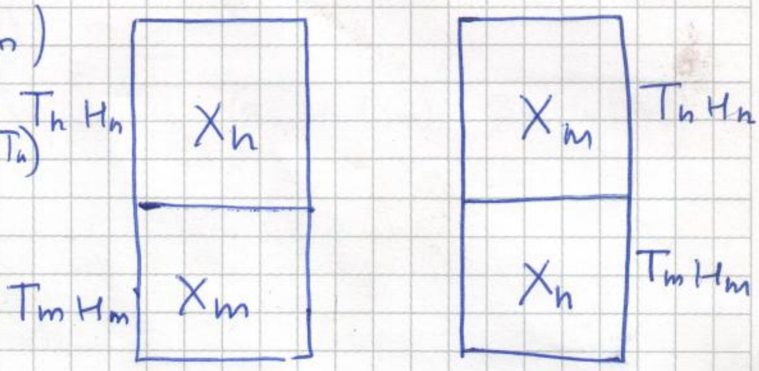
Ⓐ

Top $\left(X_m, H_m(X_m), T_m\right) \to \left(X_n, H_m(X_n), T_m\right)$

Bott $\left(X_n, H_n(X_n), T_n\right) \to \left(X_m, H_n(X_m), T_n\right)$



$$P_i = \frac{1}{Z_\beta} e^{-\beta_x H_x(x_x)}.$$

$\rho_{Bef} = P(\text{win}1) \times P(\text{win}2)$ $\quad \rho_{Af} = P(\text{win}1)\, P(\text{win}2)$

$$\text{So } \rho_{Before} = \frac{e^{-\beta_m H_m(X_m)} \cdot e^{-\beta_n H_n(X_n)}}{Z}$$

$$\rho_{Aft} = e^{-\beta_m H_m(X_n)} \cdot e^{-\beta_n H_n(X_m)}$$

For detailed balance,

$$\rho_{Before}\, \Pi(\text{Before} \to \text{After}) = \rho_{After}\, \Pi(\text{After to Before})$$

$$\Rightarrow \quad \Pi' = \frac{\rho_{Aft}}{\rho_{Bef}}$$

Under some
Metropolis condi

B

$$\frac{\pi(x_m \to x_n)}{\pi(x_n \to x_m)} = \frac{\rho(x_n)}{\rho(x_m)} = \frac{e^{-\left(\beta_m H_m(x_n) + \beta_n H_n(x_m)\right)}}{e^{-\left(\beta_m H_m(x_m) + \beta_n H_n(x_n)\right)}}$$

$$\frac{\pi(x_m \to x_n)}{\pi(x_n \to x_m)} = e^{-\Delta_{nm}} \quad , \quad \text{where}$$

$$\cancel{\Delta_{nm}}\ \Delta_{nm} = +\left[\beta_m H_m(x_n) + \beta_n H_n(x_m)\right] - \left[\beta_m H_m(x_m) + \beta_n H_n(x_n)\right]$$

With the Metropolis criteria, the probability of acceptance for $x_m \to x_n$ becomes,

$$\pi(x_m \to x_n) = \begin{cases} 1 & \text{if } \Delta_{nm} \leq 0 \\ \exp(-\Delta) & \text{if } \Delta_{nm} > 0 \end{cases}$$

$$\Delta = \left[\beta_m H_m(X_m) + \beta_n H_n(X_m)\right] - \left[\beta_m H_m(X_m) + \beta_n H_n(X_n)\right]$$

With some algebra (as show in the code an ~~dsss~~ derived below, the above equation can be written as,

$$\Delta = \beta_n \left[H_n(X_m) - H_m(X_m)\right]$$
$$+ \beta_m \left[H_m(X_n) - H_n(X_n)\right]$$
$$- (\beta_n - \beta_m)\left[H_n(X_n) - H_m(X_m)\right]$$

Algebra shown below:

$$\Delta = \Delta - \beta_m H_n(x_n) + \beta_m H_n(x_n),$$
$$- \beta_n H_m(X_m) + \beta_n H_m(X_m),$$

$$= \beta_m \left[\underline{H_m(X_n)} - \underline{H_n(X_n)}\right] + \beta_n \left[H_n(X_m) - H_m(X_m)\right]$$
$$+ \beta_m \left[\underline{H_n(X_n) - H_m(X_m)}\right] - \beta_n \left[\underline{H_n(X_n) - H_m(X_m)}\right]$$

$$= \beta_n \left[H_n(X_m) - H_m(X_m)\right]$$
$$+ \beta_m \left[H_m(X_n) - H_n(X_n)\right]$$
$$- (\beta_n - \beta_m)\left[H_n(X_n) - H_m(X_m)\right].$$

