Chapter 3

Fluctuation-dissipation relations for a kinetic Langevin equation

3.1 Introduction

Fluctuation dissipation relations (FDR) predict the response of a dynamical system to an externally applied perturbation, based on the system's internal dissipation properties. The classical Langevin equation [88, 89] supplies the best known example of such FDR. The standard formulation is to consider a scalar φ forced by a Gaussian white-noise source χ and damped at the rate γ:

$$\begin{aligned} &\frac{\partial \varphi}{\partial t} + \gamma\varphi = \chi, \\ &\langle \chi(t)\chi(t')\rangle = \varepsilon\delta(t - t'), \end{aligned} \tag{3.1}$$

where angle brackets denote the ensemble average and $\varepsilon/2$ is the mean power injected into the system by the source:

$$\frac{\partial}{\partial t}\frac{\langle\varphi^2\rangle}{2} + \gamma\langle\varphi^2\rangle = \frac{\varepsilon}{2}. \tag{3.2}$$

The steady-state mean square fluctuation level is then given by the FDR, linking the injection and the dissipation of the scalar fluctuations:

$$\langle \varphi^2 \rangle = \frac{\varepsilon}{2\gamma}. \tag{3.3}$$

The simplest physical example of such a system is a Brownian particle suspended in liquid, with φ the velocity of the particle and γ the frictional damping. More generally, Eq. (3.1) may be viewed as a generic model for systems where some perturbed quantity is randomly stirred and decays via some form of linear damping, a frequently encountered situation in, e.g., fluid dynamics.

Nearly every problem in plasma physics involves a system with driven and damped linear modes. Here we consider the prototypical such case: the behavior of perturbations of a Maxwellian equilibrium in a weakly collisional plasma in one spatial and one velocity-space dimension. In such a system (and in weakly collisional or collisionless plasmas generally), damping of the perturbed electric fields occurs via the famous Landau mechanism [4]. Landau damping, however, is different in several respects from standard "fluid" damping phenomena. It is in fact a phase mixing process: electric—and, therefore, density—perturbations are phase mixed and thus are effectively damped (see Sec. 1.1). Their (free) energy is transferred to perturbations of the particle distribution function that develop ever finer structure in velocity space and are eventually removed by collisions or, in a formally collisionless limit, by some suitable coarse-graining procedure. The electrostatic potential φ in such systems cannot in general be rigorously shown to satisfy a "fluid" equation of the form (3.1), with γ the Landau damping rate, although the idea that Eq. (3.1) or a higher-order generalization thereof is not a bad model underlies the so-called Landau-fluid closures [90–103].

It is a natural question to ask whether, despite the dynamical equations for φ (or, more generally, for the moments of the distribution function) being more complicated than Eq. (3.1), we should still expect the mean fluctuation level to satisfy Eq. (3.3), where γ is the Landau damping rate. And if that is not the case, then should the value of γ *defined* by Eq. (3.3) be viewed as the effective damping rate in a driven system, replacing the Landau rate? Plunk [39] recently considered the latter question and argued that the fact that the effective damping rate defined this way differs from the Landau rate suggests a fundamental modification of Landau response in a stochastic setting. Our take on the problem at hand differs from theirs somewhat in that we take the kinetic version of the Langevin equation (introduced in Sec. 3.2) at face value and derive the appropriate kinetic generalization of the FDR, instead of attaching a universal physical significance to the "fluid" version of it. Interestingly, the kinetic FDR does simplify to the classical fluid FDR when the Landau damping rate is small. Furthermore, we prove that in this limit (and when the system has no real frequency), the dynamics of φ is in fact described by Eq. (3.1) with γ equal precisely to the Landau rate (i.e., the simplest Landau fluid closure is a rigorous approximation in this limit). The latter result is obtained by treating the velocity-space dynamics of the system in Hermite space. We also show how phase mixing in our system can be treated as a free-energy flux in Hermite space, what form the FDR takes for the Hermite spectrum of the perturbations of the distribution function, and how collisional effects can be included. The intent of this treatment is to provide a degree of clarity as to the behavior of a very simple plasma model and thus set the stage for modelling more complex, nonlinear phenomena.

In Sec. 3.2, we describe a simple model for a weakly collisional plasma, which we call the kinetic Langevin equation, and then, in Sec. 3.3, derive the FDR for the same, including the "fluid" limit mentioned above. In Sec. 3.4, Hermite-space dynamics are treated, including the limit where Landau-fluid closures hold rigorously. An itemized summary of our findings is given in Sec. 3.5. A version of the calculation with a different random source is presented in appendix B.

3.2 Kinetic Langevin equation

We consider the following (1+1)-dimensional model of a homogeneous plasma perturbed about a Maxwellian equilibrium:

$$\frac{\partial g}{\partial t} + \underbrace{v\frac{\partial g}{\partial z}}_{\text{phase mixing}} + \underbrace{vF_0\frac{\partial \varphi}{\partial z}}_{\text{electric field}} = \underbrace{\chi(t)F_0}_{\text{source}} + \underbrace{C[g]}_{\text{collisions}}, \tag{3.4}$$

$$\varphi = \alpha\int_{-\infty}^{\infty} \mathrm{d}v\, g, \tag{3.5}$$

$$\langle\chi(t)\chi(t')\rangle = \varepsilon\delta(t-t'),$$

where $g(z, v, t)$ is the perturbed distribution function and $F_0(v)$ is the Maxwellian equilibrium distribution $F_0 = e^{-v^2}/\sqrt{\pi}$. The velocity v (in the z direction) is normalized to the thermal speed $v_{\mathrm{th}} = \sqrt{2T/m}$ (T and m are the temperature and mass of the particle species under consideration), spatial coordinate z is normalized to an arbitrary length L, and time t to $L/v_{\rm th}$. Only one species (either electrons or ions) is evolved. The second species follows the density fluctuations of the first via whatever response a particular physical situation warrants: Boltzmann, isothermal, or no response—all of these possibilities are embraced by Eq. (3.5), which determines the (suitably normalized) scalar potential φ in terms of the perturbed density associated with g; the parameter α contains all of the specific physics. For example, if g is taken to be the perturbed ion distribution function in a low-beta magnetized plasma and electrons to have Boltzmann response, then $\alpha = ZT_e/T_i$, the ratio of the electron to ion temperatures (Z is the ion charge in units of electron charge e)—the resulting system describes (Landau-damped) ion-acoustic waves; Eq. (3.5) in this case is the statement of quasineutrality. Another, even more textbook example is damped Langmuir waves, the case originally considered by Landau [4]: g is the perturbed electron distribution function, ions have no response, so $\alpha = 2/k^2\lambda_D^2$, where λ_D is the Debye length and k is the wave number of the perturbation ($\partial/\partial z = ik$); Eq. (3.5) in this case is the Gauss-Poisson law.

A particularly astrophysically and space-physically relevant example (in the sense of being accessible to measurements in the solar wind [3,30,31,34,35,104]) is the compressive perturbations in a magnetized plasma—perturbations of plasma density and magnetic-field strength at scales long compared to the ion Larmor radius (see KRMHD equations in Sec. 1.5). The model given by Eqs. (3.4–3.5) can be obtained from KRMHD by setting the Alfvén fluctuations to zero, adding a source term, and defining the parameter $\alpha = -1/\Lambda$ (see Eq. (1.20)).

Thus, Eqs. (3.4) and (3.5) correspond a variety of interesting physical situations.

