Strategic Quantizer Design via Dynamic Programming

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Abstract

This paper is concerned with the quantization setting where the encoder and the decoder have misaligned objectives. While the unconstrained variation of this problem has been well-studied under the theme of information design problems in Economics, the problem becomes more appealing and relevant to engineering applications with a constraint on the cardinality of the message space. We first motivate the problem via a toy example demonstrating the strategic quantization problem's intricacies, explicitly showing that the quantization resolution can change the nature of optimal encoding policy and the iterative optimization of the decoder and the encoder mappings may not converge to a local optimum solution. Since the problem setting involves the encoder (leader) designing the quantizer decision levels first, followed by the decoder (follower) designing the actions as a function of the decision levels, a gradient descent based solution method can be implemented. However, gradient descent solutions may be only locally optimal. As a remedy, we propose a dynamic programming-based quantizer design method inspired by the early works in the quantization literature. We then extend our approach to variable-rate (entropy-coded) setting to find the Lagrangian-optimal variable-rate strategic quantizer. We also extend the problem to communication over a noisy channel by employing a random channel index mapping, as done in prior work on classical channel-optimized quantizer design literature, combined with a dynamic programming approach to optimize quantization boundaries. We analyze the overall time complexity of

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the proposed method as well as its potential competitors and investigate several complexity reduction methods. We finally present numerical results obtained via the proposed algorithms that suggest the proposed algorithms' validity and demonstrate strategic quantization features that differentiate it from its classical counterpart.

Index Terms

I. INTRODUCTION

Consider the following problem: an encoder observes a realization of source $X \in \mathcal{X}$ with a probability distribution μ and sends a message $Z \in \mathcal{Z}$ using an injective mapping $Q : \mathcal{X} \to \mathcal{Z}$, with $|\mathcal{Z}| \leq M$. After receiving the message Z, the decoder takes action $Y \in \mathcal{Y}$. The costs that the encoder and the decoder minimize are $D_E \triangleq \mathbb{E}\{\eta_E(X, Y)\}$ and $D_D \triangleq \mathbb{E}\{\eta_D(X, Y)\}$, with $\eta_E \neq \eta_D$ (misaligned objectives). The encoder designs Q *ex-ante*, i.e., without the knowledge of the realization of X, using only the functions η_E and η_D, and the statistics of the source, $\mu(\cdot)$. The functions (η_E and η_D), the shared prior (μ), and the mapping (Q) are known to the encoder and the decoder. The problem is to design Q. We call this setup *strategic quantization*, which is the focus of this paper.

The setting without the quantization aspect (in the practical sense, if M is asymptotically large) is known in the Economics literature as the information design, or the Bayesian persuasion problem [1], [2]. These problems analyze how a communication system designer (sender) can use the information to influence the action taken by a receiver. This framework has proven beneficial in analyzing a variety of real-life applications, such as the design of transcripts when schools compete to improve their students' job prospects [3] and voter mobilization and gerrymandering [4], as well as various engineering applications, including in modeling misinformation spread over social networks [5] and privacy-constrained information processing [6], and many more [7].

For an excellent survey of the related literature in Economics, see [8], [9].

The strategic quantization problem, as described above, was discussed in a few contemporary economics and computer science studies. In [10], authors analyze the problem via a computation lens and report approximate results on this problem, relating to another problem they solved conclusively. In one of their main results, the algorithmic complexity of finding the optimum strategic quantizer was shown to be NP-hard. In a recent working paper, Aybaş and Türkel [11] analyzed this problem using the methods in [2] and provided several properties of strategic quantization. A byproduct of their analysis yields a constructive method for deriving optimal quantizers based on a search over possible posterior distributions over their feasible set. Our objective here is to leverage the rich collection of results in quantization theory, e.g., the comprehensive survey of results by Gray and Neuhoff [12], to study the same problem via the engineering lens.

Towards this objective, we consider the design problem of strategic quantization in this paper, leaving the study of theoretical aspects to a companion working paper [13]. The classical quantizer design problem is a team problem (in the game theory parlance), where iteratively enforcing the necessary conditions of optimality as in the Lloyd-Max 1 method [14] guarantees convergence to a locally optimum solution. Strategic quantizer design, however, is a game problem where the encoder and the decoder have misaligned objectives. Hence, one might expect a Lloyd-Max-like iterative approach would fail to converge to a local optimum as a natural consequence of the strategic aspect of the problem. Nevertheless, we make this point explicit with a toy example showing that iteratively enforcing necessary optimality conditions does not guarantee convergence to a local optimum strategic quantizer. Since the problem setting is Stackelberg in nature, a gradient descent optimization on the quantizer decision levels is a possible solution. However, as we show in [15], unlike in classical quantization, strategic quantization has local optima even for log-concave distributions. As a remedy, we utilize dynamic programming (DP): we decompose the optimum strategic M level quantizer into two optimum strategic quantizers at $M-1$ and 1 levels, and this observation naturally leads to a Bellman equation and an accompanying DP solution.