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$$= \beta_n \left[\mathcal{H}_n(x_m) - \mathcal{H}_m(x_m) \right]$$

$$+ \beta_m \left[\mathcal{H}_m(x_n) - \mathcal{H}_n(x_n) \right]$$

$$- (\beta_n - \beta_m) \left[\mathcal{H}_n(x_n) - \mathcal{H}_m(x_m) \right]$$

(D)

H_1

Switching to E

$$\boxed{\begin{aligned} \mathcal{U}_n &= \frac{\beta_n}{\beta_0} E_{PP} + \sqrt{\frac{\beta_m}{\beta_0}} E_{PW} + E_{WW} \\ &= \lambda_n E_{PP} + \sqrt{\lambda} E_{PW} + E_{WW} \end{aligned}}$$

$$\Delta = \beta_n \left[(\lambda_n - \lambda_m) \underline{E_{PP}(X_m)} + \left(\sqrt{\lambda_n} - \sqrt{\lambda_m}\right) E_{PW}(X_m) + \phi \right]$$

$$+ \beta_m \left[(\lambda_m - \lambda_n) \underline{E_{PP}(X_n)} + \left(\sqrt{\lambda_m} - \sqrt{\lambda_n}\right) E_{PW}(X_n) + \phi \right]$$

$$- (\beta_n - \beta_m) \left[\begin{aligned} &\lambda_n \underline{E_{PP}(X_n)} + \sqrt{\lambda_n} E_{PW}(X_n) + E_{WW}(X_n) \\ &- \lambda_m \underline{E_{PP}(X_m)} - \sqrt{\lambda_m} E_{PW}(X_m) - E_{WW}(X_m) \end{aligned} \right]$$

Coeff of E_{WW}

$$= +(\beta_n - \beta_m) \left[E_{WW}(X_m) - E_{WW}(X_n) \right]$$

$$= -(\beta_n - \beta_m) \left[E_{WW}(X_n) - E_{WW}(X_m) \right]$$

...t of $E_{pp}(x_m)$ & $E_{pp}(x_n)$

$$= \Big[\beta_n(\lambda_n - \lambda_m) + (\beta_n - \beta_m)\lambda_m\Big] E_{pp}(x_m)$$

$$+ \beta_m(\lambda_m - \lambda_n) - (\beta_n - \beta_m)\lambda_n\Big] E_{pp}(x_n)$$

$$= \left(\beta_n\lambda_n - \cancel{\beta_n\lambda_m} + \cancel{\beta_n\lambda_m} - \beta_m\lambda_m\right) E_{pp}(x_m)$$
$$+ \left(\beta_m\lambda_m - \cancel{\beta_m\lambda_n} - \beta_n\lambda_n + \cancel{\beta_m\lambda_n}\right) E_{pp}(x_n)$$

$$\checkmark \quad = (\beta_n\lambda_n - \beta_m\lambda_m)\left[E_{pp}(x_m) - E_{pp}(x_n)\right]$$

$$= -\left(\frac{\beta_n^2 - \beta_m^2}{\beta_0}\right)\left[E_{pp}(x_m) - E_{pp}(x_m)\right]$$

$$=$$

$$\lambda_n = \frac{\beta_n}{\beta_0}$$

Coeff of $E_{pw}(x_m)$ & $E_{pw}(x_n)$

$$= \left(\beta_n\left(\sqrt{\lambda_n} - \cancel{\sqrt{\lambda_m}}\right) + \left(\cancel{\beta_n} - \beta_m\right)\sqrt{\lambda_m}\right) E_{pw}(x_m)$$

$$+ \left(\beta_m\left(\cancel{\sqrt{\lambda_m}} - \cancel{\sqrt{\lambda_n}}\right) - \left(\beta_n - \cancel{\beta_m}\right)\sqrt{\lambda_n}\right) E_{pw}(x_n)$$

$$= \left(\beta_n\sqrt{\lambda_n} - \beta_m\sqrt{\lambda_m}\right)\left[E_{pw}(x_m) - E_{pw}(x_n)\right]$$

$$= -\left(\beta_n\sqrt{\lambda_n} - \beta_m\sqrt{\lambda_m}\right)\left[E_{pw}(x_n) - E_{pw}(x_m)\right]$$

$$= -\left(\frac{\beta_n^{3/2}}{\sqrt{\beta_0}} - \frac{\beta_m^{3/2}}{\sqrt{\beta_0}}\right)\left[E_{pw}(x_n) - E_{pw}(x_m)\right]$$

