end{bmatrix}$$

The projection of $\mathbf{v}$ onto S is $\begin{bmatrix} \frac{5}{2} \\ \frac{5}{2} \end{bmatrix}$.

Example 2: Projection onto a Plane

Now, consider a plane S in 3D space spanned by the basis vectors $\mathbf{u}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$ and $\mathbf{u}_2 = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$. Given a vector $\mathbf{v} = \begin{bmatrix} 2 \\ 3 \\ 4 \end{bmatrix}$, let's calculate the projection of $\mathbf{v}$ onto S:

$$P = \frac{\mathbf{u}_1 \cdot \mathbf{v}}{\|\mathbf{u}_1\|^2}\mathbf{u}_1 + \frac{\mathbf{u}_2 \cdot \mathbf{v}}{\|\mathbf{u}_2\|^2}\mathbf{u}_2 = 2 \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + 3 \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ 3 \\ 3 \end{bmatrix}$$

The projection of $\mathbf{v}$ onto S is $\begin{bmatrix} 2 \\ 3 \\ 3 \end{bmatrix}$.

15 Partitioned Matrix

- A partitioned matrix is a division of a matrix into smaller rectangular matrices called submatrices or blocks. This is why a partitioned matrix is sometimes called a block matrix.
- Partitioned matrices often appear in modern applications of linear algebra because the notation simplifies many discussions and highlights essential structure in matrix calculations, especially when dealing with matrices of great size.
- Consequently, being able to subdivide, or block, a matrix using horizontal and vertical rules creates compatible smaller matrices to make things easier for us to use.

A matrix can be partitioned in various ways. For example, suppose we are given the following matrix.

$$A = \begin{bmatrix} 1 & -3 & 0 & 4 & 2 & -5 \\ -2 & 9 & 5 & -1 & 0 & 3 \\ -4 & 7 & 1 & 3 & -6 & -8 \end{bmatrix}$$

We can partition the matrix into submatrices or blocks as follows.

$$A = \left[\begin{array}{ccc|cc|c} 1 & -3 & 0 & 4 & 2 & -5 \\ -2 & 9 & 5 & -1 & 0 & 3 \\ \hline -4 & 7 & 1 & 3 & -6 & -8 \end{array}\right]$$

Such that $A = \begin{bmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \end{bmatrix}$ whose entries are the "blocks"

$A_{11} = \begin{bmatrix} 1 & -3 & 0 \\ -2 & 9 & 5 \end{bmatrix}$, $A_{12} = \begin{bmatrix} 4 & 2 \\ -1 & 0 \end{bmatrix}$, $A_{13} = \begin{bmatrix} -5 \\ 3 \end{bmatrix}$, $A_{21} = \begin{bmatrix} -4 & 7 & 1 \end{bmatrix}$
$A_{22} = \begin{bmatrix} 3 & -6 \end{bmatrix}$, $A_{23} = \begin{bmatrix} -8 \end{bmatrix}$

Example

$$A = \left[\begin{array}{cc|c} 2 & 1 & 0 \\ -3 & 5 & -4 \\ 1 & -2 & -2 \\ \hline 0 & 3 & 7 \\ 8 & 6 & -9 \end{array}\right]$$

First, we notice that matrix is divided into 4 blocks, such that $A = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}$ where the first number in the subscript indicates the row and the second number identifies the column.
Now we can name each submatrix and indicate its dimension:

$$A_{11} = \begin{bmatrix} 2 & 1 \\ -3 & 5 \\ 1 & -2 \end{bmatrix}, 3 \times 2$$

$$\mathrm{A}_{12} = \begin{bmatrix} 0 \\ -4 \\ -2 \end{bmatrix}, 3 \times 1$$

$$\mathrm{A}_{21} = \begin{bmatrix} 0 & 3 \\ 8 & 6 \end{bmatrix}, 2 \times 2$$

$$\mathrm{A}_{22} = \begin{bmatrix} 7 \\ -9 \end{bmatrix}, 2 \times 1$$

Operations with Partitioned Matrices
If matrices are the same size and are partitioned in precisely the same way, then we can multiply by scalars and add or subtract partitioned matrices the same way we would with ordinary matrix sums and differences, except each block of $A + B$ is a matrix sum For example, let $A = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}$ and $B = \begin{bmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{bmatrix}$ be partitioned matrices, then

$$A + B = \begin{bmatrix} A_{11} + B_{11} & A_{12} + B_{12} \\ A_{21} + B_{21} & A_{22} + B_{22} \end{bmatrix}$$

$$\text{A-B} = \begin{bmatrix} A_{11} - B_{11} & A_{12} - B_{12} \\ A_{21} - B_{21} & A_{22} - B_{22} \end{bmatrix}$$

$$\text{c A+d B} = \begin{bmatrix} cA_{11} + dB_{11} & cA_{12} + dB_{12} \\ cA_{21} + dB_{21} & cA_{22} + dB_{22} \end{bmatrix}$$

Similary, we can also perform the multiplication.

