Reachability for Relative Dynamics

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1 Relative dynamics

Robot car (see **Fig. 1**):

$$\dot{x}_r = v_r \cos(\psi_r + \beta_r) \tag{1}$$
$$\dot{y}_r = v_r \sin(\psi_r + \beta_r) \tag{2}$$
$$\dot{v}_r = a_r \tag{3}$$
$$\dot{\psi}_r = \frac{v_r}{l_r} \sin(\beta_r) \tag{4}$$
$$\beta_r = tan^{-1}\left(\frac{l_r}{l_f + l_r} \tan(\delta_f)\right) \tag{5}$$

Human car (see **Fig. 2**):

$$\dot{x}_h = v_h \cos\psi_h \tag{6}$$
$$\dot{y}_h = v_h \sin\psi_h \tag{7}$$
$$\dot{v}_h = a_h \tag{8}$$
$$\dot{\psi}_h = \omega_h \tag{9}$$
$$\tag{10}$$

Referring to Karen Leung's paper, the relative state (denoted by subscript rel) between the robot car and human car is defined with respect to a coordinate system centered on and aligned with the robot car's vehicle frame.

In the new coordinate system, the relative x position aligns to the robot car's orientation, and the relative y position is perpendicular to the relative x:

$$\begin{bmatrix} x_{rel} \\ y_{rel} \end{bmatrix} = \begin{bmatrix} \cos\psi_r & \sin\psi_r \\ -\sin\psi_r & \cos\psi_r \end{bmatrix} \begin{bmatrix} x_h - x_r \\ y_h - y_r \end{bmatrix} \tag{11}$$

The relative angle is defined as $\psi_{rel} = \psi_h - \psi_r$. Since the velocity is defined with respect to the vehicle frame, we cannot define analogous relative velocity states and must include the individual velocity states of each vehicle.

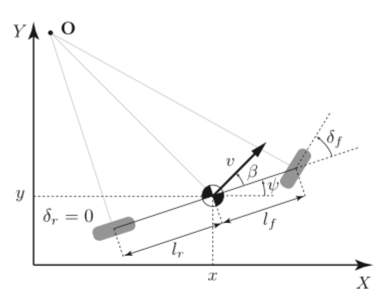
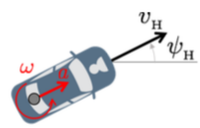


Fig. 1: Kinematic Bicycle Model



In this case, we have the dynamics for relative x as:

$$
\begin{aligned}
\dot{x}_{rel} &= -\sin\psi_r * \dot{\psi}_r * (x_h - x_r) + \cos\psi_r * (\dot{x}_h - \dot{x}_r) + \cos\psi_r * \dot{\psi}_r * (y_h - y_r) + \sin\psi_r * (\dot{y}_h - \dot{y}_r) && (12)\\
&= \dot{\psi}_r * y_{rel} + \cos\psi_r * (v_h \cos\psi_h - v_r \cos(\psi_r + \beta_r)) + \sin\psi_r * (v_h \sin\psi_h - v_r \sin(\psi_r + \beta_r)) && (13)\\
&= \frac{v_r}{l_r}\sin(\beta_r) * y_{rel} + v_h * (\cos\psi_r \cos\psi_h + \sin\psi_r \sin\psi_h) + v_r * (-\cos\psi_r \cos(\psi_r + \beta_r) - \sin\psi_r \sin(\psi_r + \beta_r)) && (14)\\
&= \frac{v_r}{l_r}\sin(\beta_r) * y_{rel} + v_h * \cos\psi_{rel} - v_r * \cos\beta_r && (15)
\end{aligned}
$$

and the dynamics for relative y as:

$$
\begin{aligned}
\dot{y}_{rel} &= -\cos\psi_r * \dot{\psi}_r * (x_h - x_r) - \sin\psi_r * (\dot{x}_h - \dot{x}_r) - \sin\psi_r * \dot{\psi}_r * (y_h - y_r) + \cos\psi_r * (\dot{y}_h - \dot{y}_r) && (16)\\
&= -\dot{\psi}_r * x_{rel} - \sin\psi_r * (v_h \cos\psi_h - v_r \cos(\psi_r + \beta_r)) + \cos\psi_r * (v_h \sin\psi_h - v_r \sin(\psi_r + \beta_r)) && (17)\\
&= -\frac{v_r}{l_r}\sin(\beta_r) * x_{rel} + v_h * (-\sin\psi_r \cos\psi_h + \cos\psi_r \sin\psi_h) + v_r * (\sin\psi_r \cos(\psi_r + \beta_r) - \cos\psi_r \sin(\psi_r + \beta_r)) && (18)\\
&= -\frac{v_r}{l_r}\sin(\beta_r) * x_{rel} + v_h * \sin\psi_{rel} - v_r * \sin\beta_r && (19)\\
& && (20)
\end{aligned}
$$

