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GATE AIR - 13

M.Tech in Data Science From IIT Guwahati

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Linear Algebra Practice - VII

Q 1

Perform the Singular Value Decomposition (SVD) of the matrix A given below. Find matrices $U, \Sigma,$ and V such that $A = U\Sigma V^T$.

The rectangular matrix A is:

$$A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix}.$$

Q 2

If A is a 10×3 matrix with an SVD $A = U\Sigma V^T$, where

$$\Sigma = \begin{bmatrix} 100 & 0 & 0 \\ 0 & 10 & 0 \\ 0 & 0 & 1 \end{bmatrix},$$

answer the following:

(i) What is the size of U?

(ii) What is the size of V?

(iii) What is the rank of A?

(iv) What are the eigenvalues of AA^T?

(v) What are the eigenvalues of A^TA?

Q 3 [MSQ]

Let M be a 3×2 real matrix having a singular value decomposition as $M = USV^T$, where the matrix

$$S = \begin{bmatrix} \sqrt{3} & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}^T,$$

U is a 3×3 orthogonal matrix, and V is a 2×2 orthogonal matrix. Then which of the following statements is/are true?

(A) The rank of the matrix M is 1.

(B) The trace of the matrix $M^T M$ is 4.

(C) The largest singular value of the matrix $(M^T M)^{-1} M^T$ is 1.

(D) The nullity of the matrix M is 1.

Q 4 [MSQ]

Which of the following statements are true?

(a) For a matrix A, in the Singular Value Decomposition (SVD) $A = U\Sigma V^T$, the factor U is an orthogonal matrix.

(b) For a symmetric positive definite matrix A, all pivots are positive.

(c) A 5×5 matrix B has eigenvalue $\lambda = 3$ with algebraic multiplicity 5 and geometric multiplicity 1, and the matrix B is therefore diagonalizable.

(d) All the eigenvalues of a real symmetric matrix are real.

Q 5

Consider the following matrix and its singular value decomposition $A = U\Sigma V^T$:

$$A = \begin{pmatrix} 1/\sqrt{10} & 1/\sqrt{15} & 1/\sqrt{2} & -1/\sqrt{3} \\ -2/\sqrt{10} & 3/\sqrt{15} & 0 & 0 \\ 2/\sqrt{10} & 2/\sqrt{15} & 0 & 1/\sqrt{3} \\ -1/\sqrt{10} & -1/\sqrt{15} & 1/\sqrt{2} & 1/\sqrt{3} \end{pmatrix} \begin{pmatrix} 3 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 1/\sqrt{3} & -1/\sqrt{6} & -1/\sqrt{2} \\ -1/\sqrt{3} & -2/\sqrt{6} & 0 \\ 1/\sqrt{3} & -1/\sqrt{6} & 1/\sqrt{2} \end{pmatrix}^T.$$

(a) A is a ___ $\times$ ___ matrix of rank $r =$ ___.

(b) Find orthonormal bases of the four fundamental subspaces of A.

Q 6

Throughout this problem, the matrix A has the following Singular Value Decomposition:

$$A = \underbrace{\frac{1}{3}\begin{bmatrix} 2 & 2 & -1 \\ x & 2 & 2 \\ 2 & -1 & 2 \end{bmatrix}}_{U} \underbrace{\begin{bmatrix} 3 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}}_{\Sigma} \underbrace{\frac{1}{5}\begin{bmatrix} 4 & -3 \\ 3 & y \end{bmatrix}}_{V^T}$$

where the matrices U and V are orthogonal and x, y denote two mystery real numbers.

1. What are the values of x and y?
2. Fill in the blanks:
   - The rank of the matrix A is ____.
   - The eigenvalues of A^TA are ____, and the eigenvalues of AA^T are ____.
   - A non-zero eigenvector of AA^T is $[\_\_, \_\_, \_\_]^T$.

Q 7 [MSQ]

Which of the following statements are true?

(a) The singular values of a diagonalizable, invertible 2×2 matrix are the absolute values of its eigenvalues.

(b) If S is symmetric, then either S or $-S$ is positive-semidefinite.

(c) If

$$A = \begin{bmatrix} -1 & 2 \\ -3 & 4 \end{bmatrix},$$

and $x \neq 0$, then $\|A^n x\| \to \infty$ as $n \to \infty$.

(d) If λ is an eigenvalue of AA^T, then λ is also an eigenvalue of $A^T A$.

(e) Any invertible matrix is diagonalizable.