The use of DP for classical (nonstrategic)[1] quantization dates back to Bruce's Ph.D. thesis in 1965 [16]. The primary motivation to develop such algorithms was the observation that the Lloyd-Max type of iterative algorithms guarantee only local optimality, and the performance gap between a local and the global optimum can be practically significant in general. The DP-based methods guarantee global optimality, which, however, comes with the penalty of high complexity. Several complexity reduction methods, developed initially for DP, have been incorporated in the subsequent works on DP-based classical quantizer design, see, e.g., [17], [18].

In this paper, building on our preliminary work [19] which outlined our DP-based approach, we present a DP-based algorithm for strategic quantization for both fixed and variable rate scenarios, inspired by the early work in classical quantization theory [16], [17]. We analyze the complexity issues in detail and provide complexity reduction strategies if the decoder distortion measure $\eta_D(\cdot,\cdot)$ is of a specific structure. We implement the proposed methods for both fixed and variable rate strategic quantization, and all our codes are publicly available [20]. The numerical results suggest the approach's validity and demonstrate the features of strategic quantization that differentiate it from its classical counterpart.

In [2], Kamenica and Gentzkow solve the information design problem using mathematical tools, such as concavification, developed in the context of repeated games [21]. While it is a conceptually valuable tool, in general, the sender may not be able to compute the concave envelope. For example, as Kamenica and Gentzkow point out in one of their subsequent papers [22], the concavification approach has limited applicability for a continuous source because then the posterior beliefs become infinite-dimensional. Even in discrete source settings, the

[1]Throughout this paper, "classical" quantization or communication refers to the settings where the encoder and the decoder have identical objectives, in contrast with the strategic settings where the aforementioned objectives are misaligned.

optimization problem quickly becomes intractable as the cardinality of the source alphabet grows beyond two. This limitation has motivated researchers to investigate constrained persuasion problems that admit tractable solutions, such as those based on duality and linear programming for a specific class of payoff functions [23], [22], [24]. Imposing an exogenous quantization constraint, as in this paper, can be viewed as another attempt in this research direction.

We note in passing that quantizers also arise as equilibrium strategies endegenously, i.e., without an external cosntraint, in a related but a distinctly different class of signaling games, namely the cheap talk [25]. In [25], the encoder chooses the mapping from the realization of the source X to message Z *after* observing it, *ex-post*, as different source realizations indicate optimality of different mappings for the encoder. The encoder's lack of commitment power in the cheap talk setting makes the notion of equilibrium a Nash equilibrium since both agents form a strategy that is the best response to each other's mapping. However, in our strategic quantization problem (and the information design problems in general as in [2], [1]), the encoder designs Q *ex-ante*, *before* seeing the source realization, and committed to the designed Q afterward. This commitment is known to the decoder and establishes a form of trust between the sender and the receiver, resulting in possibly higher payoffs for both agents. This difference also manifests itself in the notion of equilibrium we are seeking here since the encoder does not necessarily form the best response to the decoder due to its commitment to Q[2].

II. PRELIMINARIES

A. Notation

In this paper, random variables are denoted using capital letters (say X), their sample values with respective lower case letters (x), and their alphabet with respective calligraphic letters ($\mathcal{X}$).

[2]These issues are well understood in the Economics literature, see, e.g., [26] for an excellent survey. However, we emphasize them here for a reader with an engineering background; see [7] for a detailed discussion through the engineering lens.

The set of real numbers, positive integers, and non-negative integers are denoted by $\mathbb{R}$, $\mathbb{Z}^+$, and $\mathbb{Z}_{\geq 0}$, respectively. The alphabet, $\mathcal{X}$, can be finite, infinite, or a continuum like an interval $[t_1, t_2] \subset \mathbb{R}$. The expectation operator is written as $\mathbb{E}\{\cdot\}$. The operator $|\cdot|$ denotes the absolute value if the argument is a scalar real number and the cardinality if the argument is a set.

The uniform distribution over an interval $[t_1, t_2]$ and the scalar Gaussian with mean μ, variance σ^2 are denoted by $U[t_1, t_2]$ and $\mathcal{N}(\mu, \sigma^2)$ respectively. The expression $t_1 \leq i \leq t_2, i \in \mathbb{Z}_{\geq 0}$ is denoted by $i \in [t_1 : t_2]$. The notation $A[i, j]$ is used to denote the element in row i, column j of matrix A. All logarithms are based 2.

