

Mapping functions to n-sphere

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Let $f_i(x|\alpha_i)$ $i = 0(i)n$ be $n + 1$ functions we want to map on the surface of an n -sphere. The functions can be completely described by their Fisher distance matrix

$$\Delta_{ij} = \arccos(\int \sqrt{f(x|\alpha_i)f(x|\alpha_j)})$$

We suppose that $f(x|\alpha_0)$ is the nominal distribution and $f(x|\alpha_k)$ $k = 1(k)n$ are the n variational distributions. At first, we embed $f(x|\alpha_0)$ onto the south pole of the n -sphere. The position vector of the south pole of the n -sphere is $I_0 = (0, 0, \dots, x_{00})$ where $x_{00} = -1$.

Consequently, we embed the functions $f(x|\alpha_k)$ $k = 1(k)n$ recursively. We use n -dimensional vectors to denote the position of the functions mapped onto the sphere. The form of the vectors used is shown below

$$\begin{aligned} X_1 &= (0, 0, 0, \dots, x_{01}, x_{11}) \\ X_2 &= (0, 0, 0, \dots, x_{02}, x_{12}, x_{22}) \\ &\vdots \\ &\vdots \\ &\vdots \\ X_k &= (0, 0, 0, \dots, x_{0k}, x_{1k}, x_{2k}, x_{3k}, \dots, x_{k-1,k}, x_{kk}) \\ &\vdots \\ &\vdots \\ &\vdots \\ X_n &= (x_{0n}, x_{1n}, \dots, x_{n-1,n}, x_{nn}) \end{aligned} \tag{1}$$

So for X_k , the first $n - k$ coordinates are 0 and rest of them are embedded on \mathbb{R}^k . Since these vectors lie on the surface of the n -sphere they should satisfy

$$\sum_{i=0}^k x_{ik}^2 = 1 \quad \forall k \tag{2}$$

In addition to this, the position vectors must be defined in such a way that the distance between any two points X_k and X_m $k \neq m$ is their Fisher information distance i.e. Δ_{km} . Therefore we have

$$\begin{aligned} x_{0k}^2 + x_{1k}^2 + \dots + x_{m+1,k}^2 + (x_{k-(m+1),k} - x_{0m})^2 + (x_{k-m,k} - x_{1m})^2 \\ + (x_{k-(m-1),k} - x_{2m})^2 + \dots + (x_{k-1,k} - x_{m-1,m})^2 + (x_{kk} - x_{mm})^2 = \Delta_{km}^2 \end{aligned} \tag{3}$$

Since the geometry is spherical we can use Euclidean distances as a measure of the distance between the different points. Combining equations (2) and (3) we get

$$x_{k-(m+1),k} \cdot x_{0m} + x_{k-m,k} \cdot x_{1m} + \dots + x_{k-1,k} \cdot x_{m-1,m} + x_{kk} \cdot x_{mm} = 1 - \frac{\Delta_{km}^2}{2} \tag{4}$$

When we embed the points recursively using (4), for X_k , k equations of the form (4) needs to be solved. These k equations can be written as a matrix equation as shown in (5)

$$\begin{pmatrix} x_{0,k-1} & x_{1,k-1} & \cdots & \cdots & x_{k-1,k-1} \\ 0 & x_{0,k-2} & x_{1,k-2} & \cdots & x_{k-2,k-2} \\ 0 & 0 & \cdots & \cdots & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & x_{01} & x_{11} \\ 0 & 0 & \cdots & 0 & x_{00} \end{pmatrix} \cdot \begin{pmatrix} x_{1k} \\ x_{2k} \\ \vdots \\ x_{k-1,k} \\ x_{kk} \end{pmatrix} = \begin{pmatrix} 1 - \frac{\Delta_{1k}^2}{2} \\ 1 - \frac{\Delta_{2k}^2}{2} \\ \vdots \\ 1 - \frac{\Delta_{k-1,k}^2}{2} \\ 1 - \frac{\Delta_{kk}^2}{2} \end{pmatrix} \tag{5}$$

Comparing (1) and (5) we notice that the rows of the first matrix are the position vectors $X_{k-1}, X_{k-2}, \dots, X_1, X_0$ respectively which have been computed using the previous similar $k - 1$ matrix equations. These k equations preserve the distances of X_k with $X_0 \dots X_{k-1}$. The other $(n - k)$ Fisher distances are preserved by solving the consequent $(n - k)$ matrix equations for $X_{k+1} \dots X_n$. As it can be seen, (5) does not compute x_{0k} . After solving (5), we compute x_{0k} by using (2).