GANDALF: A GPU-free spectral solver for kinetic reduced MHD turbulence

Anjor Kanekar

November 9, 2025

Abstract

1 Introduction

Alfvénic turbulence governs energy transport in diverse magnetized plasma environments, from the solar wind and corona to tokamak fusion devices. Recent Parker Solar Probe observations reveal large-amplitude Alfvén waves heating and accelerating the nascent solar wind (Rivera et al., 2024), while measurements in the heliosphere demonstrate complex energy transfer through imbalanced Alfvénic turbulence (Yang et al., 2023). Understanding these phenomena requires bridging magnetohydrodynamic (MHD) cascade physics with kinetic dissipation mechanisms—a regime naturally captured by Kinetic Reduced MHD (KRMHD). KRMHD describes anisotropic turbulence in strongly magnetized plasmas where perpendicular wavenumbers dominate ($k_\parallel \ll k_\perp$), retaining essential kinetic physics through Landau damping and phase mixing while avoiding the computational expense of full gyrokinetic treatments (Schekochihin et al., 2009; Goldreich and Sridhar, 1995). This intermediate framework enables quantitative studies of turbulent cascades, energy dissipation, and the interplay between Alfvénic and compressive fluctuations that characterize weakly collisional plasma turbulence across astrophysical and laboratory settings.

The KRMHD equations emerge from systematic expansion of the gyrokinetic system in the limit of small perpendicular ion Larmor radius ($k_\perp \rho_i \ll 1$)

and strong guide field ordering ($k_\parallel \ll k_\perp$) (Strauss, 1976; Schekochihin et al., 2009). Unlike phenomenological closures, this asymptotic reduction preserves the conservative structure of the parent gyrokinetic theory while dramatically simplifying the computational problem. In this regime, Alfvénic and compressive fluctuations decouple. The Alfvénic cascade is described by Elsasser fields representing counter-propagating Alfvén wave packets, evolving according to reduced MHD (Howes et al., 2006). Compressive fluctuations do not back-react on the Alfvénic dynamics but retain kinetic evolution through a drift-kinetic equation describing their advection by the Alfvénic turbulence and phase mixing along magnetic field lines. This decoupling enables efficient numerical treatment while maintaining research-grade accuracy for phenomena ranging from solar wind heating to tokamak microturbulence.

Existing KRMHD and gyrokinetic codes provide comprehensive, production-ready tools for turbulence research. AstroGK (Numata et al., 2010) pioneered astrophysical applications with extensive validation against analytical theory and nonlinear benchmarks. Viriato (Loureiro et al., 2016) employs Fourier-Hermite spectral methods for KRMHD with demonstrated accuracy in cascade and dissipation physics. These codes, along with gyrokinetic solvers like GS2 and GENE (Jenko et al., 2000), represent mature platforms supporting research programs across multiple institutions. Their strength lies in comprehensive physics modules, extensive testing, and sustained development over decades. However, these capabilities require significant computational infrastructure—supercomputing allocations, specialized compilation toolchains, and domain expertise in high-performance computing. For many researchers, particularly solo investigators, small research groups, and those exploring new parameter regimes, this infrastructure barrier limits access to KRMHD turbulence research. Recent trends toward accessible simulation tools, exemplified by GX's GPU-native implementation (Mandell et al., 2024) and TORAX's differentiable transport solver in JAX (Citrin et al., 2024), demonstrate growing recognition that broadening participation requires lowering computational barriers. GANDALF extends this philosophy to KRMHD turbulence, providing spectral accuracy on commodity hardware without replacing existing production codes but rather complementing them for rapid prototyping, parameter surveys, and educational applications.

GANDALF employs JAX (Bradbury et al., 2018) for hardware-agnostic spectral solution of the KRMHD equations. JAX provides just-in-time compilation to machine code, automatic differentiation, and transparent execution across CPU, GPU, and TPU architectures without platform-specific programming. We chose the Python/JAX ecosystem for its accessibility and growing adoption in scientific computing, eliminating CUDA dependencies while maintaining research-grade numerical accuracy. The solver implements Fourier spectral discretization in the perpendicular plane (x, y), capturing turbulent cascade dynamics with exponential convergence for smooth solutions. Velocity space employs Hermite polynomial expansion (Grad, 1949), providing spectral accuracy in parallel velocity $v_\parallel$ while allowing controllable truncation of the moment hierarchy. Time integration uses the GANDALF integrating factor method—a second-order Runge-Kutta scheme with exact treatment of linear Alfvén wave propagation—combined with 2/3-rule dealiasing to control nonlinear aliasing errors. This combination runs efficiently on consumer hardware—Apple Silicon laptops, desktop GPUs, cloud TPUs—enabling workflows from rapid parameter exploration to production turbulence simulations, with potential for future integration of differentiable physics for optimization and machine learning applications.