B. Strategic Quantization Problem

We first define the problem in its most general (abstract) setting, then specialize in the more relevant design problem, which is inherently discrete. The encoder observes a signal $x \in \mathcal{X}$, and sends a message to the decoder, upon receiving which the decoder takes action $y \in \mathcal{Y}$. $\mathcal{X}$ is a compact metric space, and $\mathcal{Y}$ is compact. The set of Borel probabilities over $\mathcal{X}$, a compact metric space in weak topology, is denoted as $\Delta(\mathcal{X})$. The agents share a prior belief about $\mathcal{X}$, $\mu \in \Delta(\mathcal{X})$, as common knowledge.

The strategic quantizer is a measurable mapping $Q : \mathcal{X} \to \mathcal{Z}$, with $\mathcal{Z}$ as a compact metric space of messages satisfying $|\mathcal{Z}| \leq M$ for a given quantization resolution $M \in \mathbb{Z}^+$. Any quantizer induces a distribution τ over the messages, given μ.

The rate constraints are incorporated into the problem formulation as follows. In fixed-rate quantization, the resolution determines the rate (measured in bits), i.e., $\log M \leq R$ where R is the rate allowed. In variable-rate quantization, the entropy of the message determines the rate, i.e., $-\int \log \tau d(\tau) \leq R$. We note that there are no constraints on the resolution M in variable-rate quantization.

The game's timing is as follows: the encoder designs the quantizer, Q, based on common knowledge and announces it to the decoder. Nature randomly selects the signal $x \in \mathcal{X}$ using the common prior μ. The encoder observes the signal x and maps it to a message $z \in \mathcal{Z}$ using the announced quantizer Q. The message z is transmitted to the decoder noiselessly. The decoder receives z and takes action $y \in \mathcal{Y}$. The equilibrium sought after is the encoder-preferred Perfect Bayesian Equilibrium.The encoder and the decoder have (cost) measures $\eta_E(x, y)$ and $\eta_D(x, y)$ respectively. The decoder minimizes $\mathbb{E}\{\eta_D(x, y)\}$, while the encoder minimizes $\mathbb{E}\{\eta_E(x, y)\}$.

We next focus on the design problem[3] at hand. Let X take values from the source alphabet $\mathcal{X} \in [a, b]$ with probability distribution function μ. The set $\mathcal{X}$ is divided into mutually exclusive and exhaustive sets, $\mathcal{V}_1, \mathcal{V}_2, \ldots, \mathcal{V}_M$. Throughout this paper, we make the following "monotonicity" assumption.

Assumption 1 (Convex code-cells): $\mathcal{V}_i$ is convex for all $i \in [1 : M]$.

Under assumption 1, $\mathcal{V}_i$ is an interval since X is a scalar, i.e.,

$$\mathcal{V}_i = [x_{i-1}, x_i),$$

where $x_{i-1} < x_i$. The encoder chooses the boundary indices $\mathbf{q} = [x_0, x_1, \ldots, x_M]$. The decoder determines its actions $\mathbf{y} = [y_1, \ldots, y_M]$ as a best response to $\mathbf{q}$ to minimize its cost $D_D = \mathbb{E}\{\eta_D(x, y)\}$ as follows

$$y_m^* = \arg\min_{y_m \in \mathcal{Y}} \mathbb{E}\{\eta_D(x, y_m) | x \in \mathcal{V}_m\} \quad \forall m \in [1 : M]. \tag{1}$$

Hence, the decoder chooses the actions $\{y_m\}$ knowing the set of decision sets $\{\mathcal{V}_m\}$. The encoder computes what the decoder would choose as $\mathbf{y}$ given $\{\mathcal{V}_m\}$, and hence optimizes its own cost $\mathbb{E}\{\eta_E(x, y)\}$ over the choice of $\{\mathcal{V}_m\}$ accordingly:

$$\{\mathcal{V}_m^*\} = \arg\min_{\{\mathcal{V}_m\}} \sum_{m=1}^{M} \mathbb{E}\{\eta_E(x, y_m(\{\mathcal{V}_m\})) | x \in \mathcal{V}_m\} \tag{2}$$

or due to Assumption 1 equivalently over the choice of $\mathbf{q}$:

[3]Unless explicitly stated otherwise, we assume fixed-rate quantization throughout the paper.

$$\mathbf{q}^* = \arg\min_{\mathbf{q}} \sum_{m=1}^{M} \mathbb{E}\{\eta_E(x, y_m(\mathbf{q}))|x \in [x_{q_{m-1}}, x_{q_m})\}. \tag{3}$$

The action of the decoder y_m can be written as a quantization operator

$$y_m = Q(x) \quad \forall x \in \mathcal{V}_m \tag{4}$$

for all $m \in [1 : M]$.

