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GATE AIR - 13

M.Tech in Data Science From IIT Guwahati

Expertise in Machine Learning, Deep Learning, Artificial Intelligence, Probability and Statistics

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Problems of Week 2 (September)

Note : These Problems of the Day (POTD) will be the cornerstone of your entire GATE Data Science preparation. Keep in mind, that the goal is to focus on the learning experience these problems offer, rather than just finding the answers.

Start engaging with them now, and you'll notice a significant difference in your exam—trust me.

Happy learning!

POTD #28

The power W dissipated in a resistor is proportional to the square of the voltage V. That is:

$$W = rV^2$$

where r is a constant. Suppose $r = 3$, and the voltage V follows an **exponential distribution** with the probability density function:

$$f_V(v) = 0.2e^{-0.2v} \quad \text{for } v \geq 0$$

Find $E[W]$, the expected value of the power W.

POTD #29

All solutions to the system $Ax = b$ have the form:

$$x = \begin{bmatrix} 2 \\ 1 \end{bmatrix} + c \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \quad \text{for some scalar } c.$$

Which of the following statements is/are correct?

1. Matrix A can have full column rank.
2. The row rank of matrix A is 1.
3. The vector b lies in the space spanned by the first column of matrix A.
4. The vector b lies in the space spanned by all columns of matrix A.
5. Matrix A must have 2 rows.

POTD #30

Consider the system of equations $Ax = b$. Which of the following statements is/are correct?

1. Any solution to $Ax = b$ is any linear combination of a particular solution x_p and a solution in the null space.
2. The set of solutions to $Ax = b$ forms a subspace.
3. If A is invertible, there is no solution x_n in the null space.
4. If A has full column rank, there exists at least one solution x_n in the null space.
5. For any square matrix A, if the solution is unique, then A must have a zero determinant.

POTD #31

Consider the following system of equations $Ax = b$, where:

$$A = \begin{bmatrix} 1 & 1 \\ 1 & 2 \\ -2 & -3 \end{bmatrix}, \quad b = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}.$$

Which of the following statements is/are **not correct**?

1. A solution always exists for any vector b.
2. The column space of matrix A is a subspace of $\mathbb{R}^2$.
3. The null space of matrix A is a subspace of $\mathbb{R}^2$.
4. There exists a vector b such that the system has infinitely many solutions.
5. There always exists a vector b such that the system has a unique solution.

POTD #32

Which of the following subsets of $\mathbb{R}^3$ are subspaces?

1. The set of vectors $\begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$ such that $b_1 + 2b_2 + 3 = 0$.
2. The set of vectors $\begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$ such that $b_1 + b_2 + b_3 = 0$.
3. The set of all vectors that are linear combinations of $\mathbf{v} = \begin{bmatrix} 1 \\ 4 \\ 0 \end{bmatrix}$ and $\mathbf{w} = \begin{bmatrix} 2 \\ 2 \\ 2 \end{bmatrix}$.
4. The set of vectors $\begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$ such that $b_1 \leq b_2 \leq b_3$.
5. The set of vectors $\begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$ such that $b_1 b_2 b_3 = 0$.

POTD #33

Given a training dataset of 500 instances, with each input instance having 6 dimensions (features) and each output being a scalar value, we wish to apply **linear regression** to this data. The linear regression model is represented by the equation:

$$y = \beta_0 + \beta_1 x_1 + \beta_2 x_2 + \cdots + \beta_6 x_6$$

What are the dimensions of the design matrix used in applying linear regression to this data?

- (A) 500×6
- (B) 500×7
- (C) 500×6^2
- (D) None of the above

Similar Problems

Which of the following statements is/are TRUE?

Note: $\mathbb{R}$ denotes the set of real numbers.

(A) There exist $M \in \mathbb{R}^{3\times 3}, p \in \mathbb{R}^3$, and $q \in \mathbb{R}^3$ such that $Mx = p$ has a unique solution and $Mx = q$ has infinite solutions.

(B) There exist $M \in \mathbb{R}^{3\times 3}, p \in \mathbb{R}^3$, and $q \in \mathbb{R}^3$ such that $Mx = p$ has no solutions and $Mx = q$ has infinite solutions.