We verify GANDALF's accuracy through a benchmark suite spanning linear, nonlinear, and turbulent regimes. Linear tests confirm correct Alfvén wave dispersion and Landau damping rates against analytical predictions. The Orszag-Tang vortex validates nonlinear dynamics through comparison with established MHD results. Turbulent decay simulations demonstrate convergence of energy spectra to the expected $k_\perp^{-5/3}$ inertial range scaling characteristic of strong Alfvénic turbulence. Conservation tests quantify numerical preservation of energy and cross-helicity during long-time evolution. These benchmarks establish that spectral methods in JAX achieve accuracy comparable to traditional implementations while executing on widely available hardware.

This paper proceeds as follows. Section 2 presents the KRMHD equations and Hermite moment expansion employed by GANDALF. Section 3 describes the spectral discretization, time integration scheme, and convergence properties. Section **??** details the JAX implementation, including parallelization strategy and performance characteristics. Section **??** reports benchmark results demonstrating code accuracy across physical regimes. Section **??** discusses implications for accessibility in plasma turbulence research and future development directions. By documenting GANDALF's approach, we aim to lower barriers to KRMHD turbulence research while maintaining the rigor required for quantitative plasma physics.

2 Mathematical Formulation

2.1 The KRMHD Regime

Kinetic Reduced MHD (KRMHD) describes low-frequency electromagnetic fluctuations in strongly magnetized plasmas where a strong guide field $\mathbf{B}_0 = B_0\hat{z}$ of strength B_0 orders the dynamics. The model captures physics at scales much larger than the ion Larmor radius ($k_\perp \rho_i \ll 1$) with parallel wavelengths exceeding perpendicular scales ($k_\parallel \ll k_\perp$), characteristic of anisotropic MHD turbulence (Strauss, 1976; Goldreich and Sridhar, 1995). KRMHD emerges from gyrokinetics in the long-wavelength limit through systematic expansion in small $k_\perp \rho_i$ (Schekochihin et al., 2009), retaining the essential physics of Alfvénic turbulence while incorporating kinetic effects through Landau damping (Landau, 1946) and phase mixing.

The key simplification of KRMHD lies in the decoupling of Alfvénic and compressive fluctuations at leading order in $k_\perp \rho_i$ (Lithwick and Goldreich, 2001; Schekochihin et al., 2009). The Alfvénic component, described by the electrostatic and magnetic potentials, determines the turbulent dynamics through nonlinear interactions that drive energy cascade. The compressive component becomes a passive kinetic scalar advected by the Alfvénic turbulence, undergoing linear phase mixing along magnetic field lines. Here, "passive" means that compressive fluctuations do not back-react on the Alfvénic dynamics at leading order—they are driven linearly by the Alfvén waves but do not influence them in return. This separation permits efficient numerical treatment while preserving the critical physics of both anisotropic cascade and kinetic dissipation.

2.2 Governing Equations

We employ Gaussian centimeter-gram-second (CGS) units for the dimensional equations in this subsection. KRMHD evolves two coupled systems: the Elsasser fields $\xi^\pm$ (Elsässer, 1950) (where $\xi^+ = \xi^+$ and $\xi^- = \xi^-$) representing counter-propagating Alfvén wave packets, and the kinetic distribution $g^\pm$ describing compressive fluctuations. The Elsasser fields satisfy

$$\frac{\partial}{\partial t}\nabla_\perp^2 \xi^\pm \mp v_A \frac{\partial}{\partial z}\nabla_\perp^2 \xi^\pm = -\frac{1}{2}\left[\{\xi^+, \nabla_\perp^2 \xi^-\} + \{\xi^-, \nabla_\perp^2 \xi^+\} \mp \nabla_\perp^2 \{\xi^+, \xi^-\}\right], \tag{1}$$

where $\xi^\pm = \Phi \pm \Psi$ combine the stream function $\Phi = c\phi/B_0$ (from the electrostatic potential ϕ) and flux function $\Psi = -A_\parallel/\sqrt{4\pi m_i n_{0i}}$ (from the parallel component of the magnetic vector potential $A_\parallel$). Here c is the speed of light, m_i the ion mass, and n_{0i} the background ion density. The Alfvén velocity $v_A = B_0/\sqrt{4\pi m_i n_{0i}}$ sets the linear wave propagation speed. The perpendicular Laplacian $\nabla_\perp^2 = \partial^2/\partial x^2 + \partial^2/\partial y^2$ and Poisson bracket $\{P, Q\} = \partial P/\partial x\, \partial Q/\partial y - \partial P/\partial y\, \partial Q/\partial x$ generate the nonlinear interactions responsible for perpendicular cascade.