Therefore, the entire 5D relative dynamics is as follows:

$$
\begin{aligned}
\dot{x}_{rel} &= \frac{v_r}{l_r}\sin(\beta_r) * y_{rel} + v_h * \cos\psi_{rel} - v_r * \cos\beta_r && (21)\\
\dot{y}_{rel} &= -\frac{v_r}{l_r}\sin(\beta_r) * x_{rel} + v_h * \sin\psi_{rel} - v_r * \sin\beta_r && (22)\\
\dot{\psi}_{rel} &= \omega_h - \frac{v_r}{l_r}\sin(\beta_r) && (23)\\
\dot{v}_h &= a_h && (24)\\
\dot{v}_r &= a_r && (25)\\
\beta_r &= tan^{-1}\left(\frac{l_r}{l_f + l_r}\tan(\delta_f)\right) && (26)
\end{aligned}
$$

In this system, the controls are the robot car input a_r, δ_f. The disturbances are the human car input a_h, w_h

2 HJ PDE

Let V as the value function. The Hamiltonian H is:

$$H = \max_{u_r}\min_{u_h}\{\frac{\partial V}{\partial x_{rel}} * \dot{x}_{rel} + \frac{\partial V}{\partial y_{rel}} * \dot{y}_{rel} + \frac{\partial V}{\partial \psi_{rel}} * \dot{\psi}_{rel} + \frac{\partial V}{\partial v_h} * \dot{v}_h + \frac{\partial V}{\partial v_r} * \dot{v}_r\} \tag{27}$$

In our setting, the reachable set is an avoid set, indicating that given any control (from robot), there's a disturbance (from human) that lead to collision. Thus, we will have control maximizes the value function while disturbance minimizes. The corresponding HJ PDE is as follows:

$$\frac{\partial V}{\partial t} + \max_{u_r}\min_{u_h}\{\frac{\partial V}{\partial x_{rel}} * \dot{x}_{rel} + \frac{\partial V}{\partial y_{rel}} * \dot{y}_{rel} + \frac{\partial V}{\partial \psi_{rel}} * \dot{\psi}_{rel} + \frac{\partial V}{\partial v_h} * \dot{v}_h + \frac{\partial V}{\partial v_r} * \dot{v}_r\} = 0 \tag{28}$$

3 Controls and disturbances

Here we want to derive the analytical form of controls and disturbances in order to easily plug in HJ PDE.

3.1 Control

The controls are $a_r \in [a_{r,min}, a_{r,max}], \delta_f \in [\delta_{f,min}, \delta_{f,max}]$, and they maximize the Hamiltonian.

For control a_r, the situation is simple:

$$if \quad \frac{\partial V}{\partial v_r} > 0, a_r = a_{r,max}, \quad else, a_r = a_{r,min} \tag{29}$$

For control δ_f, because

$$\tan(\beta_r) = \frac{l_r}{l_f + l_r}\tan(\delta_f) \tag{30}$$

which indicates that,

$$\beta_r \propto \delta_f, \beta_r \in [\beta_{r,min}, \beta_{r,max}] \tag{31}$$

thus we can think of β_r as the intermediate control variable. To obtain the optimal control of δ_f, we can derive the optimal control of β_r first.

Inside the Hamiltonian, the following terms are related to β_r:

$$\frac{\partial V}{\partial x_{rel}} * (\frac{v_r}{l_r}\sin(\beta_r) * y_{rel} - v_r * \cos\beta_r) + \frac{\partial V}{\partial y_{rel}} * (-\frac{v_r}{l_r}\sin(\beta_r) * x_{rel} - v_r * \sin\beta_r) + \frac{\partial V}{\partial \psi_{rel}} * (-\frac{v_r}{l_r}\sin(\beta_r)) \tag{32}$$

$$= (\frac{\partial V}{\partial x_{rel}} * \frac{v_r}{l_r} * y_{rel} - \frac{\partial V}{\partial y_{rel}} * \frac{v_r}{l_r} * x_{rel} - \frac{\partial V}{\partial y_{rel}} * v_r - \frac{\partial V}{\partial \psi_{rel}} * \frac{v_r}{l_r}) * \sin(\beta_r) \tag{33}$$

$$+ (-\frac{\partial V}{\partial x_{rel}} * v_r) * \cos(\beta_r) \tag{34}$$

To simplify, we rewrite the above equation in the following form:

$$c_1 * \sin(\beta_r) + c_2 * \cos(\beta_r) \tag{35}$$

where:

