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Linear Algebra Practice - VI

Q 1 [MSQ]

Let Q be a 5×3 matrix with orthonormal columns. Which of the following statements **must be true**?

a. $\|Qx\| = \|x\|$ for $x \in \mathbb{R}^3$.

b. $\|Q^\top y\| = \|y\|$ for $y \in \mathbb{R}^5$.

c. $QQ^\top = I$.

d. $Q^\top Q = I$.

Q 2

Relate the four fundamental subspaces of $A^\top A$ to the four fundamental subspaces of a real matrix A. Fill in the blanks:

1. The nullspace of $A^\top A$ is the _______ space of A.
2. The left nullspace of $A^\top A$ is the _______ space of A.
3. The column space of $A^\top A$ is the _______ space of $A^\top$.
4. The row space of $A^\top A$ is the _______ space of $A^\top$.

Options:

a. Column

b. Row

c. Null

d. Left null

Q 3 [MSQ]

Which of the following matrices **cannot** be singular for any real square matrix A? Select all correct options:

a. $A^\top A$

b. $A^2 + I$

c. $(A + A^\top)^2 + I$

d. e^{-A}

e. $A + 10^{100} I$

f. $3A^\top A + 4I$

Q 4 [MSQ]

If the eigenvectors of A are linearly independent, then which of the following statements is/are true? Here, X is the matrix whose columns are the eigenvectors of A.

(a) A is invertible.

(b) A is diagonalizable.

(c) X is invertible.

(d) X is diagonalizable.

Q 5

Given matrices:

$$B = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 2 & 1 \\ 2 & 0 & 1 \end{bmatrix}, \quad C = \begin{bmatrix} 2 & 0 & 2 \\ 0 & 2 & 2 \\ 2 & 2 & 2 \end{bmatrix},$$

and $A = C^{-1}B$, answer the following:

a. Compute the first column of A^{-1}.

b. Compute the trace of the matrix $A^{-1}B$.

c. Given an eigenvalue of C, $\lambda_1 = 2$, find a corresponding eigenvector $\mathbf{x}_1 =$ ________.

Q 6

Given:

$$x_1 = \begin{bmatrix} 1 \\ 2 \\ -1 \\ 0 \\ 1 \end{bmatrix}, \quad b = \begin{bmatrix} 3 \\ 1 \\ 0 \\ 1 \\ 2 \end{bmatrix},$$

find the projection of b onto the **orthogonal complement** $S^{\perp}$ of the span S of x_1.

Q 7

Suppose Q is a 4×3 real matrix with orthonormal columns q_1, q_2, q_3. Find the dimensions of the following:

1. $N(Q)$
2. $N(Q^\top)$
3. $N(Q^\top Q)$
4. $N(QQ^\top)$

Q 8

The nullspace $N(A)$ of the real matrix A is spanned by the vector $v = \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix}$.

(a) Find an eigenvector and corresponding eigenvalue of the matrix:

$$B = (3I - A^\top A)(3I + A^\top A)^{-1}.$$

(Give just one eigenvalue with its eigenvector.)

(b) Aside from the eigenvalue identified in part (a), all other eigenvalues λ of B must be:

- Purely real
- Purely imaginary
- Zero
- Negative real part
- Positive real part
- $|\lambda| < 1$
- $|\lambda| > 1$
- $|\lambda| \leq 1$
- $|\lambda| \geq 1$.

Q 9

The matrix A is given as:

$$A = \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}.$$

Diagonalize A and use the formula $A^k = X\Lambda^k X^{-1}$ to compute A^k. Which of the following represents A^k?

(a) $A^k = \frac{1}{2}\begin{bmatrix} 1+3^k & 1-3^k \\ 1-3^k & 1+3^k \end{bmatrix}$

(b) $A^k = \frac{1}{3}\begin{bmatrix} 1+2^k & 1-2^k \\ 1-2^k & 1+2^k \end{bmatrix}$

(c) $A^k = \frac{1}{2}\begin{bmatrix} 1-3^k & 1+3^k \\ 1+3^k & 1-3^k \end{bmatrix}$

(d) $A^k = \frac{1}{3}\begin{bmatrix} 1+3^k & 1-3^k \\ 1-3^k & 1+3^k \end{bmatrix}$

Q 10

Suppose $A^2 = A$, meaning A is idempotent. Answer the following:

1. Which subspace contains the eigenvectors of A corresponding to the larger eigenvalue?
2. Which subspace contains the eigenvectors of A corresponding to the smaller eigenvalue?
3. Is A diagonalizable?

Q 11

Suppose the same matrix X diagonalizes both A and B, meaning they share the same eigenvectors:

$$A = X\Lambda_1 X^{-1}, \quad B = X\Lambda_2 X^{-1}.$$

Which of the following statements is true?

a. $AB = BA$.

b. $AB \neq BA$.

Q 12

Matrix A has eigenvalues and eigenvectors as follows:

- $\lambda_1 = 2$ with eigenvector $x_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$,
- $\lambda_2 = 5$ with eigenvector $x_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$.

Using the formula $A = X\Lambda X^{-1}$, where X is the matrix of eigenvectors and Λ is the diagonal matrix of eigenvalues, compute A.

Q 13,14 [MSQ]

1. If the eigenvalues of A are $2, 2, 5$, then the matrix is certainly:
   (a) Invertible
   (b) Diagonalizable
   (c) Not diagonalizable

2. If the only eigenvectors of A are multiples of $\begin{bmatrix} 1 \\ 4 \end{bmatrix}$, then A has:
   (a) No inverse
   (b) A repeated eigenvalue
   (c) No diagonalization $X \Lambda X^{-1}$

Q 15 [MSQ]

Suppose A has eigenvalues $0, 3, 5$ with independent eigenvectors u, v, w. Which of the following statements is true?

a. $Ax = u$ has no solution.

b. u and v can form a basis for the column space of A.

c. $Ax = v$ has a solution.

d. $Ax = w$ has no solution.

Q 16

Given the matrices:

$$A = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix}, \quad C = \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{bmatrix}.$$

1. Find the rank of A and C.
2. Determine the four eigenvalues of A and C.

Q 17

The block matrix A is defined as:

$$A = \begin{bmatrix} B & C \\ 0 & D \end{bmatrix} = \begin{bmatrix} 0 & 1 & 3 & 0 \\ -2 & 3 & 0 & 4 \\ 0 & 0 & 6 & 1 \\ 0 & 0 & 1 & 6 \end{bmatrix},$$

where:

- The block B has eigenvalues 1 and 2,
- The block C has eigenvalues 3 and 4,
- The block D has eigenvalues 5 and 7.

Find the eigenvalues of the 4x4 matrix A.

Q 18

A 3×3 matrix B is known to have eigenvalues $0, 1, 2$.

(a) What is the rank of B?

(b) What is the determinant of $B^T B$?

(c) What are the eigenvalues of $(B^2 + I)^{-1}$?