The energy injection in Eq. (3.4) is modelled by a white-in-time, Maxwellian-in-velocity-space source $\chi(t)F_0$ supplying fixed power $\propto \varepsilon$ to the perturbations (see below). This is a direct analog of the noise term in the "fluid" Langevin equation (3.1) and so this particular choice of forcing was made in order to enable the simplest possible comparison with the "fluid" case*. The energy injection leads to sharp gradients in the velocity space (phase mixing), which are removed by the collision operator $C[g]$. "The energy" in the context of a kinetic equation is the free energy of the perturbations [1, 16] (see Eq. (1.24)), given in this case by

$$W = \int \mathrm{d}v \, \frac{\langle g^2 \rangle}{2F_0} + \frac{\langle \varphi^2 \rangle}{2\alpha} \tag{3.6}$$

and satisfying

$$\frac{\mathrm{d}W}{\mathrm{d}t} = \frac{1+\alpha}{2}\,\varepsilon + \int \mathrm{d}v \, \frac{gC[g]}{F_0}. \tag{3.7}$$

The first term on the right-hand side is the energy injection by the source, the second, negative definite term is its thermalization by collisions. Note that the variance of φ is not by itself a conserved quantity:

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{\langle \varphi^2 \rangle}{2} + \alpha \left\langle \varphi \frac{\partial}{\partial z} \int \mathrm{d}v \, vg \right\rangle = \frac{\alpha^2}{2}\,\varepsilon. \tag{3.8}$$

*One might argue that this is not, however, the most physical form of forcing and that it would be better to inject energy by applying a random electric field to the plasma, rather than a source of density perturbations. In appendix B we present a version of our calculation for such a more physical source, and show that all the key results are similar. Note that the forcing in Eq. (3.4) does not violate particle conservation because we assume that spatial integrals of all perturbations vanish: $\int \mathrm{d}z \, g = 0$, $\int \mathrm{d}z \, \chi = 0$.

The power $\alpha^2\varepsilon/2$ injected into fluctuations of φ is transferred into higher moments of g via phase mixing. Phase mixing is precisely this process of draining free energy from the lower moments and transferring it into higher moments of the distribution function—without collisions, this is just a redistribution of free energy within Eq. (3.6), which, in the absence of source, would look like a linear damping of $\varphi^\dagger$.

In the presence of a source, the system described by Eqs. (3.4) and (3.5) is a driven-damped system much like the Langevin equation (3.1). The damping of φ in the kinetic case is provided by Landau damping (phase mixing) as opposed to the explicit dissipation term in Eq. (3.1). It is an interesting question whether in the steady state, the second term on the left-hand side of Eq. (3.8) can be expressed as $\gamma_{\text{eff}}\langle\varphi^2\rangle$, leading an analogue of the FDR (Eq. (3.3)), and if so, whether the "effective damping rate" γ_{eff} in this expression is equal to the Landau damping rate γ_L. The answer is that an analogue of the FDR does exist, γ_{eff} is non-zero for vanishing collisionality, but in general, $\gamma_{\text{eff}} \neq \gamma_L$.

3.3 Kinetic Fluctuation-Dissipation Relations

Ignoring collisions in Eq. (3.4) and Fourier-transforming it in space in time, we get

$$g_{k\omega} = -\varphi_{k\omega}\frac{vF_0}{v-\omega/k} - \frac{i\chi_{k\omega}}{k}\frac{F_0}{v-\omega/k}. \tag{3.9}$$

---

†Note that $\alpha = -1$ corresponds to an effectively undriven system; the Landau damping rate for this case is zero (Eq. (3.16)). We will see in Sec. 3.4.1 that in this case the driven density moment decouples from the rest of the perturbed distribution function; see Eq. (3.27). For $\alpha < -1$ the system is no longer a driven-damped system; this parameter regime never occurs physically.

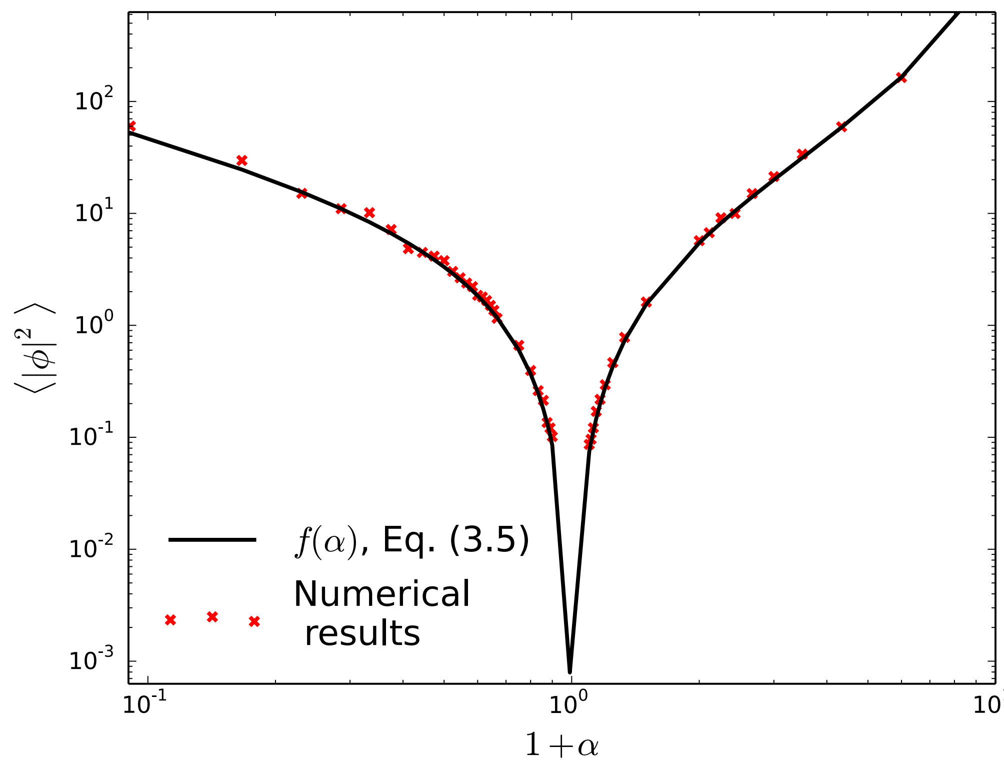

Figure 3.1: Normalized steady-state amplitude $2\pi|k|\langle|\varphi_k|^2\rangle/\varepsilon_k = f(\alpha)$ vs. $1+\alpha$: the solid line is the analytical prediction ($f(\alpha)$ as per Eq. (3.13)), the crosses are computed from the long-time limit of $\langle|\varphi_k|^2\rangle$ obtained via direct numerical solution of Eqs. (3.4) and (3.5).

Introducing the plasma dispersion function $Z(\zeta) = \int \mathrm{d}v F_0/(v-\zeta)$, where the integration is along the Landau countour [105], we find from Eqs. (3.9) and (3.5):

$$\varphi_{k\omega} = -\frac{i\chi_{k\omega}}{|k|}\frac{Z(\omega/|k|)}{D_\alpha(\omega/|k|)}, \tag{3.10}$$

$$D_\alpha\left(\frac{\omega}{|k|}\right) = 1 + \frac{1}{\alpha} + \frac{\omega}{|k|}Z\left(\frac{\omega}{|k|}\right). \tag{3.11}$$

Note that $D_\alpha(\omega/|k|) = 0$ is the dispersion relation for the classic Landau problem [4]. We now inverse Fourier transform Eq. (3.10) back into the time domain,

$$\varphi_k(t) = \int \mathrm{d}\omega\, e^{-i\omega t}\varphi_{k\omega} = -\frac{i}{|k|}\int \mathrm{d}\omega\, e^{-i\omega t}\chi_{k\omega}\frac{Z(\omega/|k|)}{D_\alpha(\omega/|k|)}, \tag{3.12}$$

and compute $\langle|\varphi_k|^2\rangle$ in the steady state. In order to do this, we use the fact that $\chi_{k\omega} \equiv \int \mathrm{d}t e^{i\omega t}\chi_k(t)/2\pi$ satisfies $\langle\chi_{k\omega}\chi^*_{k\omega'}\rangle = \varepsilon_k\delta(\omega-\omega')/2\pi$ because $\langle\chi_k(t)\chi^*_k(t')\rangle = \varepsilon_k\delta(t-t')$, where ε_k is the source power at wave number k. The result is

$$\langle|\varphi_k|^2\rangle = \frac{\varepsilon_k}{2\pi|k|}f(\alpha), \quad f(\alpha) = \int_{-\infty}^{+\infty}\mathrm{d}\zeta\left|\frac{Z(\zeta)}{D_\alpha(\zeta)}\right|^2, \tag{3.13}$$

where we have changed the integration variable to $\zeta = \omega/|k|$. This is the fluctuation-dissipation relation for our kinetic system that predicts the long-time behavior of the electrostatic potential. The function $f(\alpha)$, computed numerically as per Eq. (3.13), is plotted in Fig. 3.1, together with the results of direct numerical solution of Eqs. (3.4) and (3.5), in which $f(\alpha)$ is found by computing the saturated fluctuation level $\langle|\varphi_k|^2\rangle$.

