Continued Fractions and Transcendental Numbers

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Introduction

Continued fractions are expressions that can either be in two forms:

Finite

$$a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{\ddots + \cfrac{1}{a_n}}}}$$

for a finite sequence $\{a_i\}_{i=0}^{n}$. It is denoted by $[a_0; a_1, \ldots, a_n]$.

Infinite

$$a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{\ddots}}}$$

for an infinite sequence $\{a_i\}_{i=0}^{\infty}$. It is denoted by $[a_0; a_1, \ldots]$.

We will only consider $a_0 \in \mathbb{Z}$ and $a_i \in \mathbb{N} = \{1, 2, \ldots\}$ for $i \geq 1$.

Notation and Basic Properties

- Every finite continued fraction can be written as a **rational** number. Denote the numerator p and the denominator q, so we assign the fraction $[a_0; a_1, \dots a_n]$ to the value $\frac{p}{q}$, where $\frac{p}{q}$ is irreducible.
- Consider a truncation of an infinite sequence of length k, $\{a_i\}_{i=0}^{k}$. These truncations give finite continued fractions — $[a_0; a_1, \dots, a_k] = \frac{p_k}{q_k}$.
- Let $\frac{p_k}{q_k}$ be called the k-th ordered *convergent*. Naturally, this leads us to consider the sequence of convergents $\{\frac{p_k}{q_k}\}$.

Fundamental Fact

The sequence of convergents converges for all continued fractions!

- Infinite continued fractions are assigned to the value $\lim\limits_{k\to\infty} \frac{p_k}{q_k}$, where the limit is always **irrational**.

Notation and Basic Properties

Fun Fact

Every convergent of odd order is greater than any convergent of even order!

Theorem 1

For any real number α, there is a unique continued fraction representation equal to α. The fraction is finite if α is rational and infinite if α is irrational.

Proof Idea. Let α be a real number.

If we have $\alpha = a_0 + \frac{1}{r_1}$ for some $r_1 \geq 1$, then a_0 must be the largest integer not exceeding α since $\frac{1}{r_1} \leq 1$.

Next, we need $r_1 = a_1 + \frac{1}{r_2}$ for some $r_2 \geq 1$, so a_1 must be the largest integer not exceeding r_1. Continue on using the same logic.

Fundamental Fact

$p_k = a_k p_{k-1} + p_{k-2}$ $\qquad$ $q_k = a_k q_{k-1} + q_{k-2}$

Approximation

A fraction $\frac{a}{b}$ is a best approximation of a number α if for any fraction $\frac{c}{d}$, if $0 < d \leq b$ we have

$$|d\alpha - c| > |b\alpha - a|$$

Note: we can see the definition above as a stricter version of this inequality:

$$\left|\alpha - \frac{c}{d}\right| > \left|\alpha - \frac{a}{b}\right|$$

Theorem 2

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1. Every best approximation is a convergent
2. Every convergent is a best approximation, besides the trivial case

$$\alpha = a_0 + \frac{1}{2}, \quad \frac{p_0}{q_0} = \frac{a_0}{1}$$

Approximation

We know $\{\frac{p_k}{q_k}\}_{k=0}$ converges to α — but at what rate? Note that for given $a_0, \ldots, a_k$, the size of q_{k+1} relative to q_k is directly related to the size of a_{k+1}. Indeed, we have:

$$\frac{q_{k+1}}{q_k} = \frac{a_{k+1}q_k + q_{k-1}}{q_k} \approx a_{k+1}$$

In addition, we can find the inequality

$$\frac{1}{q_k(q_k + q_{k+1})} < \left|\alpha - \frac{p_k}{q_k}\right| \leq \frac{1}{q_k q_{k+1}}$$

This means the greater a_{k+1} is, the better the approximation of $\frac{p_k}{q_k}$ is with respect to the size of q_k. This also tells us how to create numbers with bad rates of approximation — the worst being the golden ratio!

$$\phi = \frac{1+\sqrt{5}}{2} = 1 + \cfrac{1}{1 + \cfrac{1}{\ddots}}$$

Approximation

Theorem 3

We can strengthen our inequality with some restrictions.

1. For all indices k, we have
$$\left|\alpha - \frac{p_k}{q_k}\right| \leq \frac{1}{q_k^2}$$
2. Out of any two consecutive indices $\{k-1, k\}$, we can find an index i with
$$\left|\alpha - \frac{p_i}{q_i}\right| \leq \frac{1}{2q_i^2}$$
3. Out of any three consecutive indices $\{k-2, k-1, k\}$, we can find an index i with
$$\left|\alpha - \frac{p_i}{q_i}\right| \leq \frac{1}{\sqrt{5}q_i^2}$$

.

but we cannot continue this pattern, since for any $c > \sqrt{5}$, we have $\left|\alpha - \frac{p_k}{q_k}\right| > \frac{1}{cq_k^2}$ for $\alpha = \phi = \frac{1+\sqrt{5}}{2}$ for all sufficiently large k.