(C) There exist $M \in \mathbb{R}^{2\times 3}, p \in \mathbb{R}^2$, and $q \in \mathbb{R}^2$ such that $Mx = p$ has a unique solution and $Mx = q$ has infinite solutions.

(D) There exist $M \in \mathbb{R}^{3\times 3}, p \in \mathbb{R}^3$, and $q \in \mathbb{R}^3$ such that $Mx = p$ has a unique solution and $Mx = q$ has no solutions.

Similar Problems

Select all choices that are subspaces of $\mathbb{R}^3$.

Note: $\mathbb{R}$ denotes the set of real numbers.

(A)

$$\left\{ \mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \in \mathbb{R}^3 : \mathbf{x} = \alpha \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + \beta \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \alpha, \beta \in \mathbb{R} \right\}$$

(B)

$$\left\{ \mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \in \mathbb{R}^3 : \mathbf{x} = \alpha^2 \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix} + \beta^2 \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \alpha, \beta \in \mathbb{R} \right\}$$

(C)

$$\left\{ \mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \in \mathbb{R}^3 : 5x_1 + 2x_3 = 0, 4x_1 - 2x_2 + 3x_3 = 0 \right\}$$

(D)

$$\left\{ \mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \in \mathbb{R}^3 : 5x_1 + 2x_3 + 4 = 0 \right\}$$

Answers of book questions

Sam will read either one chapter of his probability book or one chapter of his history book. If the number of misprints in a chapter of his probability book is Poisson distributed with mean 2 and if the number of misprints in his history chapter is Poisson distributed with mean 5, then assuming Sam is equally likely to choose either book, what is the expected number of misprints that Sam will come across? Answer - 3.5

Suppose that $p(x, y)$, the joint probability mass function of X and Y, is given by

$$p(1,1) = 0.5, \quad p(1,2) = 0.1, \quad p(2,1) = 0.1, \quad p(2,2) = 0.3$$

Answer – 5/6,1/6

Calculate the conditional probability mass function of X given that $Y = 1$.

Answers of book questions

The joint density of X and Y is given by

$$f(x,y) = \begin{cases} \frac{1}{2} y e^{-xy}, & 0 < x < \infty,\ 0 < y < 2 \\ 0, & \text{otherwise} \end{cases}$$

What is $E[e^{X/2} \mid Y = 1]$? Answer - 2

A miner is trapped in a mine containing three doors. The first door leads to a tunnel that takes him to safety after two hours of travel. The second door leads to a tunnel that returns him to the mine after three hours of travel. The third door leads to a tunnel that returns him to his mine after five hours. Assuming that the miner is at all times equally likely to choose any one of the doors, what is the expected length of time until the miner reaches safety?

Answer - 10

Answers of book questions

The average salary of newly graduated students with bachelor's degrees in chemical engineering is \$53,600, with a standard deviation of \$3,200. Approximate the probability that the average salary of a sample of 12 recently graduated chemical engineers exceeds \$55,000.

When we attempt to locate a target in two-dimensional space, suppose that the coordinate errors are independent normal random variables with mean 0 and standard deviation 2. Find the probability that the distance between the point chosen and the target exceeds 3.

If X and Y are independent Poisson random variables with respective means λ_1 and λ_2, calculate the conditional expected value of X given that $X + Y = n$.

Answers of book questions

There are n components. On a rainy day, component i will function with probability p_i; on a nonrainy day, component i will function with probability q_i, for $i = 1, \ldots, n$. It will rain tomorrow with probability α. Calculate the conditional expected number of components that function tomorrow, given that it rains.

Suppose the joint density of X and Y is given by

$$f(x, y) = \begin{cases} 4y(x-y)e^{-(x+y)}, & 0 < x < \infty,\ 0 \leq y \leq x \\ 0, & \text{otherwise} \end{cases}$$

Compute $E[X \mid Y = y]$.