Eq. (3.13) can be written in the form

$$\langle|\varphi_k|^2\rangle = \frac{\alpha^2 \varepsilon_k}{2\gamma_{\rm eff}}, \quad \gamma_{\rm eff}(\alpha) = \frac{\pi\alpha^2}{f(\alpha)}|k|, \tag{3.14}$$

but the "effective damping rate" $\gamma_{\rm eff}$ is not in general the same as the Landau damping rate γ_L. This is illustrated in Fig. 3.2, where we plot the real (ω_L) and imaginary ($-\gamma_L$) parts of the slowest-damped root(s) of $D_\alpha(\omega/|k|) = 0$ together with $\gamma_{\rm eff}(\alpha)$ for $\alpha < 0$ and $\gamma_{\rm eff}(\alpha)/2$ for $\alpha > 0$. In the latter case, the linear modes of the system have real frequencies and the analogy with the Langevin equation (3.1) is not apt—a better mechanical analogy is a damped oscillator, as explained at the end of Sec. 3.3.2; the FDR in this case acquires an extra factor of 1/2, which is why we plot $\gamma_{\rm eff}/2$ (see Eq. (3.23)). Remarkably, $\gamma_{\rm eff}(\alpha)$ does asymptote to γ_L in the limit $1 + \alpha \ll 1$ and to $2\gamma_L$ in the limit $\alpha \to \infty$, i.e., when the damping is weak. These asymptotic results can be verified analytically.

3.3.1 Zero real frequency, weak damping ($\alpha \to -1$)

When $\alpha+1 \ll 1$, the solution of the dispersion relation will satisfy $\zeta = \omega/|k| \ll 1$. In this limit,

$$Z(\zeta) \approx i\sqrt{\pi}, \quad D_\alpha(\zeta) \approx 1 + \frac{1}{\alpha} + i\zeta\sqrt{\pi} \approx i\sqrt{\pi}\left(\zeta - i\frac{1+\alpha}{\sqrt{\pi}}\right). \tag{3.15}$$

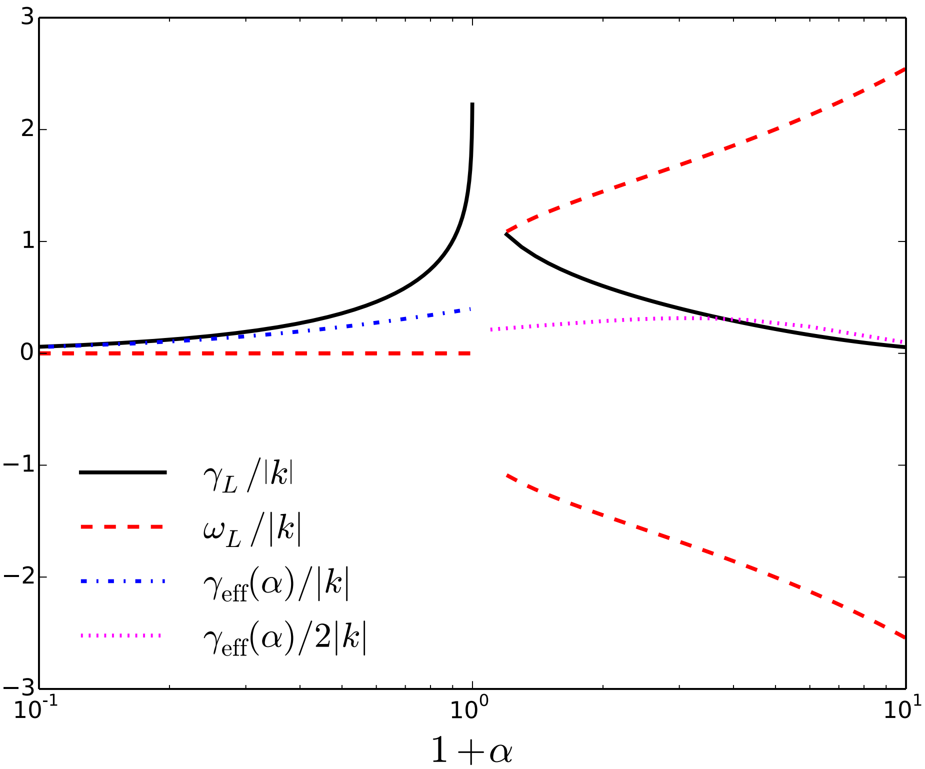

Figure 3.2: Slowest-damped solutions of the dispersion relation $D_\alpha(\omega/|k|) = 0$: normalized frequency $\omega_L/|k|$ (red dashed line) and damping rate $\gamma_L/|k|$ (black soloid line) vs. $1+\alpha$. Also shown are $\gamma_{\rm eff}(\alpha)$ for $\alpha < 0$ (blue dash-dotted line) and $\gamma_{\rm eff}(\alpha)/2$ for $\alpha > 0$ (magenta dotted line), as per Eq. (3.14). The two asymptotic limits in which these match γ_L are discussed in Secs. 3.3.1 and 3.3.2.

Therefore, the solution of $D_\alpha(\omega/|k|) = 0$ is

$$\omega \approx -i\gamma_L, \quad \gamma_L = \frac{1+\alpha}{\sqrt{\pi}}|k|. \tag{3.16}$$

A useful physical example of Landau damping in this regime is the Barnes damping [38] of compressive fluctuations in high-beta plasmas, where $1 + \alpha \approx 1/\beta_i$ (see Schekochihin et al [1], their eq. (190)).

Since the zeros of $D_\alpha(\zeta)$ and $D^*_\alpha(\zeta)$, which are poles of the integrand in the expression for $f(\alpha)$ (Eq. (3.13)), lie very close to the real line in this case, the integral is easily computed by using the approximate expressions (3.15) for $Z(\zeta)$ and $D_\alpha(\zeta)$ and applying Plemelj's formula, to obtain

$$f(\alpha) \approx \frac{\pi\sqrt{\pi}}{1+\alpha} = \frac{\pi|k|}{\gamma_L} \quad \Rightarrow \quad \langle|\varphi_k|^2\rangle \approx \frac{\sqrt{\pi}\varepsilon_k}{2(1+\alpha)|k|} = \frac{\varepsilon_k}{2\gamma_L}. \tag{3.17}$$

Noting that $\alpha^2 \approx 1$, this is the same as Eq. (3.14) with $\gamma_{\text{eff}} = \gamma_L$, so the "fluid" FDR is recovered. Note, however, that this recovery of the exact form of the "fluid" FDR is a property that is not universal with respect to the exact form of energy injection: as shown in appendix B, it breaks down for a different forcing (see Eq. (B.14)).

3.3.2 Large real frequency, weak damping ($\alpha \to \infty$)

Another analytically tractable limit is $\alpha \gg 1$, in which case the solutions of the dispersion relation have $\zeta = \omega/|k| \gg 1$. In this limit,

$$Z(\zeta) \approx i\sqrt{\pi}\, e^{-\zeta^2} - \frac{1}{\zeta} - \frac{1}{2\zeta^3}, \quad D_\alpha(\zeta) \approx \frac{1}{\alpha} - \frac{1}{2\zeta^2} + i\sqrt{\pi}\,\zeta e^{-\zeta^2}. \tag{3.18}$$

The solutions of $D_\alpha(\omega/|k|) = 0$ are

$$\omega \approx \pm\sqrt{\frac{\alpha}{2}}\,|k| - i\gamma_L, \quad \gamma_L = \sqrt{\pi}\,\frac{\alpha^2}{4}\,e^{-\alpha/2}|k|. \tag{3.19}$$

Two textbook examples of Landau-damped waves in this regime are ion acoustic waves at $\beta_i \ll 1$, $T_i \ll T_e$ (cold ions), for which $\alpha = ZT_e/T_i$, and long-wavelength Langmuir waves, for which $\alpha = 2/k^2\lambda_D^2$ [4].