$$= \frac{\beta_m^2 - \beta_n^2}{\beta_0}\left[E_{pp}(x_n) - E_{pp}(x_m)\right]$$

$$+ \frac{\beta_m^{3/2} - \beta_n^{3/2}}{\sqrt{\beta_0}}\left[E_{pw}(x_n) - E_{pw}(x_m)\right]$$

$$+ (\beta_m - \beta_n)\left[E_{ww}(x_n) - E_{ww}(x_m)\right]$$

Algebra
$$\frac{\beta_m^{3/2} - \beta_n^{3/2}}{\sqrt{\beta_0}} = \frac{\beta_m^{3/2} - \beta_n^{3/2}}{\sqrt{\beta_0}} \times \frac{\sqrt{\beta_m} + \sqrt{\beta_n}}{(\beta_m + \sqrt{\beta_n})}$$

$$= \frac{\beta_m^2 - \beta_n^{3/2}\beta_m^{1/2} + \beta_m^{3/2}\beta_n^{1/2} - \beta_n^2}{\sqrt{\beta_0}\left(\sqrt{\beta_m} + \sqrt{\beta_n}\right)}$$

See rough scratch

$$= \frac{\beta_m^2 - \beta_n\sqrt{\beta_m\beta_n} + \beta_m\sqrt{\beta_n\beta_m} - \beta_n^2}{\sqrt{\beta_0}\left(\sqrt{\beta_m} + \sqrt{\beta_n}\right)} = \frac{(\beta_m - \beta_n)\left[\beta_m + \beta_n + \sqrt{\beta_m\beta_n}\right]}{\sqrt{\beta_0}\left(\sqrt{\beta_m} + \sqrt{\beta_n}\right)}$$

$$\therefore \frac{\Delta}{\beta_m - \beta_n} = \left[\left(\frac{\beta_m + \beta_n}{\beta_0}\right)\left(E_{pp}(x_n) - E_{pp}(x_m)\right)\right]$$

$$+ \frac{\beta_m + \beta_n + \sqrt{\beta_n\beta_m}}{\left(\sqrt{\beta_0}\sqrt{\beta_m} + \sqrt{\beta_n}\right)}\left[E_{pw}(x_n) - E_{pw}(x_m)\right)$$

$$+ E_{ww}(x_n) - E_{ww}(x_m)$$

$$\Delta = (\beta_m - \beta_n) \left[\frac{\beta_m+\beta_n}{\beta_0}\Big(E_{pp}(x_n) - E_{pp}(x_m) \Big) + \frac{\beta_m+\beta_n+\sqrt{\beta_m \beta_n}}{\sqrt{\beta_0}\left(\sqrt{\beta_m}+\sqrt{\beta_n}\right)} \Big(E_{pw}(x_n) - E_{pw}(x_m) \Big) + E_{ww}(x_n) - E_{ww}(x_m) \right]$$

$$\Delta = (\beta_m - \beta_n) \left[\frac{\beta_m+\beta_n}{\beta_0}\Big(E_{pp}(x_n) - E_{pp}(x_m) \Big) + \frac{\beta_m^{3/2} - \beta_n^{3/2}}{\sqrt{\beta_0}\cdot(\beta_m-\beta_n)} \Big(E_{pw}(x_n) - E_{pw}(x_m) \Big) + E_{ww}(x_n) - E_{ww}(x_m) \right]$$

$$\Delta = \left(\frac{\beta_m^2 - \beta_n^2}{\cancel{\sqrt{}}\beta_0}\right)\Big[E_{pp}(x_n) - E_{pp}(x_m) \Big] + \frac{\beta_m^{3/2} - \beta_n^{3/2}}{\sqrt{\beta_0}} \Big(E_{pw}(x_n) - E_{pw}(x_m) \Big) + (\beta_m - \beta_n)\Big(E_{ww}(x_n) - E_{ww}(x_m) \Big)$$

H1

$$\Delta_{nm} = \frac{\beta_n^2 - \beta_m^2}{\beta_0}\left(E_{pp}(x_n) - E_{pp}(x_m)\right)$$

$$+ \frac{\beta_n^{3/2} - \beta_m^{3/2}}{\sqrt{\beta_0}}\left(E_{pw}(x_n) - E_{pw}(x_m)\right)$$

$$+ \frac{\beta_n - \beta_m}{\beta}\left(E_{ww}(x_n) - E_{ww}(x_m)\right)$$

$$\frac{\beta_n^2 - \beta_n^2}{\beta_0} \quad \left(\beta_n^{3/2} - \beta_m^{3/2}\right.$$