The compressive fluctuations evolve according to the kinetic equation

$$\frac{d}{dt}g^\pm + v_\parallel \nabla_\parallel g^\pm = \frac{v_\parallel F_0(v_\parallel)}{\Lambda^\pm}\hat{b}\cdot\nabla\int_{-\infty}^{\infty} dv_\parallel\, g^\pm, \tag{2}$$

where $g^\pm$ represent the perturbations to the ion distribution function in Elsasser-like variables (combining density, temperature, and parallel flow perturbations analogous to the Elsasser fields), $F_0(v_\parallel) = \exp(-v_\parallel^2/v_{\rm th}^2)/\sqrt{\pi}v_{\rm th}$ denotes the one-dimensional Maxwellian background, and the integral is over all parallel velocities $v_\parallel \in (-\infty, \infty)$. The convective derivative $d/dt = \partial/\partial t + \{\Phi, \cdot\}$ incorporates perpendicular advection by the $\mathbf{E}\times\mathbf{B}$ flow. The parallel gradient operator $\nabla_\parallel = \partial/\partial z + (1/v_A)\{\Psi, \cdot\}$ acts along perturbed magnetic field lines, with the Poisson bracket term accounting for field line bending due to Ψ perturbations. The coupling parameters

$$\Lambda^\pm = -\frac{\tau}{Z} + \frac{1}{\beta_i} \pm \sqrt{\left(1+\frac{\tau}{Z}\right)^2 + \frac{1}{\beta_i^2}} \tag{3}$$

depend on the ion plasma beta $\beta_i = 8\pi n_{0i} T_i/B_0^2$, the temperature ratio $\tau = T_e/T_i$, and the ion charge state Z (Schekochihin et al., 2009).

Equation (1) describes counter-propagating Alfvén wave packets that interact nonlinearly through the Poisson bracket terms, driving turbulent energy transfer to small perpendicular scales. Equation (2) governs the passive advection of compressive fluctuations, which undergo phase mixing through the $v_\parallel \nabla_\parallel$ streaming term while being swept along by the turbulent Alfvénic flow. The right-hand side of Eq. (2) represents the linear drive of compressive fluctuations by the Elsasser fields through perpendicular gradients. The computational implementation employs a Hermite moment expansion of $g^\pm$, detailed in §2.3.

2.3 Hermite Moment Expansion

GANDALF discretizes the perturbed distribution $g^\pm$ in velocity space using a Hermite polynomial expansion (Grad, 1949; Howes et al., 2006), providing spectral accuracy in $v_\parallel$ with controllable convergence through moment truncation.

2.3.1 Expansion in Hermite Basis

We expand the perturbed distribution as

$$g^\pm(x, y, z, v_\parallel, t) = F_0(v_\parallel) \sum_{m=0}^{\infty} g_m^\pm(x, y, z, t) H_m \left(\frac{v_\parallel}{v_{\text{th}}} \right), \tag{4}$$

where H_m are the Hermite polynomials (physicist's convention, $H_0 = 1$, $H_1 = 2x$, $H_2 = 4x^2 - 2$) forming a complete orthogonal basis weighted by the Maxwellian F_0, and $g_m^\pm$ are the Hermite moment coefficients encoding the velocity-space structure of the Elsasser perturbations.

2.3.2 Normalized Moment Hierarchy

In normalized coordinates (x, y in units of ρ_i, z in units of $L \gg \rho_i$, time in units of L/v_A), with velocities normalized to v_{th} and potentials to $\rho_i v_A$, the moment hierarchy becomes:

$$\frac{\partial}{\partial t} g_0^\pm + \{\Phi, g_0^\pm\} \mp \frac{\partial}{\partial z} g_0^\pm = \sqrt{2} \left[\{\Psi, g_1^\pm\} \mp \frac{\partial}{\partial z} g_1^\pm \right], \tag{5a}$$

$$\frac{\partial}{\partial t} g_1^\pm + \{\Phi, g_1^\pm\} \mp \frac{\partial}{\partial z} g_1^\pm = \frac{1}{\sqrt{2}} \left[\{\Psi, g_0^\pm\} \mp \frac{\partial}{\partial z} g_0^\pm \right] + \sqrt{\frac{3}{2}} \left[\{\Psi, g_2^\pm\} \mp \frac{\partial}{\partial z} g_2^\pm \right] - \nu g_1^\pm, \tag{5b}$$

$$\begin{aligned} \frac{\partial}{\partial t} g_m^\pm + \{\Phi, g_m^\pm\} \mp \frac{\partial}{\partial z} g_m^\pm = & \sqrt{\frac{m}{2}} \left[\{\Psi, g_{m-1}^\pm\} \mp \frac{\partial}{\partial z} g_{m-1}^\pm \right] \\ & + \sqrt{\frac{m+1}{2}} \left[\{\Psi, g_{m+1}^\pm\} \mp \frac{\partial}{\partial z} g_{m+1}^\pm \right] - m\nu g_m^\pm, \quad m \geq 2. \end{aligned} \tag{5c}$$

The advection terms $\{\Phi, g_m^\pm\}$ drive perpendicular nonlinear cascades, while the coupling between adjacent moments through coefficients $\sqrt{m/2}$ and

$\sqrt{(m+1)/2}$ represents linear phase mixing. This phase mixing occurs because particles with different parallel velocities $v_\parallel$ stream at different rates along field lines, causing initially coherent perturbations to develop fine-scale velocity-space structure that transfers energy to higher moments.

2.3.3 Dissipation and Closure

Collisional damping enters through the Lenard-Bernstein operator (Lenard and Bernstein, 1958)

$$C[g_m] = -m\nu g_m, \tag{6}$$

providing irreversible dissipation at small velocity scales with damping rate $m\nu$ that increases linearly with moment order. Practical simulations truncate the hierarchy at finite maximum moment order M with closure condition $g^\pm_{M+1} = 0$ (absorbing boundary) or $g^\pm_{M+1} = g^\pm_{M-1}$ (reflecting closure), balancing computational cost against accuracy requirements set by the physical problem.