Answers of book questions

The following table uses data concerning the percentages of teenage male and female full-time workers whose annual salaries fall in different salary groupings. Suppose random samples of 1,000 men and 1,000 women were chosen. Use the table to approximate the probability that

(a) at least half of the women earned less than $20,000

(b) more than half of the men earned $20,000 or more

(c) more than half of the women and more than half of the men earned $20,000 or more

(d) 250 or fewer of the women earned at least $25,000

(e) at least 200 of the men earned $50,000 or more

(f) more women than men earned between $20,000 and $24,999

Earnings Range	Percentage of Women	Percentage of Men
$4,999 or less	2.8	1.8
$5,000 to $9,999	10.4	4.7
$10,000 to $19,999	41.0	23.1
$20,000 to $24,999	16.5	13.4
$25,000 to $49,999	26.3	42.1
$50,000 and over	3.0	14.9

Book Problems to Practice

We are given the system of equations:

$$x_1 + 2x_2 + 3x_3 + 5x_4 = b_1$$

$$2x_1 + 4x_2 + 8x_3 + 12x_4 = b_2$$

$$3x_1 + 6x_2 + 7x_3 + 13x_4 = b_3$$

The matrix A has rank 2.

1. Find the condition on b_1, b_2, b_3 for $Ax = b$ to have a solution.
2. Describe the column space of A. Which plane in $\mathbb{R}^3$?
3. Describe the nullspace of A. Which special solutions in $\mathbb{R}^4$?

Book Problems to Practice

Suppose you have this information about the solutions to $Ax = b$ for a specific b. What does that tell you about m, n, and r (and A itself)? And possibly about b.

1. There is exactly one solution.
2. All solutions to $Ax = b$ have the form $x = \begin{bmatrix} 2 \\ 1 \end{bmatrix} + c \begin{bmatrix} 1 \\ 1 \end{bmatrix}$.
3. There are no solutions.
4. All solutions to $Ax = b$ have the form $x = \begin{bmatrix} 1 \\ 0 \end{bmatrix} + c \begin{bmatrix} 1 \\ 0 \end{bmatrix}$.
5. There are infinitely many solutions.

Book Problems to Practice

Under what condition on b_1, b_2, b_3 is this system solvable? Include b as a fourth column in the elimination process. Find all solutions when that condition holds:

$$x + 2y - 2z = b_1$$

$$2x + 5y - 4z = b_2$$

$$4x + 9y - 8z = b_3$$

Find the complete solution (also called the *general solution*) to the following system of equations:

$$\begin{bmatrix} 1 & 3 & 1 & 2 \\ 2 & 6 & 4 & 8 \\ 0 & 0 & 2 & 4 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ t \end{bmatrix} = \begin{bmatrix} 1 \\ 3 \\ 1 \end{bmatrix}$$

Book Problems to Practice

Suppose column 5 of matrix U has no pivot.

- Then, x_5 is a _____ variable.
- The zero vector **(is / is not)** the only solution to $Ax = 0$.
- If $Ax = b$ has a solution, then it has _____ solutions.

Suppose row 3 of matrix U has no pivot.

- Then, that row is _____.
- The equation $Ux = c$ is solvable only if _____.
- The equation $Ax = b$ **(is / might not be)** solvable.

Book Problems to Practice

The largest possible rank of a 3 by 5 matrix is _____.

- Then, there is a pivot in every _____ of U and R.
- The solution to $Ax = b$ **(always exists / is unique)**.
- The column space of A is _____.
- An example of such a matrix A is _____.

The largest possible rank of a 6 by 4 matrix is _____.

- Then, there is a pivot in every _____ of U and R.
- The solution to $Ax = b$ **(always exists / is unique)**.
- The null space of A is _____.
- An example of such a matrix A is _____.

Book Problems to Practice

Explain why the following statements are all false:

(a) The complete solution is any linear combination of x_p and x_n.

(b) A system $Ax = b$ has at most one particular solution.

(c) The solution x_p with all free variables set to zero is the shortest solution (minimum length $||x||$).

Find a 2 by 2 counterexample.

(d) If A is invertible, there is no solution x_n in the null space.

Book Problems to Practice

True or False (with reason if true or example to show it is false):

(a) A square matrix has no free variables.

(b) An invertible matrix has no free variables.

(c) An $m \times n$ matrix has no more than n pivot variables.

(d) An $m \times n$ matrix has no more than m pivot variables.

Free Solutions Link-

https://unacademy.com/class/potd-week-2-september/UQHISNKZ

Discuss your Doubts here - https://t.me/ManojGateDA