In the integral in Eq. (3.13), the poles are again very close to the real line and so in the integrand, we may approximate, in the vicinity of one of the two solutions (3.19)

$$Z(\zeta) \approx \mp\sqrt{\frac{2}{\alpha}}, \quad D_\alpha(\zeta) \approx \pm\left(\frac{2}{\alpha}\right)^{3/2}\left(\zeta \mp \sqrt{\frac{\alpha}{2}} + i\,\frac{\gamma_L}{|k|}\right). \tag{3.20}$$

Using again Plemelj's formula and noting that equal contributions arise from each of the two roots, we find

$$f(\alpha) \approx 2\sqrt{\pi}\,e^{\alpha/2} = \frac{\pi\alpha^2|k|}{2\gamma_L} \quad \Rightarrow \quad \langle|\varphi_k|^2\rangle \approx \frac{\alpha^2\varepsilon_k}{4\gamma_L}, \tag{3.21}$$

which is the same as Eq. (3.14) with $\gamma_{\text{eff}} = 2\gamma_L$.

Despite the apparently discordant factor of 2, this, in fact, is again consistent with a non-kinetic, textbook FDR. However, since we are considering a system with a large frequency, the relevant mechanical analogy is not Eq. (3.1), but the equally standard (and more general) equation for a forced and damped oscillator:

$$\ddot{\varphi} + \gamma\dot{\varphi} + \omega^2\varphi = \dot{\chi}, \tag{3.22}$$

where overdots mean time derivatives.

We continue to consider χ a Gaussian white noise satisfying $\langle\chi(t)\chi(t')\rangle = \varepsilon\delta(t - t')$. For $\omega = 0$, Eq. (3.22) then precisely reduces to Eq. (3.1). For $\omega \neq 0$, it is not hard to show (by Fourier transforming in time, solving, then inverse Fourier transforming and squaring the amplitude) that the stationary mean square amplitude $\langle\varphi^2\rangle$ for Eq. (3.22) still satisfies Eq. (3.3). However, the relationship between the actual linear damping rate γ_L of φ and the parameter γ depends on the frequency: $\gamma_L = \gamma$ when $\omega \ll \gamma$ and $\gamma_L = \gamma/2$ when $\omega \geq \gamma/2$. In the latter case, which is the one with which we are preoccupied here, Eq. (3.3) becomes, in terms of γ_L:

$$\langle\varphi^2\rangle = \frac{\varepsilon}{4\gamma_L}. \tag{3.23}$$

The required extra factor of 2 is manifest.‡

‡As in Sec. 3.3.1, this very simple mechanical analogy also breaks down for a different choice of forcing; see appendix B (Eq. (B.15)).

3.4 Velocity-space structure

The kinetic FDR derived in the previous section was concerned with the rate of removal of free energy from the density moment of the perturbed distribution function. This free energy flows into higher moments, i.e., is "phase mixed" away. In this section, we diagnose the velocity-space structure of the fluctuations and extend the FDR to compute their amplitude.

3.4.1 Kinetic equation in Hermite space

The emergence of ever finer velocity-space scales is made explicit by recasting the kinetic equation (3.4) in Hermite space, a popular approach for many years [40, 106–114]. The distribution is decomposed into Hermite moments as follows

$$g(v) = \sum_{m=0}^{\infty} \frac{H_m(v) F_0}{\sqrt{2^m m!}} g_m, \quad g_m = \int \mathrm{d}v \, \frac{H_m(v)}{\sqrt{2^m m!}} \, g(v), \tag{3.24}$$

where $H_m(v)$ is the Hermite polynomial of order m. In terms of Hermite moments, Eq. (3.5) becomes

$$\varphi = \alpha g_0, \tag{3.25}$$

while Eq. (3.4) turns into a set of equations for the Hermite moments g_m, where phase mixing is manifested by the coupling of higher-m moments to the lower-m

ones:

$$\frac{\partial g_0}{\partial t} + \frac{\partial}{\partial z}\frac{g_1}{\sqrt{2}} = \chi, \tag{3.26}$$

$$\frac{\partial g_1}{\partial t} + \frac{\partial}{\partial z}\left(g_2 + \frac{1+\alpha}{\sqrt{2}}\,g_0\right) = 0, \tag{3.27}$$

$$\frac{\partial g_m}{\partial t} + \frac{\partial}{\partial z}\left(\sqrt{\frac{m+1}{2}}\,g_{m+1} + \sqrt{\frac{m}{2}}\,g_{m-1}\right) = -\nu m g_m, \quad m \geq 2, \tag{3.28}$$

where ν is the collision frequency and we have used the [80] collision operator, a natural modelling choice in this context because its eigenfunctions are Hermite polynomials.

The free energy (3.6) in these terms is

$$W = \frac{1+\alpha}{2}\langle g_0^2\rangle + \frac{1}{2}\sum_{m=1}^{\infty}\langle g_m^2\rangle \tag{3.29}$$

and satisfies

$$\frac{\mathrm{d}W}{\mathrm{d}t} = \frac{1+\alpha}{2}\,\varepsilon - \nu\sum_{m=2}^{\infty} m\langle g_m^2\rangle. \tag{3.30}$$

3.4.2 Fluctuation-Dissipation Relations in Hermite space

It is an obvious generalization of the FDR to seek a relationship between the fluctuation level in the m-th Hermite moment, $\langle|g_m|^2\rangle$ (the "Hermite spectrum"), and the injected power ε. This can be done in exactly the same manner as the kinetic FDR was derived in Sec. 3.3. Hermite-transforming Eq. (3.9) gives

$$g_{m,k\omega} = -\frac{i\chi_{k\omega}}{|k|}\frac{1+\alpha}{\alpha}\frac{(-\operatorname{sgn} k)^m}{\sqrt{2^m m!}}\frac{Z^{(m)}(\omega/|k|)}{D_\alpha(\omega/|k|)}, \quad m \geq 1, \tag{3.31}$$

where we have used

$$Z^{(m)}(\zeta) \equiv \frac{\mathrm{d}^m Z}{\mathrm{d}\zeta^m} = (-1)^m \int \mathrm{d}v\, \frac{H_m(v) F_0(v)}{v-\zeta} \tag{3.32}$$

and $Z^{(m)}(\omega/k) = (\operatorname{sgn} k)^{m+1} Z^{(m)}(\omega/|k|)$. The mean square fluctuation level in the statistical steady state is then derived similarly to Eq. (3.13):

$$C_{m,k} \equiv \langle |g_{m,k}|^2 \rangle = \frac{\varepsilon_k}{2\pi |k|} \left(\frac{1+\alpha}{\alpha}\right)^2 \frac{1}{2^m m!} \int_{-\infty}^{+\infty} \mathrm{d}\zeta \left| \frac{Z^{(m)}(\zeta)}{D_\alpha(\zeta)} \right|^2, \quad m \geq 1. \tag{3.33}$$

This is the extension of the kinetic FDR, Eq. (3.13), to the fluctuations of the perturbed distribution function. The "Hermite spectrum" $C_{m,k}$ characterizes the distribution of free energy in phase space.