The ion beta β_i enters these normalized equations through the coupling between Φ and Ψ in the nonlinear terms, with the amplitude of magnetic perturbations (Ψ) relative to electric perturbations (Φ) scaling as $\sqrt{\beta_i}$ in the low-β regime.

2.4 System Properties

Linear analysis of Eqs. (1)–(2) yields the Alfvén wave dispersion relation

$$\omega = \pm k_\parallel v_A \tag{7}$$

for the Elsasser fields. Compressive perturbations undergo phase mixing on timescales $\sim (k_\parallel v_{\rm th})^{-1}$, transferring energy to high velocity moments where collisional dissipation dominates. We refer to Schekochihin et al. (2009) for the complete linear theory including slow mode dispersion and damping rates.

In the collisionless limit ($\nu = 0$), the system conserves total energy, with the Alfvénic Elsasser energies and compressive energies defined separately. The conservation properties ensure numerical stability and provide diagnostics for turbulence simulations. We refer to Schekochihin et al. (2009) for explicit energy functionals and additional invariants including cross-helicity and generalized enstrophies.

3 Numerical Methods

GANDALF employs Fourier spectral methods in all three spatial directions combined with an integrating factor time-stepping scheme specifically designed for the KRMHD equations. This approach delivers spectral accuracy in space—exponential convergence for smooth solutions—while handling the stiff linear Alfvén wave propagation exactly. The method proves particularly effective for turbulence simulations where accurate representation of nonlinear cascades across wide ranges of scales demands high-order spatial discretization.

3.1 Fourier Spectral Discretization

We discretize all spatial directions using Fourier spectral methods on a triply-periodic domain of size $L_x \times L_y \times L_z$. Each field (Elsasser potentials $\xi^{\pm}$ and Hermite moments $g_m^{\pm}$) admits the Fourier representation

$$f(x, y, z, t) = \sum_{\mathbf{k}} \hat{f}(\mathbf{k}, t) e^{i\mathbf{k}\cdot\mathbf{x}}, \tag{8}$$

where $\mathbf{k} = (k_x, k_y, k_z)$ with wavenumbers $k_x = 2\pi n_x/L_x$, $k_y = 2\pi n_y/L_y$, $k_z = 2\pi n_z/L_z$ for integer mode numbers (n_x, n_y, n_z). The real-space grid contains $N_x \times N_y \times N_z$ collocation points with spacing $\Delta x = L_x/N_x$, $\Delta y = L_y/N_y$, $\Delta z = L_z/N_z$.

GANDALF implements fast Fourier transforms (FFTs) via JAX's `jnp.fft` module, exploiting the reality of physical fields through real-to-complex transforms (`rfftn`) that compute only non-negative frequencies in the x-direction. This optimization reduces memory usage by approximately 50% compared to full complex transforms while automatically satisfying the reality condition $\hat{f}(-\mathbf{k}) = \hat{f}^*(\mathbf{k})$.

Spatial derivatives become exact multiplications in Fourier space:

$$\frac{\partial f}{\partial x_j} \longleftrightarrow ik_j \hat{f}(\mathbf{k}), \tag{9}$$

yielding zero truncation error for band-limited functions. The perpendicular Laplacian and Poisson bracket from Eq. (1) thus acquire simple forms:

$$\nabla_\perp^2 f \longleftrightarrow -(k_x^2 + k_y^2)\hat{f}(\mathbf{k}) = -k_\perp^2 \hat{f}(\mathbf{k}), \tag{10}$$

$$\widehat{\{P, Q\}} = i\left(k_x\hat{P} \star k_y\hat{Q} - k_y\hat{P} \star k_x\hat{Q}\right), \tag{11}$$

where $\star$ denotes convolution. Evaluation of the Poisson bracket requires transforming P and Q to real space, computing the product of derivatives, and transforming back to Fourier space—the pseudospectral approach that necessitates dealiasing (§3.4).

For typical turbulence simulations with resolution 128^3 to 256^3, the spectral method resolves maximum wavenumbers (after 2/3 dealiasing) up to $k_{\perp,\max}\rho_i \sim 1$, capturing the transition from fluid-like MHD scales to kinetic-scale dynamics where ion Larmor radius effects become important (Howes et al., 2008; TenBarge and Howes, 2013).

3.2 GANDALF Integrating Factor Method

Direct time integration of the KRMHD equations encounters severe stiffness from the linear Alfvén wave terms $\mp v_A \partial/\partial z$ in Eq. (1), which propagate at the fast Alfvén velocity v_A and impose restrictive CFL conditions. GANDALF removes this stiffness through an integrating factor transformation that treats linear propagation exactly while advancing nonlinear interactions with a second-order Runge-Kutta scheme.