Transcendental Numbers

Algebraic numbers are numbers that can be written as a root of an algebraic equation, e.g., $\sqrt[3]{2}$ is algebraic, since it is the root of $x^3 = 2$. The degree of an algebraic number is the highest power of its algebraic equation. Transcendental numbers are numbers that are not algebraic.

Fun Fact

Almost every number is transcendental!

Theorem 4 (Liouville's Approximation Theorem)

For an irrational algebraic number α of degree n, there exists a C such that for any p, q, we have

$$\left|\alpha - \frac{p}{q}\right| > \frac{C}{q^n}$$

Transcendental Numbers

We can use Theorem 4 cleverly to create transcendental numbers. If we create α such that for any C and any n, we can find p, q with

$$\left|\alpha - \frac{p}{q}\right| \leq \frac{C}{q^n}$$

then α is transcendental.
For given $K \in \mathbb{N}$ and $a_0, \ldots, a_K$, if we choose

$$a_{k+1} > q_k^{k-1}$$

for infinitely many $k \geq K$, then we have

$$\left|\alpha - \frac{p_k}{q_k}\right| \leq \frac{1}{q_k q_{k+1}} < \frac{1}{q_k(a_{k+1}q_k)} = \frac{1}{q_k^2 a_{k+1}} < \frac{1}{q_k^{k+1}} \leq^* \frac{C}{q_k^k}$$

This happens for infinitely many k such that $q_k > \frac{1}{C}$.
Numbers that satisfy this condition are called Liouville numbers.

Transcendental Numbers

A periodic continued fraction is a fraction that, for some k and h, can be written as $\alpha = [a_0; a_1, \ldots, a_k, \overline{a_{k+1}, \ldots, a_{k+h}}]$, in which the overline represents repetition of terms $a_{k+1}, \ldots, a_{k+h}$.

Theorem 5

A number is represented by a periodic continued fraction if and only if it is a quadratic irrational number.

Proof Idea: periodic $\implies$ quadratic. For simplicity's sake, we omit $a_1, \ldots, a_k$. Let $x = [\overline{a_{k+1}; a_{k+2}, \ldots, a_{k+h}}]$. Then

$$x = a_{k+1} + \cfrac{1}{\ddots + \cfrac{1}{a_{k+h} + \frac{1}{x}}} = \frac{Ax + B}{Cx + D} \longrightarrow x \text{ is quadratic}$$

It can then easily be shown that α is quadratic.

Palindromic Continued Fractions

This section is based on *Palindromic continued fractions* by B. Adamczewski and Y. Bugeaud.

A palindrome is a finite word $a_1a_2 \dots a_n$ where $a_j = a_{n+1-j}$ for $j \geq 1$, e.g. 41044014.

Theorem 6

If $\alpha = [a_0; a_1, \dots, a_n, \dots]$, where $a_1 \dots a_n$ is an palindrome for arbitrary large n, then α is either quadratic irrational or transcendental.

What do these sequences $\{a_i\}$ look like? 410440145678 876541044014

This theorem can be generalized. The generalization comes down to loosening the palindromic condition on the sequence, as to allow a certain amount of noise. In all cases, we end up with a transcendental/quadratic irrational number.

Palindromic Continued Fractions

Proof sketch.

1. We bring in a theorem obtained by Wolfgang M. Schmidt.
   Theorem A: Let α be a non-rational, non-quadratic number. If there exists a $\omega > \frac{3}{2}$ and infinitely many (p, q, r) triples such that

$$\max\left\{\left|\alpha - \frac{p}{q}\right|, \left|\alpha^2 - \frac{r}{q}\right|\right\} < \frac{1}{q^\omega}$$

   then α is transcendental.

2. We have $\frac{q_{n-1}}{q_n} = [0; a_n, \ldots, a_1]$. When $a_1 \ldots a_n$ is a palindrome, $\frac{p_n}{q_n} = [0; a_1, \ldots, a_n] = [0; a_n, \ldots, a_1] = \frac{q_{n-1}}{q_n}$. This gives $p_n = q_{n-1}$.

3. $$\alpha^2 \approx \left(\frac{p_n}{q_n}\right)\left(\frac{p_{n-1}}{q_{n-1}}\right) = \left(\frac{\cancel{p_n}}{q_n}\right)\left(\frac{p_{n-1}}{\cancel{q_{n-1}}}\right) = \frac{p_{n-1}}{q_n}$$

   For other continued fractions, it is harder to find an approximation of α^2 with the same denominator as the approximations of α.

Thank You