Mapping functions to n-sphere

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Let $f_i(x|\alpha_i)$ $i = 0(i)n$ be $n+1$ functions we want to map on the surface of an n-sphere. The functions can be completely described by their Fisher distance matrix

$\Delta_{ij} = \arccos(\int \sqrt{f(x|\alpha_i)f(x|\alpha_j)})$

We suppose that $f(x|\alpha_0)$ is the nominal distribution and $f(x|\alpha_k)$ $k = 1(k)n$ are the n variational distributions. At first, we embed $f(x|\alpha_0)$ onto the south pole of the n-sphere. The position vector of the south pole of the n-sphere is $I_0 = (0, 0, ..., x_{00})$ where $x_{00} = -1$.

Consequently, we embed the functions $f(x|\alpha_k)$ $k = 1(k)n$ recursively. We use n-dimensional vectors to denote the position of the functions mapped onto the sphere. The form of the vectors used is shown below

$$
\begin{aligned}
X_1 &= (0, 0, 0, ..., x_{01}, x_{11}) \\
X_2 &= (0, 0, 0, ..., x_{02}, x_{12}, x_{22}) \\
\cdot &= \cdots \\
\cdot &= \cdots \\
\cdot &= \cdots \\
X_k &= (0, 0, 0, ..., x_{0k}, x_{1k}, x_{2k}, x_{3k}, ..., x_{k-1,k}, x_{kk}) \\
\cdot &= \cdots \\
\cdot &= \cdots \\
\cdot &= \cdots \\
X_n &= (x_{0n}, x_{1n}, ..., x_{n-1,n}, x_{nn})
\end{aligned} \tag{1}
$$

So for X_k, the first $n-k$ coordinates are 0 and rest of them are embedded on $\mathbb{R}^k$. Since these vectors lie on the surface of the n-sphere they should satisfy

$$\sum_{i=0}^{k} x_{ik}^2 = 1 \;\forall\; k \tag{2}$$

In addition to this, the position vectors must be defined in such a way that the distance between any two points X_k and X_m $k \neq m$ is their Fisher information distance i.e. Δ_{km}. Therefore we have

$$
\begin{gathered}
x_{ok}^2 + x_{1k}^2 + ... + x_{m+1,k}^2 + (x_{mk} - x_{0m})^2 + (x_{m+1,k} - x_{1m})^2 \\
+ ...... + (x_{kk} - x_{mm})^2 + (x_{k-1,k} - x_{m-1,m})^2 = \Delta_{km}^2
\end{gathered} \tag{3}
$$

Since the geometry is spherical we can use Euclidean distances as a measure of the distance between the different points. Combining equations (2) and (3) we get

$$x_{mk} \cdot x_{0m} + x_{m+1,k} \cdot x_{1m} + ... + x_{k-1,k} \cdot x_{m-1,m} + x_{kk} \cdot x_{mm} = 1 - \frac{\Delta_{km}^2}{2} \tag{4}$$

When we embed the points recursively using (4), for X_k, k equations of the form (4) needs to be solved. These k equations can be written as a matrix equation as shown in (5)

$$
\begin{pmatrix}
x_{0,k-1} & x_{1,k-1} & \cdots & \cdots & x_{k-1,k-1} \\
0 & x_{0,k-2} & x_{1,k-2} & \cdots & x_{k-2,k-2} \\
0 & 0 & \cdots & \cdots & \cdots \\
\vdots & \vdots & \vdots & \vdots & \vdots \\
0 & 0 & \cdots & x_{01} & x_{11} \\
0 & 0 & \cdots & 0 & x_{00}
\end{pmatrix}
\cdot
\begin{pmatrix}
x_{0k} \\ x_{1k} \\ x_{2k} \\ \vdots \\ x_{k-1,k} \\ x_{kk}
\end{pmatrix}
=
\begin{pmatrix}
1 - \frac{\Delta_{0k}^2}{2} \\
1 - \frac{\Delta_{1k}^2}{2} \\
1 - \frac{\Delta_{2k}^2}{2} \\
\vdots \\
1 - \frac{\Delta_{k-1,k}^2}{2} \\
1 - \frac{\Delta_{kk}^2}{2}
\end{pmatrix} \tag{5}
$$

Comparing (1) and (5) we notice that the rows of the first matrix are the position vectors $X_{k-1}, X_{k-2}, \cdots, X_1, X_0$ respectively which have been computed using the previous similar $k-1$ matrix equations. These k equations preserve the distances of X_k with $X_0 \cdots X_{k-1}$. The other $(n-k)$ Fisher distances are preserved by solving the consequent $(n-k)$ matrix equations for $X_{k+1}...X_n$.