3.4.3 Hermite spectrum

It is interesting to derive the asymptotic form of this spectrum at $m \gg 1$. Using in Eq. (3.32) the asymptotic form of the Hermite polynomials at large m [115],

$$e^{-v^2/2} H_m(v) \approx \left(\frac{2m}{e}\right)^{m/2} \sqrt{2} \cos\left(v\sqrt{2m} - \pi m/2\right), \tag{3.34}$$

and remembering that the v integration is over the Landau contour (i.e., along the real line, cicumnavigating the pole at $v = \zeta$ from below), we find

$$Z^{(m)}(\zeta) \approx i^{m+1} \sqrt{2\pi} \left(\frac{2m}{e}\right)^{m/2} e^{-\zeta^2/2 + i\zeta\sqrt{2m}}, \tag{3.35}$$

provided $\zeta \ll \sqrt{2m}$ (this result is obtained by expressing the cosine in Eq. (3.34) in terms of exponentials, completing the square in the exponential function appearing in the integral (3.32) and moving the integration contour to $v = \pm i\sqrt{2m}$; the dominant contribution comes from the Landau pole). Finally, in Eq. (3.33),

$$\frac{|Z^{(m)}(\zeta)|^2}{2^m m!} \approx \sqrt{\frac{2\pi}{m}}\, e^{-\zeta^2}, \tag{3.36}$$

and so the Hermite spectrum has a universal scaling at $m \gg 1$:

$$C_{m,k} \approx \left[\frac{\varepsilon_k}{\sqrt{2\pi}|k|}\left(\frac{1+\alpha}{\alpha}\right)^2 \int_{-\infty}^{+\infty} \frac{\mathrm{d}\zeta\, e^{-\zeta^2}}{|D_\alpha(\zeta)|^2}\right]\frac{1}{\sqrt{m}} = \frac{\varepsilon_k(1+\alpha)}{\sqrt{2}|k|}\frac{1}{\sqrt{m}}. \tag{3.37}$$

The universal $1/\sqrt{m}$ scaling was derived in a different way by Zocco et al. [112] (see Sec. 3.4.4; [111, 114]). The integral in (3.37) was evaluated using the Kramers–Kronig relations [116, 117] for the function $h(\zeta) = 1/D_\alpha(\zeta) - \alpha$ (which is analytic in the upper half plane and decays at least as fast as $1/|\zeta|^2$ at large ζ):

$$\int_{-\infty}^{+\infty} \frac{\mathrm{d}\zeta\, e^{-\zeta^2}}{|D_\alpha(\zeta)|^2} = -\sqrt{\pi}\left[\frac{1}{\pi}\mathcal{P}\int_{-\infty}^{+\infty}\frac{\mathrm{d}\zeta\,\mathrm{Im}\, h(\zeta)}{\zeta-\zeta'}\right]_{\zeta'=0} = -\sqrt{\pi}\,\mathrm{Re}\, h(0) = \frac{\alpha^2}{1+\alpha}\sqrt{\pi}. \tag{3.38}$$

Note that in the limit of high frequency ($\alpha \gg 1$, Sec. 3.3.2), the approximation (3.35) requires $\omega_L/|k| \ll \sqrt{2m}$, or $\alpha \ll 4m$, but there is also a meaningful intermediate range of m for which $1 \leq m \ll \alpha/4$. In this range, we can approximate $Z(\zeta) \approx -1/\zeta$

and, since $\zeta \approx \pm\sqrt{\alpha/2}$, we have in Eq. (3.33):

$$\frac{|Z^{(m)}(\zeta)|^2}{2^m m!} \approx \frac{2m!}{\alpha^{m+1}} \quad \Rightarrow \quad C_{m,k} \approx \frac{\varepsilon_k}{\sqrt{\pi}|k|}\frac{m!}{\alpha^m}\, e^{\alpha/2}. \tag{3.39}$$

This spectrum decays with m up to $m \sim \alpha$, where it transitions into the universal spectrum (3.37).

3.4.4 Free-energy flux, the effect of collisions and the FDR for the total free energy

Observe that the total free energy in our system, with its $1/\sqrt{m}$ Hermite spectrum, is divergent. The regularization in Hermite space (removal of fine velocity-space scales) is provided by collisions. If ν is infinitesimal, these are irrelevant at finite m, but eventually become important as $m \to \infty$. To take account of their effect and to understand the free-energy flow in Hermite space, we consider Eq. (3.28), which it is convenient to Fourier transform in z and rewrite in terms of $\tilde{g}_{m,k} \equiv (i\,\mathrm{sgn}\,k)^m g_{m,k}$:

$$\frac{\partial \tilde{g}_{m,k}}{\partial t} + \frac{|k|}{\sqrt{2}}\left(\sqrt{m+1}\,\tilde{g}_{m+1,k} - \sqrt{m}\,\tilde{g}_{m-1,k}\right) = -\nu m \tilde{g}_{m,k}. \tag{3.40}$$

The Hermite spectrum $C_{m,k} = \langle |g_{m,k}|^2\rangle = \langle |\tilde{g}_{m,k}|^2\rangle$ therefore satisfies

$$\frac{\partial C_{m,k}}{\partial t} + \Gamma_{m+1/2,k} - \Gamma_{m-1/2,k} = -2\nu m C_{m,k}, \tag{3.41}$$

where $\Gamma_{m+1/2,k} = |k|\sqrt{2(m+1)}\mathrm{Re}\langle\tilde{g}_{m+1,k}\tilde{g}^*_{m,k}\rangle$ is the free-energy flux in Hermite space. If we make an assumption (verified in Sec. 3.4.5) that for $m \gg 1$ the Hermite moments $\tilde{g}_{m,k}$ are continuous in m, i.e., $\tilde{g}_{m+1,k} \approx \tilde{g}_{m,k}$, then

$$\Gamma_{m+1/2,k} \approx |k|\sqrt{2(m+1)}\, C_{m+1,k} \tag{3.42}$$

and Eq. (3.41) turns into a closed evolution equation for the Hermite spectrum [112]:

$$\frac{\partial C_{m,k}}{\partial t} + |k|\frac{\partial}{\partial m}\sqrt{2m}\, C_{m,k} = -2\nu m C_{m,k}. \tag{3.43}$$

The universal $C_{m,k} \propto 1/\sqrt{m}$ spectrum derived in Sec. 3.4.3 is now very obviously a constant-flux spectrum, reflecting steady pumping of free energy towards higher m's (phase mixing). The full steady-state solution of Eq. (3.43) including the collisional cutoff is

$$C_{m,k} = \frac{A_k}{\sqrt{m}} \exp\left(-\frac{2\sqrt{2}}{3}\frac{\nu}{|k|}m^{3/2}\right), \tag{3.44}$$

where A_k is an integration constant, which must be determined by matching this high-m solution with the Hermite spectrum at low m. This we are now in a position to do: for $1 \ll m \ll (\nu/|k|)^{-2/3}$, $C_{m,k} \approx A_k/\sqrt{m}$ and comparison with Eq. (3.37) shows that the constant A_k is the same as the constant $A_k(\alpha)$ in that equation. Thus, Eq. (3.44) with A_k given by Eq. (3.37) provides a uniformly valid expression for the Hermite-space spectrum, including the collisional cutoff (modulo the Hermite-space continuity assumption (3.42), which we will justify in Sec. 3.4.5).

As a check of consistency of our treatment, let us calculate the collisional dissipation rate of the free energy. This is the second term on the right-hand side of Eq. (3.30). Since $C_{m,k} \propto 1/\sqrt{m}$ before the collisional cutoff is reached, the sum over m will be dominated by $m \sim (\nu/|k|)^{-2/3}$ and can be approximated by an integral:

$$\nu \sum_{m,k} m C_{m,k} \approx \sum_k \nu \int_0^\infty \mathrm{d}m \, m C_{m,k} = \sum_k \frac{A_k |k|}{\sqrt{2}}. \tag{3.45}$$

On the other hand, in steady state, Eq. (3.30) implies

$$\nu \sum_{m,k} m C_{m,k} = \frac{1+\alpha}{2} \varepsilon. \tag{3.46}$$

If energy injection is into a single k mode, $\varepsilon = \varepsilon_k$, comparing these two expressions implies

$$A_k = \frac{\varepsilon_k (1+\alpha)}{\sqrt{2}|k|}, \tag{3.47}$$

which, of course, is consistent with Eq. (3.37).

