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**GATE AIR - 13**

**M.Tech in Data Science From IIT Guwahati**

**Expertise in Linear Algebra, Probability and Statistics, Machine Learning, Deep Learning**

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# Vector Spaces, Projection

## Problems

## Problem #1 [MSQ]

Consider the matrix $A$ and the equation $A\mathbf{x} = \mathbf{b}$, where:

$$A = \begin{bmatrix} 1 & 2 \\ -1 & -2 \\ -2 & -4 \end{bmatrix}$$

Which of the following statements are **correct** regarding the system $A\mathbf{x} = \mathbf{b}$?

**(A)** The system does not have a solution for any vector $\mathbf{b}$.

**(B)** For matrix $A$, all columns are on the same line and all rows are on the same line.

**(C)** For a vector $\mathbf{b}$ that is in the opposite direction to the first column of $A$, no solution exists.

**(D)** The column space of the zero matrix is a line.

**(E)** All rows and columns of matrix $A$ lie on the same line.

## Problem #2 [MSQ]

Given a matrix $A$ of size $3 \times 4$, determine which of the following statements are correct:

1. The equation $A\mathbf{x} = 0$ will have a solution only when $\mathbf{x}$ is the zero vector.
2. If matrix $A$ with rank $r$ is broken into two matrices: $C$, which contains the first $r$ independent columns of $A$, and $R$, then the number of dependent columns in $A$ and $R$ would be the same.
3. If matrix $A$ is broken into $C$ and $R$ as described in option 2, then the rank of $R$ is always equal to the number of columns in $C$.
4. If the matrix $A$ is multiplied by another matrix of size $4 \times 4$ with full rank, the rank of the resulting matrix could be greater than the rank of $A$.

## Problem #3 [MSQ]

For a square matrix $A$, the system $Ax = b$ is transformed into an upper triangular system $Ux = c$ by applying a series of row operations to both $A$ and $b$. Based on this transformation, which of the following statements are correct?

1. $A^{-1}$ exists if and only if $c$ is in the column space of $U$.
2. $A^{-1}$ may not exist even if $b$ is in the column space of $A$.
3. $A^{-1}$ will exist if at least one pivot in $U$ is non-zero.
4. The column space of $A$ and the column space of $A^T$ are always identical.
5. The dimensions of the column space of $A$ and the column space of $A^T$ are always equal.

## Problem #4 [MSQ]

Consider a square matrix $A$ and the system $Ax = b$. Which of the following statements are correct?

1. For the same matrix $A$, there can be two different vectors $b$ such that one results in a unique solution, while the other has no solution.
2. For the same matrix $A$, there can be two different vectors $b$ such that one results in a unique solution, while the other leads to infinitely many solutions.
3. For the same matrix $A$, there can be two different vectors $b$ such that one results in infinitely many solutions, while the other has no solution.
4. If the vector $b$ lies in the column space of $A$, the system $Ax = b$ will always have a unique solution.

## Problem #5 [MSQ]

All solutions to the system $Ax = b$ have the form:

$$x = \begin{bmatrix} 2 \\ 1 \end{bmatrix} + c \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \quad \text{for some scalar } c.$$

Which of the following statements is/are correct?

1. Matrix $A$ can have full column rank.
2. The row rank of matrix $A$ is 1.
3. The vector $b$ lies in the space spanned by the first column of matrix $A$.
4. The vector $b$ lies in the space spanned by all columns of matrix $A$.
5. Matrix $A$ must have 2 rows.

## Problem #6 [MSQ]

Consider the system of equations $Ax = b$. Which of the following statements is/are correct?

1. Any solution to $Ax = b$ is any linear combination of a particular solution $x_p$ and a solution in the null space.
2. The set of solutions to $Ax = b$ forms a subspace.
3. If $A$ is invertible, there is no solution $x_n$ in the null space.
4. If $A$ has full column rank, then there will exist a solution $x_n$ in the null space.
5. For any square matrix $A$, if the solution is unique, then $A$ must have a zero determinant.

## Problem #7 [MSQ]

Consider the following system of equations $Ax = b$, where:

$$A = \begin{bmatrix} 1 & 1 \\ 1 & 2 \\ -2 & -3 \end{bmatrix}, \quad b = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}.$$

Which of the following statements is/are not correct?

1. A solution always exists for any vector $b$.
2. The column space of matrix $A$ is a subspace of $\mathbb{R}^2$.
3. The null space of matrix $A$ is a subspace of $\mathbb{R}^2$.
4. There exists a vector $b$ such that the system has infinitely many solutions.
5. For any vector $b$, the system has a unique solution.