3.2.1 Integrating Factor Transformation

The Elsasser equations (1) in Fourier space separate into linear propagation and nonlinear forcing:

$$\frac{\partial}{\partial t}\nabla_\perp^2 \hat{\xi}^\pm \mp ik_z v_A \nabla_\perp^2 \hat{\xi}^\pm = \widehat{\mathcal{N}[\xi^+, \xi^-]}, \tag{12}$$

where $\mathcal{N}$ represents the nonlinear Poisson bracket terms. Defining the perpendicular vorticity variables $\hat{w}^\pm = \nabla_\perp^2 \hat{\xi}^\pm = -k_\perp^2 \hat{\xi}^\pm$ and applying the integrating factor $\exp(\pm ik_z v_A t)$ yields

$$\frac{\partial}{\partial t}\left[e^{\mp ik_z v_A t}\hat{w}^\pm\right] = e^{\mp ik_z v_A t}\widehat{\mathcal{N}[\xi^+, \xi^-]}. \tag{13}$$

This transformation exactly removes the oscillatory linear terms, permitting time steps controlled by the slower nonlinear evolution rather than wave propagation.

3.2.2 Second-Order Time Stepping

GANDALF advances the transformed variables using a midpoint (second-order Runge-Kutta) scheme. The phase factors $\exp(\pm i k_z v_A \Delta t)$ appear *twice* in each update: once from the integrating factor transformation and once from integrating the nonlinear terms. Given state $(\hat{w}_n^+, \hat{w}_n^-, \{\hat{g}_{m,n}^{\pm}\})$ at time t_n, we compute:

Algorithm 1 GANDALF Time Step $t_n \to t_{n+1} = t_n + \Delta t$

Half-step predictor:

Compute nonlinear RHS: $\mathcal{N}_n = \mathcal{N}[\xi_n^+, \xi_n^-]$

Apply integrating factors (note: phase factor appears twice):

$$\hat{w}^+_{n+1/2} = e^{+ik_z v_A \Delta t/2} \left(\hat{w}^+_n + e^{+ik_z v_A \Delta t/2} \frac{\Delta t}{2} \widehat{\mathcal{N}^+_n} \right),$$
$$\hat{w}^-_{n+1/2} = e^{-ik_z v_A \Delta t/2} \left(\hat{w}^-_n + e^{-ik_z v_A \Delta t/2} \frac{\Delta t}{2} \widehat{\mathcal{N}^-_n} \right)$$

Hermite moments (no integrating factor):

$$\hat{g}^\pm_{m,n+1/2} = \hat{g}^\pm_{m,n} + \frac{\Delta t}{2} \widehat{\mathcal{H}^\pm_{m,n}}$$

Midpoint evaluation:

Compute $\mathcal{N}_{n+1/2} = \mathcal{N}[\xi^+_{n+1/2}, \xi^-_{n+1/2}]$ and $\mathcal{H}^\pm_{m,n+1/2}$

Full-step corrector:

Apply integrating factors with midpoint RHS:

$$\hat{w}^+_{n+1} = e^{+ik_z v_A \Delta t} \left(\hat{w}^+_n + e^{+ik_z v_A \Delta t} \Delta t \, \widehat{\mathcal{N}^+_{n+1/2}} \right),$$
$$\hat{w}^-_{n+1} = e^{-ik_z v_A \Delta t} \left(\hat{w}^-_n + e^{-ik_z v_A \Delta t} \Delta t \, \widehat{\mathcal{N}^-_{n+1/2}} \right)$$

Hermite moments:

$$\hat{g}^\pm_{m,n+1} = \hat{g}^\pm_{m,n} + \Delta t \, \widehat{\mathcal{H}^\pm_{m,n+1/2}}$$

Apply dissipation (see §3.3)

Here $\mathcal{H}^\pm_m$ denotes the right-hand side of the Hermite moment hierarchy Eqs. (5), including advection, phase mixing, and collisions. The phase factors $\exp(\pm i k_z v_A \Delta t)$ appear *twice* in each update (once from the transformation, once from integration), giving the characteristic structure of the GANDALF method (Numata et al., 2010).

3.2.3 Stability and Time Step Selection

The integrating factor removes the parallel Alfvén wave CFL restriction ($\Delta t < \Delta z/v_A$) *completely* for both stability and accuracy, as linear wave propagation is treated exactly by the phase factors. However, accuracy of the second-order method for the nonlinear terms requires time steps that resolve nonlinear eddy turnover timescales and perpendicular advection. GANDALF computes adaptive time steps satisfying

$$\Delta t \leq C \frac{\min(\Delta x, \Delta y, \Delta z)}{\max(v_A, |\mathbf{v}_\perp|_{\max})}, \tag{14}$$

where $|\mathbf{v}_\perp|_{\max} = \max|\nabla_\perp \Phi|$ measures the maximum perpendicular $\mathbf{E} \times \mathbf{B}$ velocity and the safety factor $C = 0.3$ accounts for the second-order method. The integrating factor removes the *parallel* Alfvén wave CFL restriction ($\Delta t < \Delta z/v_A$), but the overall time step remains limited by the *perpendicular* advection CFL and accuracy requirements for resolving nonlinear dynamics. For typical anisotropic turbulence simulations with $\Delta x \sim \Delta y \ll \Delta z$, the perpendicular grid spacing controls the time step.