Finally, we use Eq. (3.44) to calculate (approximately) the total steady-state amount of free energy across the phase space:

$$\frac{1}{2} \sum_{m=1}^{\infty} C_{m,k} = \frac{\Gamma(1/3)}{\sqrt{2}\, 3^{2/3}} \frac{A_k}{(\nu/|k|)^{1/3}} = \frac{\Gamma(1/3)}{2 \cdot 3^{2/3}} \frac{1+\alpha}{\nu^{1/3}|k|^{2/3}} \varepsilon_k \tag{3.48}$$

(we have again approximated the sum with an integral, assumed energy injection into a single k and used Eq. (3.47)). Eq. (3.48) can be thought of as the FDR for the total free energy. The fact that this diverges as $\nu \to 0$ underscores the principle that the "true" dissipation (in the sense of free energy being thermalized) is always collisional—a consequence of Boltzmann's H theorem.

3.4.5 Continuity in Hermite space

In this section, we make a somewhat lengthy formal digression to justify the assumption of continuity of Hermite moments in m at large m, which we need for the approximation (3.42). The formalism required for this will have some interesting features which are useful in framing one's thinking about energy flows in Hermite space.

Returning to Eq. (3.40) and considering $1 \ll m \ll (\nu/|k|)^{-2}$, we find that to lowest approximation, the $\sqrt{m}$ terms are dominant and must balance, giving $\tilde{g}_{m+1,k} \approx \tilde{g}_{m-1,k}$. This is consistent with continuity in m, viz., $\tilde{g}_{m+1,k} \approx \tilde{g}_{m,k}$, but there is also a solution allowing the consecutive Hermite moments to alternate sign: $\tilde{g}_{m+1,k} \approx -\tilde{g}_{m,k}$. Thus, there are, formally speaking, two solutions: one for which $\tilde{g}_{m,k}$ is continuous and one for which $(-1)^m\tilde{g}_{m,k}$ is. To take into account both of them, we introduce the following decomposition [118]:

$$\tilde{g}_{m,k} = \tilde{g}^+_{m,k} + (-1)^m \tilde{g}^-_{m,k}, \tag{3.49}$$

where the "+" ("continuous") and the "−" ("alternating") modes are

$$\tilde{g}^+_{m,k} = \frac{\tilde{g}_{m,k} + \tilde{g}_{m+1,k}}{2}, \quad \tilde{g}^-_{m,k} = (-1)^m \frac{\tilde{g}_{m,k} - \tilde{g}_{m+1,k}}{2}. \tag{3.50}$$

The Hermite spectrum and the flux of the free energy can be expressed in terms of the spectra of these modes as follows:

$$C_{m,k} \equiv \langle|\tilde{g}_{m,k}|^2\rangle = C^+_{m,k} + C^-_{m,k}, \tag{3.51}$$

$$\Gamma_{m+1/2,k} \equiv |k|\sqrt{2(m+1)}\mathrm{Re}\langle\tilde{g}_{m+1,k}\tilde{g}^*_{m,k}\rangle \approx |k|\sqrt{2m}\left(C^+_{m,k} - C^-_{m,k}\right), \tag{3.52}$$

where $C^\pm_{m,k} \equiv \langle|\tilde{g}^\pm_{m,k}|^2\rangle$ and the last expression in Eq. (3.52) is an approximation valid for $m \gg 1$.

The functions $\tilde{g}^\pm_{m,k}$ can both be safely treated as continuous in m for $m \gg 1$. Treating them so in Eq. (3.40) and working to lowest order in $1/m$, we find that they satisfy the following *decoupled* evolution equations:

$$\frac{\partial\tilde{g}^\pm_{m,k}}{\partial t} \pm \sqrt{2}|k|m^{1/4}\frac{\partial}{\partial m}m^{1/4}\tilde{g}^\pm_{m,k} = -\nu m\tilde{g}^\pm_{m,k}, \tag{3.53}$$

or, for their spectra,

$$\frac{\partial C^\pm_{m,k}}{\partial t} \pm |k|\frac{\partial}{\partial m}\sqrt{2m}\,C^\pm_{m,k} = -2\nu m C^\pm_{m,k}. \tag{3.54}$$

Manifestly, the "+" mode propagates from lower to higher m and the "−" mode from higher to lower m—they are the "phase-mixing" and the "un-phase-mixing" collisionless solutions, respectively.[§]

Taking the collisional term into account and noting that energy is injected

[§]The existence of un-phase-mixing solutions has been known for a long time: e.g., [108] treated them as forward and backward propagating waves in a mechanical analogy of Eq. (3.40) with a row of masses connected by springs. The un-phase mixing solutions are also what allows the phenomenon of plasma echo [9], including in stochastic nonlinear systems [118].

into the system at low, rather than high, m, the solution satisfying the boundary condition $\tilde{g}_{m,k} \to 0$ as $m \to \infty$ has $\tilde{g}^-_{m,k} = 0$ and so $\tilde{g}_{m,k} = \tilde{g}^+_{m,k}$. Thus, $\tilde{g}_{m,k}$ is continuous in m. With $C^-_{m,k} = 0$, Eq. (3.52) is the same as our earlier approximation (3.42) (to lowest order in the $m \gg 1$ expansion).

As $\tilde{g}^+_{m,k}$ and $\tilde{g}^-_{m,k}$ are decoupled at large m, if we start with a $\tilde{g}^-_{m,k} = 0$ solution, no $\tilde{g}^-_{m,k}$ will be produced. However, both the decoupling property and the interpretation of $\tilde{g}^\pm_{m,k}$ as the phase-mixing and un-phase-mixing modes are only valid to lowest order in $1/m$. It is useful to know how well this approximation holds.

Let us use Eq. (3.31) to calculate (in the collisionless limit)

$$R_{m+1} \equiv \frac{\tilde{g}_{m+1,k\omega}}{\tilde{g}_{m,k\omega}} = i \operatorname{sgn} k \, \frac{g_{m+1,k\omega}}{g_{m,k\omega}} = -\frac{i}{\sqrt{2(m+1)}} \frac{Z^{(m+1)}(\zeta)}{Z^{(m)}(\zeta)}. \tag{3.55}$$

Taking $m \gg 1, \zeta^2/4$ and using Eq. (3.35), we find¶

$$R_{m+1} = 1 + \frac{i\zeta}{\sqrt{2m}} - \frac{1}{4m} + O\left(\frac{1}{m^{3/2}}\right). \tag{3.56}$$

Therefore, to lowest order in $1/\sqrt{m}$,

$$\tilde{g}^-_{m,k\omega} = (-1)^m \tilde{g}_{m,k\omega} \frac{1 - R_{m+1}}{2} \approx (-1)^{m+1} \frac{i\zeta}{2\sqrt{2m}} \tilde{g}_{m,k\omega}. \tag{3.57}$$

¶The same lowest-order expression can be found by Fourier-transforming Eq. (3.40) in time, ignoring collisions, writing $R_{m+1} = R_m^{-1}\sqrt{m/(m+1)} + i\zeta\sqrt{2/(m+1)}$, approximating $R_m \approx R_{m+1}$, solving the resulting quadratic equation for R_{m+1}, expanding in powers of $1/\sqrt{m}$ and choosing the solution for which $R_{m+1} = 1$ to lowest order. This last step is the main difference between the two methods: if we work with Eq. (3.40) in the manner just described, we have to make an explicit choice between the continuous and alternating solutions ($R_{m+1} = 1$ and $R_{m+1} = -1$); on the other hand, Eq. (3.31) already contains the choice of the former (which is ultimately traceable to Landau's prescription guaranteeing damping rather than growth of the perturbations).