## Problem #8 [MSQ]

Which of the following subsets of $\mathbb{R}^3$ are subspaces?

1. The set of vectors $\begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$ such that $b_1 + 2b_2 + 3 = 0$.
2. The set of vectors $\begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$ such that $b_1 + b_2 + b_3 = 0$.
3. The set of all vectors that are linear combinations of $\mathbf{v} = \begin{bmatrix} 1 \\ 4 \\ 0 \end{bmatrix}$ and $\mathbf{w} = \begin{bmatrix} 2 \\ 2 \\ 2 \end{bmatrix}$.
4. The set of vectors $\begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$ such that $b_1 \le b_2 \le b_3$.
5. The set of vectors $\begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$ such that $b_1 b_2 b_3 = 0$.

## Problem #9 [MSQ]

Given five vectors in $\mathbb{R}^7$, which of the following statements is/are correct regarding finding a basis for the space they span?

1. Place the vectors as columns in matrix $A$, convert $A$ to its reduced row echelon form (RREF). The pivot columns of $R$ form a basis for the space they span.
2. Place the vectors as rows in matrix $A$, convert $A$ to its reduced row echelon form (RREF). The nonzero rows in $R$ form a basis for the space they span.
3. Place the vectors as rows in matrix $A$, convert $A$ to its reduced row echelon form (RREF). The pivot columns of $R$ form a basis for the space they span.
4. Place the vectors as rows in matrix $A$, convert $A$ to its reduced row echelon form (RREF). The rows in $A$ corresponding to the nonzero rows in $R$ form a basis for the space they span, assuming no row swaps are performed.
5. Place the vectors as columns in matrix $A$, convert $A$ to its reduced row echelon form (RREF). The corresponding columns of $A$ (corresponding to the pivot columns of $R$) form a basis for the space they span.

## Problem #10 [MSQ]

**Which of the following statements are correct? (Multiple correct answers)**

1. $A$ and $A^T$ have the same number of pivots.
2. $A$ and $A^T$ have the same left null space.
3. If the row space equals the column space, then $A^T = A$.
4. If $A^T = -A$, then the row space of $A$ equals the column space.
5. If matrices $A$ and $B$ share the same four fundamental subspaces, then $A$ is a multiple of $B$.

## Problem #11 [MSQ]

Given an $m \times n$ matrix $A$ such that $A^T Ax = 0$, which of the following statements are correct?

1. $Ax$ is in the null space of $A$.
2. $Ax$ is in the column space of $A$.
3. $Ax$ is always a zero vector.
4. The rank of matrix $A$ is $n$.

## Problem #12 [MSQ]

Given an $n \times n$ matrix $A$ where $a_{ij}$ represents the element at the $i$-th row and $j$-th column of $A$, and $a_{ij} = i^2 + j^2$, which of the following statements are correct?

1. The column space of $A$ is perpendicular to its null space.
2. The column space of $A$ is perpendicular to its left null space.
3. The row space of $A$ is perpendicular to its null space.
4. The row space of $A$ is perpendicular to the null space of $A^T$.

Problem #13 [MSQ]

Let $\mathbb{R}$ be the set of real numbers, $U$ be a subspace of $\mathbb{R}^3$, and $M \in \mathbb{R}^{3\times 3}$ be the matrix corresponding to the projection onto the subspace $U$.

Which of the following statements is/are **TRUE**?

1. If $U$ is a 1-dimensional subspace of $\mathbb{R}^3$, then the null space of $M$ is a 1-dimensional subspace.
2. If $U$ is a 2-dimensional subspace of $\mathbb{R}^3$, then the null space of $M$ is a 1-dimensional subspace.
3. $M^2 = M$
4. $M^3 = M$

## Problem #14 [MSQ]

Consider the matrix $A = \begin{bmatrix} 1 & 0 \\ 1 & 1 \\ 1 & 2 \end{bmatrix}$ and the vector $b = \begin{bmatrix} 6 \\ 0 \\ 0 \end{bmatrix}$.

Which of the following statements are correct?

1. The orthogonal projection of vector $b$ onto the column space of $A$ is $\begin{bmatrix} 5 \\ 2 \\ -1 \end{bmatrix}$.
2. If $P$ is the projection matrix, then $(I - P)^2 = I - 2P$.
3. If $P$ is the projection matrix that projects onto the column space of $A$, then $I - P$ projects onto the null space of $A$.
4. The projection matrix given by $A(A^T A)^{-1} A^T$ only exists if $A$ has linearly independent columns.

## Problem #15 [MSQ]

Which of the following statements is/are correct?