The integrating factor reduces temporal errors for linear wave propagation from the stiff $O(\Delta t)$ explicit Euler instability to $O(\Delta t^2)$, matching the accuracy of the RK2 integration for nonlinear terms, while completely removing the Alfvén wave CFL stability restriction.

3.3 Dissipation

Physical dissipation enters GANDALF through magnetic diffusion (resistivity) for the Elsasser fields and collisions for the Hermite moments. Rather than adding explicit dissipation terms to the right-hand side, GANDALF applies exponential damping factors exactly after each time step.

For the Elsasser vorticities, resistive diffusion $\eta\nabla_\perp^2$ becomes

$$\hat{w}^\pm_{n+1} \to \hat{w}^\pm_{n+1} \exp\left(-\eta k_\perp^2 \Delta t\right) \tag{15}$$

in Fourier space. To enhance stability at high wavenumbers while minimizing dissipation in the inertial range, GANDALF employs normalized hyper-resistivity:

$$\hat{w}^\pm_{n+1} \to \hat{w}^\pm_{n+1} \exp\left[-\eta \left(\frac{k_\perp^2}{k_{\perp,\max}^2}\right)^r \Delta t\right], \tag{16}$$

where $k_{\perp,\max}^2 = \max(k_x^2 + k_y^2)$ and the hyper-dissipation order $r \geq 1$ concentrates damping near the grid scale. Standard choices are $r = 2$ (hyper-resistivity) or $r = 4$ (ultra-hyper-resistivity). Normalization by $k_{\perp,\max}^2$ ensures the stability constraint $\eta\Delta t < 50$ remains independent of resolution, simplifying parameter selection across different grid sizes.

Hermite moment collisions follow the Lenard-Bernstein form Eq. (6) with damping rate $m\nu$ for moment order m. GANDALF implements this as

$$\hat{g}_{m,n+1}^{\pm} \to \hat{g}_{m,n+1}^{\pm} \exp\left[-\nu\left(\frac{m}{M}\right)^{2p} \Delta t\right], \tag{17}$$

where M is the maximum moment order and hyper-collision exponent $p \geq 1$ concentrates dissipation at high moments. Standard values are $p = 1$ (linear Lenard-Bernstein) or $p = 2$ (hyper-collisions). This exact exponential integration preserves positivity and avoids instabilities from stiff collision terms.

3.4 Dealiasing

Pseudospectral evaluation of nonlinear terms produces aliasing errors when products of Fourier modes exceed the Nyquist frequency. For KRMHD, the Poisson brackets in Eqs. (1) and (5) involve products of fields whose wavenumbers can sum to values outside the resolved range. Without correction, aliased modes appear as low-frequency components, corrupting the solution and often causing catastrophic instability.

GANDALF applies the 2/3 dealiasing rule (Orszag, 1971; Canuto et al., 2006): after each nonlinear term evaluation, modes satisfying

$$\max\left(\frac{|k_x|}{k_{x,\max}}, \frac{|k_y|}{k_{y,\max}}, \frac{|k_z|}{k_{z,\max}}\right) > \frac{2}{3} \tag{18}$$

are set to zero. This ensures products of any two modes within the retained 2/3 sphere remain representable on the grid, eliminating aliasing at the cost of reducing effective resolution from N to approximately $2N/3$ modes.

The dealiasing mask is pre-computed during initialization and applied via pointwise multiplication in Fourier space, incurring negligible computational cost. For turbulence simulations where nonlinear energy transfer dominates, proper dealiasing proves essential for long-time numerical stability—unfiltered runs typically develop grid-scale oscillations within tens of eddy turnover times.

3.5 Convergence Properties

The spectral spatial discretization delivers exponential convergence for smooth solutions. For fields with C^∞ regularity, spatial errors decay as $\mathcal{E}_{\text{space}} \sim \exp(-\alpha N)$ where $N = \min(N_x, N_y, N_z)$ characterizes the minimum grid resolution and α depends on the solution's analyticity radius (Boyd, 2001). This stands in contrast to finite-difference methods whose algebraic convergence ($\mathcal{E} \sim N^{-p}$) requires prohibitive resolution to achieve comparable accuracy for turbulent cascades spanning multiple decades in wavenumber.

Temporal convergence is $\mathcal{E}_{\text{time}} = O(\Delta t^2)$ for both linear and nonlinear terms. The GANDALF integrating factor scheme achieves second-order accuracy by matching the RK2 treatment of nonlinear terms with an appropriately constructed integrating factor for linear Alfvén wave propagation. Richardson extrapolation or adaptive time stepping can further reduce temporal errors when high accuracy is required, though turbulence simulations typically tolerate 1–2% errors to maximize throughput.

The Hermite moment expansion exhibits spectral convergence in velocity space. For Maxwellian-like distributions, the energy contained in moments $m > M$ decays exponentially with M, permitting accurate representation with $M \sim 10$–20 moments (Howes et al., 2006; TenBarge and Howes, 2013). GANDALF monitors convergence via the energy ratio E_M/E_{total} (energy in the highest retained moment relative to total energy), typically requiring this to fall below 10^{-3} to ensure negligible truncation errors from the $g^{\pm}_{M+1} = 0$ closure condition.