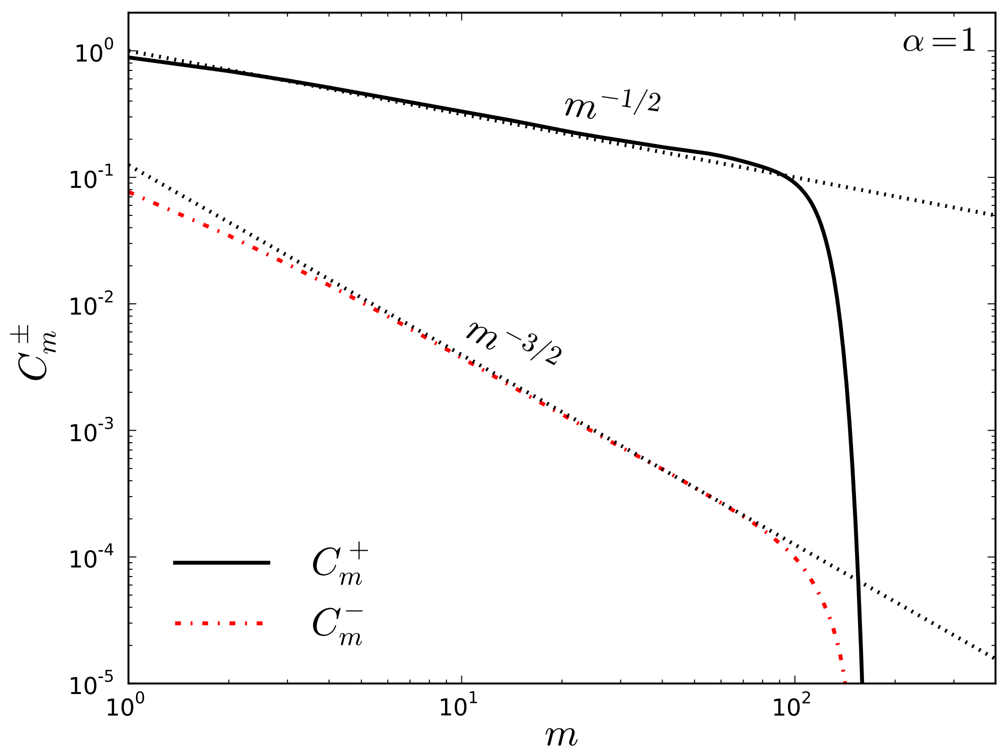

Figure 3.3: The free-energy spectra $C_m^\pm$ obtained via direct numerical solution of Eqs. (3.26–3.28) with $\alpha = 1.0$ followed by decomposing the solution according to Eq. (3.50). In the code, rather than using the Lenard–Bernstein collision operator (as per Eq. (3.28)), hypercollisional regularization, $-\nu m^6 g_{m,k}$, was used to maximize the utility of the velocity-space resolution, hence the very sharp cut off. The dotted lines show the collisionless approximation: Eq. (3.37) for $C_{m,k}^+$ (the phase-mixing "+" mode predominates, so $C_{m,k} \approx C_{m,k}^+$) and Eq. (3.58) for $C_{m,k}^-$.

Following the same steps as those that led to Eq. (3.37)[‖], we get

$$C^-_{m,k} \approx \left[\frac{\varepsilon_k}{8\sqrt{2\pi}|k|} \left(\frac{1+\alpha}{\alpha} \right)^2 \int_{-\infty}^{+\infty} \frac{\mathrm{d}\zeta\, \zeta^2 e^{-\zeta^2}}{|D_\alpha(\zeta)|^2} \right] \frac{1}{m^{3/2}} = \frac{\varepsilon_k (1+\alpha)^2}{16\sqrt{2}|k|} \frac{1}{m^{3/2}}, \qquad (3.58)$$

so both the energy (~ 1, while the total is $\sim \nu^{-1/3}$; see Eq. (3.48)) and the dissipation ($\sim \nu \sum_m m C^-_{m,k} \sim \nu^{2/3}$) associated with the "−" modes is small.

The steady-state spectra $C^\pm_{m,k}$ obtained via direct numerical solution of Eqs. (3.4) and (3.5) are shown in Fig. 3.3, where they are also compared with the analytical expressions (3.37) and (3.58).

Note that we could have, without further ado, simply taken Eq. (3.56) to be the proof of continuity in Hermite space. We have chosen to argue this point via the decomposition (3.49) because it provided us with a more intuitive understanding of the connection between this continuity and the direction of the free-energy flow (phase mixing rather than un-phase mixing).

3.4.6 The simplest Landau-fluid closure

Simplistically described, the idea of Landau-fluid closures is to truncate the Hermite hierarchy of Eqs. (3.26–3.28) at some finite m and to replace in the last retained equation

$$g_{m+1,k}(t) = -(i\,\mathrm{sgn}\,k) R_{m+1} g_{m,k}(t), \qquad (3.59)$$

[‖]The integral is again calculated via Kramers–Kronig relations, this time for the function $h(\zeta) = \zeta^2/D_\alpha(\zeta) - \alpha\zeta^2 - \alpha^2/2$, so $\int_{-\infty}^{+\infty} \mathrm{d}\zeta\, \zeta^2 e^{-\zeta^2}/|D_\alpha(\zeta)|^2 = \alpha^2\sqrt{\pi}/2$.

where R_{m+1}, which in general depends on the complex frequency ζ (Eq. (3.55)), is approximated by some suitable frequency-independent expression leading to the correct recovery of the linear physics from the truncated system. A considerable level of sophistication has been achieved in making these choices and we are not proposing to improve on the existing literature [90–93,95,98,99,102]. It is, however, useful, in the context of the result of Sec. 3.3.1 that the "fluid" version of FDR is recovered in the limit of low frequency and weak damping, to show how the same conclusion can be arrived at via what is probably the simplest possible Landau-fluid closure.

In the limit $\zeta \to 0$, the ratio R_{m+1}, given by Eq. (3.55), becomes independent of ζ and so a closure in the form (3.59) becomes a rigorous approximation. It is not hard to show that

$$Z^{(m)}(0) = \frac{i^{m+1}\sqrt{\pi}\, m!}{\Gamma(m/2+1)}. \tag{3.60}$$

Therefore, for $\zeta \ll 1$ and $m \geq 1$,[**]

$$R_{m+1} = \frac{m}{\sqrt{2(m+1)}} \frac{\Gamma(m/2)}{\Gamma((m+1)/2)}. \tag{3.61}$$

If we wish to truncate at $m = 1$, then $R_2 = \sqrt{\pi}/2$, and so in Eq. (3.27),

$$g_{2,k} = -i \operatorname{sgn} k \frac{\sqrt{\pi}}{2} g_{1,k}. \tag{3.62}$$

[**]The same result can be obtained by inferring $R_{m+1} \approx R_m^{-1}\sqrt{m/(m+1)}$ from Eq. (3.40) (provided $m \ll 1/\zeta^2$), then iterating this up to some Hermite number M such that $1 \ll M \ll 1/\zeta^2$, and approximating $R_M \approx 1$ (Eq. (3.56)). The condition $m, M \ll 1/\zeta^2$ is necessary so that the ζ terms in R_{m+1} are not just small compared to unity but also compared to the next-order $1/m$ terms (see Eq. (3.56)).

On the basis of Eq. (3.26), we must order $g_{1,k} \sim O(\zeta) g_{0,k}$. Thefore, $\partial g_{1,k}/\partial t \sim O(\zeta^2) g_{0,k}$ must be neglected in Eq. (3.27), from which we then learn that

$$g_{1,k} \approx -i \operatorname{sgn} k \sqrt{\frac{2}{\pi}} (1+\alpha) \, g_{0,k}. \tag{3.63}$$

Finally, substituting this into Eq. (3.26), we get

$$\frac{\partial g_{0,k}}{\partial t} + \frac{1+\alpha}{\sqrt{\pi}} |k| g_{0,k} = \chi_k. \tag{3.64}$$

This is a Langevin equation (3.1) with a damping rate that is precisely the Landau damping rate γ_L in the limit $1+\alpha \ll 1$ (and so $\zeta \ll 1$), given by Eq. (3.16). In this limit, $\varphi = -g_0$ (Eq. (3.25), $\alpha \approx -1$) and we recover the standard "fluid" FDR (Eq. (3.17)). As we discussed in Sec. 3.2, a useful application of this regime is to compressive fluctuations in high-beta plasmas: in this case $1+\alpha \approx 1/\beta_i \ll 1$ and the damping is the Barnes damping (also known as transit-time damping) [38], well known in space and astrophysical contexts [1, 119, 120].