Partitioned Matrices and Identity or Zero Matrix

For example, suppose matrix A is as shown below. $A = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 1 & 1 & 1 & 0 & 0 & 1 \end{bmatrix}$
Just looking at this matrix can make your eyes glass over. But if we partition it just right, we start to realize that it's really not as bad as we first imagined. Look below:

$$A = \left[\begin{array}{ccc|cc|c} 1 & 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 & 1 \\ \hline 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ \hline 1 & 1 & 1 & 0 & 0 & 1 \end{array}\right]$$

$$= \begin{bmatrix} I_{3\times3} & 0_{3\times2} & A_{3\times1} \\ 0_{2\times3} & I_{2\times2} & 0_{2\times1} \\ A^T_{1\times3} & 0_{1\times2} & 1 \end{bmatrix}$$

$$\text{where} \quad \text{A} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

Block Diagonal Matrices and Inverses of Partitioned Matrices A partitioned matrix A is "block diagonal" if the matrices on the main diagonal are square and all other position matrices are zero.

$$\text{A} = \begin{bmatrix} D_1 & 0 & \cdots & 0 \\ 0 & D_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & D_k \end{bmatrix}$$

And since all the matrices $D_1 D_2, \ldots, D_k$ are square matrices, then matrix is invertible if and only if each matrix on the diagonal is invertible

$$\text{A}^{-1} = \begin{bmatrix} D_1^{-1} & 0 & \cdots & 0 \\ 0 & D_2^{-1} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & D_k^{-1} \end{bmatrix}$$

Example:

Find A inverse if A is a block diagonal matrix.

$$A = \left[\begin{array}{cc|cc|c} 1 & -1 & 0 & 0 & 0 \\ 1 & 2 & 0 & 0 & 0 \\ \hline 0 & 0 & 5 & 4 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ \hline 0 & 0 & 0 & 0 & 2 \end{array}\right]$$

All we have to do is find the inverse of each block along the main diagonal, as the other blocks are 0.

$$A = \left[\begin{array}{cc|cc|c} 1 & -1 & 0 & 0 & 0 \\ 1 & 2 & 0 & 0 & 0 \\ \hline 0 & 0 & 5 & 4 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ \hline 0 & 0 & 0 & 0 & 2 \end{array}\right]$$

$$= \begin{bmatrix} A_{11} & 0 & 0 \\ 0 & A_{22} & 0 \\ 0 & 0 & A_{33} \end{bmatrix}$$

$$A_{11} = \begin{pmatrix} 1 & -1 \\ 1 & 2 \end{pmatrix}$$

$$\Rightarrow A_{11}^{-1} = \tfrac{1}{3}\begin{pmatrix} 2 & 1 \\ -1 & 2 \end{pmatrix}$$

$$= \begin{pmatrix} 2/3 & 1/3 \\ -1/3 & 2/3 \end{pmatrix}$$

$$A_{22} = \begin{pmatrix} 5 & 4 \\ 1 & 1 \end{pmatrix}$$

$$\Rightarrow A_{22}^{-1} = \tfrac{1}{1}\begin{pmatrix} 1 & -4 \\ -1 & 5 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & -4 \\ -1 & 5 \end{pmatrix}$$

$$A_{33} = (2)$$

$$\Rightarrow A_{33}^{-1} = (\tfrac{1}{2})$$

Therefore,

$$A^{-1} = \left[\begin{array}{cc|cc|c} 2/3 & 1/3 & 0 & 0 & 0 \\ -1/3 & 2/3 & 0 & 0 & 0 \\ \hline 0 & 0 & 1 & -4 & 0 \\ 0 & 0 & -1 & 5 & 0 \\ \hline 0 & 0 & 0 & 0 & 1/2 \end{array}\right].$$