1. Any set of $n$ independent vectors in $\mathbb{R}^n$ must span $\mathbb{R}^n$.
2. Any set of $n$ vectors in $\mathbb{R}^m$ must be linearly dependent if $n < m$.
3. If the $n$ columns of a matrix $A$ span $\mathbb{R}^n$, then there will always exist a unique solution for $A\mathbf{x} = \mathbf{b}$.
4. The solution of $A\mathbf{x} = \mathbf{b}$ is a linear combination of vectors in the row space and null space of $A$.
5. If for any two square matrices $A$ and $B$, $AB = I$, then $BA = I$.

# Answer Key

| Question | Answer |
| --- | --- |
| 1 | B |
| 2 | 2,3 |
| 3 | 2,5 |
| 4 | 3 |
| 5 | 2,3,4 |
| 6 | 4 |
| 7 | 3 |
| 8 | 2,3 |

| Question | Answer |
| --- | --- |
| 9 | 2,4,5 |
| 10 | 1,4 |
| 11 | 2,3 |
| 12 | 1,2,3,4 |
| 13 | 2,3,4 |
| 14 | 1,4 |
| 15 | 1,3,4,5 |
| | |

# Gate DA 2024 Question 1

Which of the following statements is/are TRUE?

**Note:** $\mathbb{R}$ denotes the set of real numbers.

**Options:**

(A) There exist $M \in \mathbb{R}^{3\times 3}$, $\mathbf{p} \in \mathbb{R}^3$, and $\mathbf{q} \in \mathbb{R}^3$ such that $M\mathbf{x} = \mathbf{p}$ has a unique solution and $M\mathbf{x} = \mathbf{q}$ has infinite solutions.

(B) There exist $M \in \mathbb{R}^{3\times 3}$, $\mathbf{p} \in \mathbb{R}^3$, and $\mathbf{q} \in \mathbb{R}^3$ such that $M\mathbf{x} = \mathbf{p}$ has no solutions and $M\mathbf{x} = \mathbf{q}$ has infinite solutions.

(C) There exist $M \in \mathbb{R}^{2\times 3}$, $\mathbf{p} \in \mathbb{R}^2$, and $\mathbf{q} \in \mathbb{R}^2$ such that $M\mathbf{x} = \mathbf{p}$ has a unique solution and $M\mathbf{x} = \mathbf{q}$ has infinite solutions.

(D) There exist $M \in \mathbb{R}^{3\times 2}$, $\mathbf{p} \in \mathbb{R}^3$, and $\mathbf{q} \in \mathbb{R}^3$ such that $M\mathbf{x} = \mathbf{p}$ has a unique solution and $M\mathbf{x} = \mathbf{q}$ has no solutions.

# Gate DA 2024 Question 2

Select all choices that are subspaces of $\mathbb{R}^3$.

**Note:** $\mathbb{R}$ denotes the set of real numbers.

**Options:**

(A) $\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \in \mathbb{R}^3 : \mathbf{x} = \alpha \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + \beta \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \alpha, \beta \in \mathbb{R}$

(B) $\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \in \mathbb{R}^3 : \mathbf{x} = \alpha^2 \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix} + \beta^2 \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \alpha, \beta \in \mathbb{R}$

(C) $\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \in \mathbb{R}^3 : 5x_1 + 2x_3 = 0, \quad 4x_1 - 2x_2 + 3x_3 = 0$

(D) $\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \in \mathbb{R}^3 : 5x_1 + 2x_3 + 4 = 0$

## Gate DA 2024- Question 3

Let $\mathbb{R}$ be the set of real numbers, $U$ be a subspace of $\mathbb{R}^3$, and $M \in \mathbb{R}^{3\times 3}$ be the matrix corresponding to the projection onto the subspace $U$.

Which of the following statements is/are **TRUE**?

1. If $U$ is a 1-dimensional subspace of $\mathbb{R}^3$, then the null space of $M$ is a 1-dimensional subspace.
2. If $U$ is a 2-dimensional subspace of $\mathbb{R}^3$, then the null space of $M$ is a 1-dimensional subspace.
3. $M^2 = M$
4. $M^3 = M$

# Gate DA Sample paper- Question 4

Given a matrix $A_{m\times n}$, the following statements are made regarding the matrix $A$:

**P.** The column space is orthogonal to the row space.

**Q.** The column space is orthogonal to the left null space.

**R.** The row space is orthogonal to the null space.

**T.** The null space is orthogonal to the left null space.

Which of the statement(s) is/are true?

A) P and Q

B) P and R

C) Q and R

D) P and T

# Answer Key

| Question | Answer |
| --- | --- |
| 1 | B,D |
| 2 | A,C |
| 3 | 2,3,4 |
| 4 | Q,R |