3.5 Conclusions and discussion

We have provided a reasonably complete treatment of the simplest generalization of the Langevin problem to plasma kinetic systems. While we have focused on the simplest Langevin problem, in which the source term is a white noise, there is an obvious route towards generalizing this by considering source terms with more coherent time dependence (longer correlation times, prescribed frequency spectra;

see [39]). One such calculation was recently undertaken by Plunk [40], who considered a coherent oscillating source and found that when the frequency of the source is large, the amount of energy that can be absorbed by the kinetic system is exponentially small. Another straightforward generalization (or variation) of our model (as treated in this chapter) is energy injection into momentum, rather than density fluctuations—which can be interpreted as forcing by an externally imposed random electric field. Whereas some of the more literal parallels with the Langevin problem are lost in this case, the results are fundamentally the same (appendix B). Let us itemize the main results and conclusions.

- Eq. (3.13) is the fluctuation-dissipation relation for the kinetic system (Eqs. (3.4) and (3.5)), expressing the relationship between the fluctuation level $\langle|\varphi_k|^2\rangle$ and the injected power. This can be expressed in terms of an "effective" damping rate $\gamma_{\rm eff}$ in a way that resembles the standard "fluid" version of the fluctuation-dissipation relation (Eq. (3.14)), but $\gamma_{\rm eff}$ is not in general equal to the Landau damping rate γ_L. We stress that this result is not a statement of any kind of surprising "modification" of Landau damping in a system with a random source, but rather a clarification of what the linear response in the statistical steady state of such a system actually is. The system, in general, is not mathematically equivalent to the Langevin equation (3.1) and so the fluctuation-dissipation relation for it need not have the same form.

- In the limit of zero real frequency and weak Landau damping, the effective and

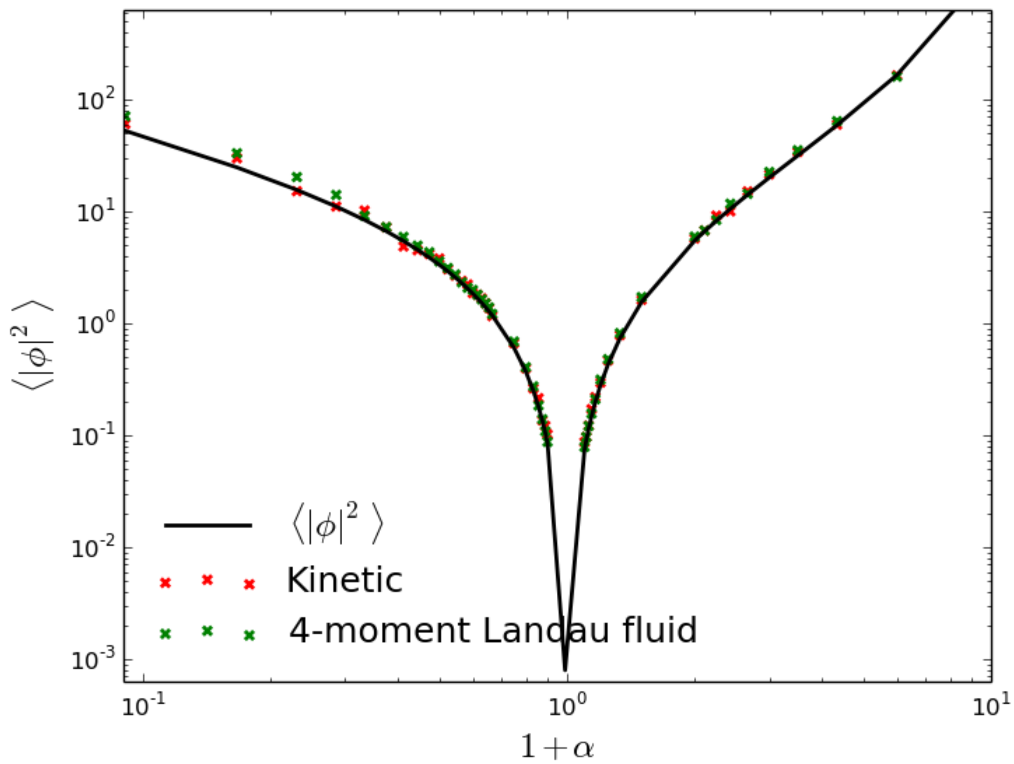

Figure 3.4: Reproduction of Fig. 3.1 along with the normalized saturated amplitude calculated using a 4-moment Landau fluid model (green crosses).

the Landau damping rates do coincide (Eq. (3.17)). Another way to view this result is by noting that this is a regime in which the simplest possible Landau-fluid closure becomes a rigorous approximation and the evolution equation for the electrostatic potential can be written as a Langevin equation with the Landau damping rate γ_L (Eq. (3.64)). It is crucial to note, however, that a more realistic 4-moment Landau-fluid model reproduces the kinetic results with near-perfect accuracy as can be seen in Fig. 3.4.

- Another limit in which the fluctuation-dissipation relation for the kinetic system can be interpreted in "fluid" (in fact, mechanical) terms is one of high real frequency and exponentially Landau small damping, although the correct analogy is not the Langevin equation but a forced-damped oscillator (Sec. 3.3.2; this analogy, however, ceases to hold in such a simple form for a different choice of forcing, as shown in appendix B).

- The damping of the perturbations of φ (which are linearly proportional to the density perturbations) occurs via phase mixing, which transfers the free energy originally injected into φ away from it and into higher moments of the perturbed distribution function. This process can be described as a free-energy flow in Hermite space. The generalization of the FDR to higher-order Hermite moments takes the form of an expression for the Hermite spectrum $C_{m,k}$ (Eq. (3.33)), which at high Hermite numbers $m \gg 1$ has a universal scaling $C_{m,k} \propto 1/\sqrt{m}$ (Eq. (3.37)). This scaling corresponds to a constant free-energy flux from low to high m (Eq. (3.42)). Analysis of the solutions of the kinetic equation making use of a formal decomposition of these solutions into phase mixing and un-phase mixing modes underscores the predominance of the former (Sec. 3.4.5).

- A solution for the Hermite spectrum including the collisional cutoff is derived (Eq. (3.44)). The fluctuation-dissipation relation for the total free energy stored in the phase space (Eq. (3.48)) shows that it diverges $\propto \nu^{-1/3}$ in the limit of vanishing collisionality ν, a result that underscores the fact that ultimately all dissipation (i.e., all entropy production in the system) is collisional.

In the process of deriving these results, we have made an effort to explain the simple connections between the Landau formalism (solutions of the kinetic equation expressed via the plasma dispersion function) and the Hermite-space one. We are not aware of any work where the results presented here are adequately explained—although implicitly they underlie the thinking behind both Landau-fluid closures [90–93,95,98,99,102] and Hermite-space treatments for plasma kinetics [40,106–114].

Besides providing a degree of clarity on an old topic in the linear theory of collisionless plasmas, our findings lay the groundwork for a study of the much more complicated nonlinear problem of the role of Landau damping and phase mixing in turbulent collisionless plasma systems [118,121], which is carried out in the following chapters.

Chapter 4

Kinetic passive scalar advection by 2D velocity

4.1 Introduction

Advection of a passive scalar by a turbulent velocity field is a fundamental and well studied problem in hydrodynamic turbulence [14, 54–78]. Investigations into passive scalar turbulence have helped develop the basic ideas underlying hydrodynamic turbulence theory (Refs. [69, 77, 78] give thorough reviews of this topic). Recently, a few authors have carried out numerical investigations of passive scalar advection in magnetohydrodynamic turbulence [122–125]. However, to the best of our knowledge, kinetic passive scalar turbulence has not been studied before.

A kinetic passive scalar is slaved to a turbulent cascade while simultaneously being phase mixed. Since the particle distribution functions for such systems may develop non-trivial velocity space structure, it is unclear if the results regarding passive scalar advection derived in the fluid limit will still be valid in the kinetic regime. In the fluid limit, the passive scalar acquires the same energy spectrum as the advecting velocity field. However, in the kinetic limit, if phase mixing turns out to be the dominant process, the turbulent cascade of the scalar will terminate. This will result in an exponentially attenuated spectrum. The key question then is whether a kinetic passive scalar has a power law spectrum, or an exponentially decaying spectrum.