Q 8 [MSQ]

Which of the following statements are true?

(a) If A is a 3×3 matrix that has eigenvalues 1 and -1, both of algebraic multiplicity one, then A is diagonalizable (over the real numbers).

(b) Let V be a subspace of $\mathbb{R}^n$ and let P_V be the matrix for projection onto V. Then P_V is diagonalizable.

(c) Any eigenvector of A with a nonzero eigenvalue is contained in the column space of A.

(d) A positive definite symmetric matrix has positive numbers on the main diagonal.

Q 9

The matrix A has a nullspace $N(A)$ spanned by

$$\begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix},$$

and a left nullspace $N(A^T)$ spanned by

$$\begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}, \quad \begin{bmatrix} 1 \\ 1 \\ -1 \\ -1 \end{bmatrix}.$$

(a) What is the shape of the matrix A and its rank?

(b) If we consider the vector

$$b = \begin{bmatrix} -1 \\ \alpha \\ 0 \\ \beta \end{bmatrix},$$

for what value(s) of α and β (if any) is $Ax = b$ solvable? Will the solution (if any) be unique?

(c) Give the orthogonal projections of

$$y = \begin{bmatrix} 1 \\ 2 \\ -3 \end{bmatrix}$$

onto two of the four fundamental subspaces of A.

Q 10

You are given a matrix

$$A = \begin{bmatrix} 1 & 2 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 1 \\ 1 & 0 & 1 \end{bmatrix}.$$

(a) Determine the **number of non-zero singular values** of A, A^T, and $A^T A$.

(b) Give bases for the column space ($C(A)$), null space ($N(A)$), and the null space of the transpose ($N(A^T)$).

Q 11

Let A be a real 3×3 matrix. The matrix $B = A + A^T$ has eigenvalues $\lambda_1 = 2$, $\lambda_2 = 0$, and $\lambda_3 = 1$, with corresponding eigenvectors:

$$x_1 = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}, \quad x_2 = \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix}, \quad x_3 = \begin{bmatrix} 1 \\ 2 \\ -5 \end{bmatrix}.$$

(a) Find the matrix e^B.

(b) Let $C = (I - B)(I + B)^{-1}$. What are the eigenvalues of C?

Q 12

The matrix A has the diagonalization $A = X\Lambda X^{-1}$ with

$$X = \begin{pmatrix} 1 & 1 & -1 & 0 \\ & 1 & 2 & 1 \\ & & 1 & 0 \\ & & & 1 \end{pmatrix}, \quad \Lambda = \begin{pmatrix} 1 & & & \\ & 2 & & \\ & & -2 & \\ & & & -1 \end{pmatrix}.$$

Give a basis for the nullspace $N(M)$ of the matrix $M = A^4 - 2A^2 - 8I$.

Q 13,14

Let $A = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$ and I_3 be the 3×3 identity matrix. Determine the nullity of $5A(I_3 + A + A^2)$.

Let A be the 2×2 real matrix having eigenvalues 1 and -1, with corresponding eigenvectors

$$\begin{bmatrix} \frac{\sqrt{3}}{2} \\ \frac{1}{2} \end{bmatrix} \text{ and } \begin{bmatrix} -\frac{1}{2} \\ \frac{\sqrt{3}}{2} \end{bmatrix},$$

respectively. If $A^{2021} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, then $a + b + c + d$ equals ______.

Q 15

Let M be the collection of all 3×3 real symmetric positive definite matrices. Consider the set

$$S = \left\{ A \in M : A^{50} - \frac{1}{4} A^{48} = 0 \right\},$$

where 0 denotes the 3×3 zero matrix. Then the number of elements in S equals:

(A) 0

(B) 1

(C) 8

(D) ∞

Q 16

Let M be a 3×3 non-zero idempotent matrix ($M^2 = M$) and let I_3 denote the 3×3 identity matrix. Determine which of the following statements is **FALSE**:

1. The eigenvalues of M are 0 and 1.
2. $\text{Rank}(M) = \text{Trace}(M)$.
3. $I_3 - M$ is idempotent.
4. $(I_3 + M)^{-1} = I_3 - 2M$.

Options:

(A) 1

(B) 2

(C) 3

(D) 4

Q 17

For real numbers a, b, and c, let

$$M = \begin{bmatrix} a & ac & 0 \\ 1 & c & 0 \\ b & bc & 1 \end{bmatrix}.$$

Which of the following statements is **TRUE**?