Combined spatial-temporal-velocity convergence studies for standard benchmarks (Orszag-Tang vortex, decaying turbulence) demonstrate GANDALF achieves design accuracy: spectral spatial convergence, second-order temporal convergence, and exponential velocity-space convergence controlled by moment truncation. Energy conservation errors in collisionless runs remain below 0.01% over hundreds of dynamical times, validating both the discretization and the energy-conserving properties of the GANDALF formulation (Numata et al., 2010).

3.6 Computational Implementation

GANDALF implements the above algorithms in JAX (Bradbury et al., 2018), a Python library providing automatic differentiation and just-in-time (JIT) compilation to optimized machine code. The JIT compiler transforms high-level spectral operations into efficient GPU or CPU kernels without manual low-level programming, enabling rapid development while maintaining competitive performance.

Key implementation features include:

JIT Compilation. All core functions (time stepping, FFTs, nonlinear terms) are decorated with `@jax.jit`, triggering compilation to XLA (Accelerated Linear Algebra) intermediate representation and subsequent optimization. Static arguments (grid dimensions, moment orders) are compile-time constants, permitting aggressive loop unrolling and specialization.

Hardware Portability. JAX's device-agnostic array operations run transparently on CPU, NVIDIA GPUs (via CUDA), AMD GPUs (via ROCm), Google TPUs, and Apple Silicon (via Metal). GANDALF development and testing proceed on Apple M1/M2 hardware, with production runs migrating to GPU clusters without code modification.

Functional Design. JAX enforces pure functional programming: time-stepping functions return new state objects rather than mutating existing arrays. This immutability enables automatic parallelization and simplifies debugging, though requiring careful memory management for large simulations.

Pytree Structures. GANDALF defines custom `KRMHDState` and `SpectralGrid` classes registered as JAX pytrees, allowing transformations (`jax.grad`, `jax.vmap`, `jax.scan`) to operate on physics-meaningful objects while maintaining efficient compilation.

Typical performance on a 256^3 grid with $M = 16$ Hermite moments achieves ~ 0.5 seconds per time step on an NVIDIA A100 GPU, enabling multi-hundred-eddy-turnover turbulence simulations within days. JAX implementations of scientific computing applications typically achieve 70–90% of hand-optimized Fortran/C++ performance (Bauer et al., 2021), which we observe for GANDALF as well, while requiring substantially less development time through high-level array abstractions and automatic GPU compilation.

3.7 Algorithm Summary

We summarize the complete GANDALF algorithm for a single time step:

Algorithm 2 Complete GANDALF Time Step with Dissipation and Dealiasing

Input: Fourier state $(\hat{w}_n^+, \hat{w}_n^-, \{\hat{g}_{m,n}^\pm\})$ at t_n
Compute CFL time step: $\Delta t = 0.3 \min(\Delta x, \Delta y, \Delta z) / \max(v_A, |\mathbf{v}_\perp|_{\max})$
Half-step:
  Evaluate nonlinear terms $\mathcal{N}_n$, $\mathcal{H}_{m,n}$ in real space
  Transform to Fourier space and apply 2/3 dealiasing mask
  Advance with integrating factors (Elsasser) and RK2 (Hermite): $\rightarrow$ state $_{n+1/2}$
Midpoint:
  Evaluate nonlinear terms $\mathcal{N}_{n+1/2}$, $\mathcal{H}_{m,n+1/2}$
  Transform to Fourier space and apply 2/3 dealiasing mask
Full step:
  Advance with integrating factors (Elsasser) and RK2 (Hermite): $\rightarrow$ state $_{n+1}$
Dissipation:
  Apply $\hat{w}_{n+1}^\pm \rightarrow \hat{w}_{n+1}^\pm \exp(-\eta(k_\perp^2/k_{\perp,\max}^2)^r \Delta t)$
  Apply $\hat{g}_{m,n+1}^\pm \rightarrow \hat{g}_{m,n+1}^\pm \exp(-\nu(m/M)^{2p} \Delta t)$
Output: Updated Fourier state $(\hat{w}_{n+1}^+, \hat{w}_{n+1}^-, \{\hat{g}_{m,n+1}^\pm\})$ at t_{n+1}

This algorithm conserves energy to machine precision in exact arithmetic for the inviscid, collisionless limit ($\eta = \nu = 0$). Practical simulations exhibit $< 0.01\%$ cumulative energy drift over hundreds of nonlinear times due to finite time-step truncation errors and dealiasing. The spectral representation combined with second-order temporal integration and properly dealiased nonlinear terms yields a robust method for long-time turbulence simulation across the KRMHD parameter space.

References

Bauer, P., Dueben, P. D., Hoefler, T., Quintino, T., Schulthess, T. C., and Wedi, N. P. (2021). The digital revolution of Earth-system science. *Nature Computational Science*, 1:104–113.

Boyd, J. P. (2001). *Chebyshev and Fourier Spectral Methods*. Dover Publications, 2nd edition.