$$c_1 = \frac{\partial V}{\partial x_{rel}} * \frac{v_r}{l_r} * y_{rel} - \frac{\partial V}{\partial y_{rel}} * \frac{v_r}{l_r} * x_{rel} - \frac{\partial V}{\partial y_{rel}} * v_r - \frac{\partial V}{\partial \psi_{rel}} * \frac{v_r}{l_r} \tag{36}$$

$$c_2 = -\frac{\partial V}{\partial x_{rel}} * v_r \tag{37}$$

When $c_1 > 0$,

$$c_1 * \sin(\beta_r) + c_2 * \cos(\beta_r) = \sqrt{c_1^2 + c_2^2} * \sin(\beta_r + \tan^{-1}(\frac{c_2}{c_1})) \tag{38}$$

Because control maximizes the Hamiltonian, we should select $(\beta_r + \tan^{-1}(\frac{c_2}{c_1}))$ as close as to $\pi/2$, which indicates that β_r should be as close as to $(-\tan^{-1}(\frac{c_2}{c_1}) + \pi/2)$. Note that $(-\tan^{-1}(\frac{c_2}{c_1})) \in (-\pi/2, \pi/2)$.

Thus, if $(-\tan^{-1}(\frac{c_2}{c_1}) + \pi/2) \in [\beta_{r,min}, \beta_{r,max}]$, then

$$\beta_r = -\tan^{-1}(\frac{c_2}{c_1}) + \pi/2 \tag{39}$$

else:

$$if \quad (-\tan^{-1}(\frac{c_2}{c_1}) + \pi/2) \geq \beta_{r,max}, \quad \beta_r = \beta_{r,max} \tag{40}$$

$$if \quad (-\tan^{-1}(\frac{c_2}{c_1}) + \pi/2) \leq \beta_{r,min}, \quad \beta_r = \beta_{r,min} \tag{41}$$

When $c_1 < 0$, if $(-\tan^{-1}(\frac{c_2}{c_1}) - \pi/2) \in [\beta_{r,min}, \beta_{r,max}]$, then

$$\beta_r = -\tan^{-1}(\frac{c_2}{c_1}) - \pi/2 \tag{42}$$

else:

$$if \quad (-\tan^{-1}(\frac{c_2}{c_1}) - \pi/2) \geq \beta_{r,max}, \quad \beta_r = \beta_{r,max} \tag{43}$$

$$if \quad (-\tan^{-1}(\frac{c_2}{c_1}) - \pi/2) \leq \beta_{r,min}, \quad \beta_r = \beta_{r,min} \tag{44}$$

If $c_1 = 0$, in (35) we only have $c_2 * cos(\beta_r)$. Then if $c_2 >= 0$

$$if \quad 0 \in [\beta_{r,min}, \beta_{r,max}], \beta_r = 0, \quad else, \beta_r = min(|\beta_{r,min}|, |\beta_{r,max}|) \tag{45}$$

else if $c_2 < 0$:

$$if \quad |\beta_{r,min}| >= |\beta_{r,max}|, \beta_r = |\beta_{r,min}|, \quad else, \beta_r = |\beta_{r,max}| \tag{46}$$

3.2 Disturbances

The disturbances are $a_h \in [a_{h,min}, a_{h,max}]$, $w_h \in [w_{h,min}, w_{h,max}]$, and they minimize the Hamiltonian.

For disturbances a_h:

$$if \quad \frac{\partial V}{\partial v_h} > 0, a_h = a_{r,min}, \quad else, a_h = a_{h,max} \tag{47}$$

For disturbances w_h:

$$if \quad \frac{\partial V}{\partial \psi_{rel}} > 0, w_h = w_{r,min}, \quad else, w_h = w_{h,max} \tag{48}$$

4 Reachable set

The target set Γ is a cylinder located at the center of the robot car.

$$\Gamma = \{(x_{rel}, y_{rel}) \mid x_{rel}^2 + y_{rel}^2 \leq r^2\} \tag{49}$$

5 Parameter setup

$$l_r = 1.738m, l_f = 1.058m \tag{50}$$

$$\beta_{r,min} = -0.2, \beta_{r,max} = 0.2 \tag{51}$$

$$a_{r,min} = a_{h,min} = -5m/s^2, a_{r,max} = a_{h,max} = 3m/s^2, \tag{52}$$

$$\omega_{h,min} = -0.34rad/s^2, \omega_{h,max} = 0.34rad/s^2 \tag{53}$$

The computation range for $(x_{rel}, y_{rel}, \psi_{rel}, v_h, v_r)$ is $[-10, -10, -\pi, 0, 0] \times [10, 10, \pi, 17, 17]$.