1. $\text{Rank}(M) = 3$ for every $a, b, c \in \mathbb{R}$.
2. If $a + c = 0$, then M is diagonalizable for every $b \in \mathbb{R}$.
3. M has a pair of orthogonal eigenvectors for every $a, b, c \in \mathbb{R}$.
4. If $b = 0$ and $a + c = 1$, then M is **NOT** idempotent.

Q 18

Let A be a 2×2 real matrix such that $AB = BA$ for all 2×2 real matrices B. If the trace of A equals 5, determine the determinant of A.

Q 19

Let M be a 2×2 real matrix such that $(I + M)^{-1} = I - \alpha M$, where α is a non-zero real number and I is the 2×2 identity matrix. If the trace of the matrix M is 3, then the value of α is:

Q 20

Which of the following statements are true?

a. If v is an eigenvector of $A^T A$, then Av is also an eigenvector of AA^T.

b. If v is an eigenvector of $A^T A$, then v is an eigenvector of AA^T.

c. The trace of $A^T A$ is equal to the sum of all a_{ij}^2, where a_{ij} are the elements of matrix A.

d. For every rank-one matrix A, the square of the singular value is equal to the sum of all a_{ij}^2, where a_{ij} are the elements of A.

Q 21

Given the real matrix A and two of its eigenvectors:

$$x_1 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \quad x_2 = \begin{bmatrix} 0 \\ i \\ 1 \end{bmatrix},$$

with corresponding eigenvalues $\lambda_1 = 1$ and $\lambda_2 = 2 + i$, answer the following:

(a) Determine the third eigenvalue λ_3 of A and construct the matrix A in terms of its eigenvalues and eigenvectors.

(b) Compute the determinant and trace of A.

(c) Derive the characteristic polynomial of A, $\det(A - \lambda I)$, in terms of λ. Simplify your answer to a polynomial expression.

(d) For the matrix $A^2 - 2I\ e^{A^{-1}}$, determine its eigenvalues and eigenvectors.

Q 22

If A is a 3×3 matrix with $\det(A) = 3$, calculate $\det(A^T A^{-1}) + \det(2A)$.

Q 23

Consider the matrix

$$A = \begin{bmatrix} 3 & 1 \\ 2 & 2 \end{bmatrix},$$

which has an eigenvalue $\lambda_1 = 1$ and a corresponding eigenvector $x_1 = \begin{bmatrix} 1 \\ -2 \end{bmatrix}$.

(a) Determine the other eigenvalue λ_2 and find a corresponding eigenvector $x_2 = \begin{bmatrix} 1 \\ ? \end{bmatrix}$.

(b) Let B be a 2×2 matrix such that $Bx_k = (1 - \lambda_k + \lambda_k^2)x_k$ for the two eigenvectors x_k (where $k = 1, 2$). Compute the matrix B.

Q 24

Let A be a square matrix such that the null space of $A - I$ is spanned by

$$\begin{bmatrix} 1 \\ 2 \end{bmatrix},$$

and the null space of $A - 5I$ is spanned by

$$\begin{bmatrix} 1 \\ -2 \end{bmatrix}.$$

(a) Without performing detailed calculations, determine whether A is Hermitian or not.

(b) Construct the matrix A.

(c) Compute e^{A+I}.

Q 25

For each of the following statements about matrices, determine whether it **must be true**, **may be true**, or **cannot be true**.

(a) If a matrix is diagonalizable, it must/may/cannot have orthogonal eigenvectors.

(b) If a matrix A is not diagonalizable, then $\det(A - \lambda I)$ must/may/cannot have repeated roots.

(c) If $A^n x$ goes to zero as $n \to \infty$ for some x, then A must/may/cannot have an eigenvalue λ with $|\lambda| > 1$.

(d) If $e^{At}x$ goes to zero as $t \to \infty$ for every x, then A must/may/cannot have an eigenvalue λ with $|\lambda| > 1$.

(e) If A has an eigenvector $\begin{bmatrix}1\\2\\3\end{bmatrix}$, then it must/may/cannot have an eigenvector $\begin{bmatrix}-3\\-6\\-9\end{bmatrix}$.

GATE DA 2024 [2 M]

Let

$$u = \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \\ 5 \end{bmatrix},$$

and let $\sigma_1, \sigma_2, \sigma_3, \sigma_4, \sigma_5$ be the singular values of the matrix

$$M = uu^T,$$

where u^T is the transpose of u.

The value of $\sum_{i=1}^{5} \sigma_i$ is ______.