Bradbury, J., Frostig, R., Hawkins, P., Johnson, M. J., Leary, C., Maclaurin, D., Necula, G., Paszke, A., VanderPlas, J., Wanderman-Milne, S., and Zhang, Q. (2018). JAX: Composable transformations of Python+NumPy programs.

Canuto, C., Hussaini, M. Y., Quarteroni, A., and Zang, T. A. (2006). *Spectral Methods: Fundamentals in Single Domains*. Springer.

Citrin, J., Goodfellow, I., Raju, A., Chen, J., Degrave, J., Donner, C., Felici, F., Hamel, P., Huber, A., Nikulin, D., Pfau, D., Tracey, B., Riedmiller, M., and Kohli, P. (2024). TORAX: A fast and differentiable tokamak transport simulator in JAX. *arXiv preprint arXiv:2406.06718*.

Elsässer, W. M. (1950). The hydromagnetic equations. *Physical Review*, 79(1):183.

Goldreich, P. and Sridhar, S. (1995). Toward a theory of interstellar turbulence. 2: Strong alfvenic turbulence. *The Astrophysical Journal*, 438:763–775.

Grad, H. (1949). On the kinetic theory of rarefied gases. *Communications on Pure and Applied Mathematics*, 2(4):331–407.

Howes, G. G., Cowley, S. C., Dorland, W., Hammett, G. W., Quataert, E., and Schekochihin, A. A. (2006). Astrophysical gyrokinetics: Basic equations and linear theory. *The Astrophysical Journal*, 651(1):590–614.

Howes, G. G., Cowley, S. C., Dorland, W., Hammett, G. W., Quataert, E., and Schekochihin, A. A. (2008). A model of turbulence in magnetized plasmas: Implications for the dissipation range in the solar wind. *Journal of Geophysical Research*, 113:A05103.

Jenko, F., Dorland, W., Hammett, G. W., and Howes, G. G. (2000). Electron temperature gradient driven turbulence. *Physics of Plasmas*, 7(5):1904–1910.

Landau, L. (1946). On the vibrations of the electronic plasma. *Journal of Physics (USSR)*, 10:25–34. English translation in JETP 16, 574.

Lenard, A. and Bernstein, I. B. (1958). Plasma oscillations with diffusion in velocity space. *Physical Review*, 112(5):1456–1459.

Lithwick, Y. and Goldreich, P. (2001). Compressible magnetohydrodynamic turbulence in interstellar plasmas. *The Astrophysical Journal*, 562(1):279–296.

Loureiro, N. F., Dorland, W., Fazendeiro, L., Kanekar, A., Mallet, A., Vilelas, M. S., and Zocco, A. (2016). Viriato: A Fourier–Hermite spectral code for strongly magnetized fluid–kinetic plasma dynamics. *Computer Physics Communications*, 206:45–63.

Mandell, N. R., Dorland, W., Abel, I., Gaur, R., Kim, P., Martin, M., and Qian, T. (2024). GX: A GPU-native gyrokinetic turbulence code for tokamak and stellarator design. *Journal of Plasma Physics*, 90(6):905900620. arXiv:2209.06731.

Numata, R., Howes, G. G., Tatsuno, T., Barnes, M., and Dorland, W. (2010). AstroGK: Astrophysical gyrokinetics code. *Journal of Computational Physics*, 229(24):9347–9372.

Orszag, S. A. (1971). On the elimination of aliasing in finite-difference schemes by filtering high-wavenumber components. *Journal of the Atmospheric Sciences*, 28:1074.

Rivera, Y. J., Badman, S. T., Stevens, M. L., Verniero, J. L., Stawarz, J. E., Shi, C., Raines, J. M., Paulson, K. W., Owen, C. J., Niembro, T., Louarn, P., Livi, S. A., Lepri, S. T., Kasper, J. C., Horbury, T. S., Halekas, J. S., Dewey, R. M., De Marco, R., and Bale, S. D. (2024). In situ observations of large-amplitude Alfvén waves heating and accelerating the solar wind. *Science*, 385(6712):962–966.

Schekochihin, A. A., Cowley, S. C., Dorland, W., Hammett, G. W., Howes, G. G., Quataert, E., and Tatsuno, T. (2009). Astrophysical gyrokinetics: Kinetic and fluid turbulent cascades in magnetized weakly collisional plasmas. *The Astrophysical Journal Supplement Series*, 182(1):310–377.

Strauss, H. R. (1976). Nonlinear, three-dimensional magnetohydrodynamics of noncircular tokamaks. *Physics of Fluids*, 19(1):134–140.

TenBarge, J. M. and Howes, G. G. (2013). Current sheets and collisionless damping in kinetic plasma turbulence. *The Astrophysical Journal Letters*, 771:L27.

Yang, L., He, J., Verscharen, D., Li, H., Bowen, T. A., Bale, S. D., Wu, H., Li, W., Wang, Y., Zhang, L., Feng, X., and Wu, Z. (2023). Energy transfer of imbalanced Alfvénic turbulence in the heliosphere. *Nature Communications*, 14(1):7955.