Remark 1: Assumption 1 is the first of the two regularity conditions commonly employed in the classical quantization literature, cf.[14]. Note that the second regularity condition, $y_m \in \mathcal{V}_m$, is not included in Assumption 1. For classical quantization, both regularity conditions hold for common settings in engineering. For example, they are satisfied for fixed-rate quantization of any source and variable-rate quantization of any discrete source if the distortion measure is $\eta(x, y) = \rho(|x - y|))$, where ρ is a strictly increasing convex function, see, e.g., [27]. In passing, we note that for the strategic quantization problem, the second regularity condition may not hold for even simple costs, such as $\eta_E(x, y) = \rho(|x - y + \epsilon|)$ and $\eta_E(x, y) = \rho(|x - y|)$ for any $\epsilon \neq 0$ where ρ again is an increasing convex function.

In the Economics parlance, Assumption 1 is referred as the "monotonicity" condition. In [28], [29] sufficient conditions on η_E and η_D for the monotonicity of optimal encoder strategies are characterized within the unconstrained (without quantization) variation of the same problem. We note that here we have the quantization constraint in the problem formulation as an exogenous constraint on the message set, hence it is not clear apriori whether the results [28], [29] would be applicable here. We leave the study of similar characterizations for the strategic quantization problem to an ongoing theoretical work [13].

Remark 2: A key consideration here is that the encoder is committed to its choice of $\mathbf{q}$, it cannot determine $\mathbf{q}$ as the best response to $\mathbf{y}$. Hence, while the decoder can optimize its action $\mathbf{y}$ as the best response to $\mathbf{q}$, the encoder cannot choose $\mathbf{q}$ as the best response to $\mathbf{y}$, but that to a function of itself, i.e., the best response to $\mathbf{y}(\mathbf{q})$. This aspect of the problem introduces a hierarchy in the game play (the encoder plays first, and the decoder responds, which is referred to as the "Stackelberg equilibrium" in the computer science and control literature, and more formally constitutes an instance of subgame perfect Bayesian Nash equilibrium) and naturally is not a Nash equilibrium since $\mathbf{q}$ may not be the best response to $\mathbf{y}$. In cheap talk [25], Nash equilibria are sought after and the equilibria achieving strategies happen to be injective mappings, i.e., quantizers, without an exogenous rate constraint. It is essential to note the substantial difference between the problem formulation in this paper and the cheap talk literature [25].

While the analysis is done for continuous X, for the tractability of the algorithm, the source is required to be discrete. If the source is not purely discrete, we approximate it via uniformly quantizing $[a, b]$ to N points to obtain an ordered set of N points i.e.,

$$\mathcal{X} = \{a + (1+2i)\delta\}, \quad i = 0, \dots, N-1$$

with the probability mass function

$$P(x_i) = \int_{x_i-\delta}^{x_i+\delta} d\mu, \quad i \in [0 : N-1], \tag{5}$$

where $\delta = \frac{b-a}{2N}$. The integrals are transformed to summations over $\mathcal{V}_i$. Obviously, this approximation induces a performance loss in the designed quantizer which can be reduced by increasing N. Hence, reducing the computational complexity of the proposed method, particularly for continuous sources, is of paramount practical importance. To this effect, we analyze several complexity reduction methods in Section IV in detail.

To facilitate the computations, we take $\mathcal{Y}$ (as defined in the previous section, the reconstruction space) as the discretized real line at a suitable resolution (in practice, this choice depends on the distortion functions η_E and η_D).

We formally present the design problems addressed in this paper as follows:

*Problem 1 (***Fixed-rate***):* For a given rate R, scalar source X with a probability distribution function μ find the decision boundaries $\mathbf{q} = [x_0, x_1, \dots, x_M]$ and actions $\mathbf{y}(\mathbf{q}) = [y_1, \dots, y_M]$ as a function of boundaries that satisfy:

$$\mathbf{q}^* = \arg\min_{\mathbf{q}} \sum_{m=1}^{M} \mathbb{E}\{\eta_E(x, y_m(\mathbf{q}))|x \in [x_{m-1}, x_m)\},$$

where actions $\mathbf{y}(\mathbf{q})$ are given as

$$y_m^*(\mathbf{q}) = \arg\min_{y_m \in \mathcal{Y}} \mathbb{E}\left\{\eta_D(x, y_m)|x \in [x_{m-1}, x_m)\right\} \quad \forall m \in [1 : M],$$

and the rate satisfies

$$\log M \leq R.$$

For the variable-rate variation, we take the Shannon entropy, of the quantization intervals as the total rate allowed, i.e., $-\sum_m p_m \log p_m \leq R$, noting that the probability of source realization lying in the i-th interval is $p_i = \int \mathrm{d}\mu$. The optimal variable-rate quantizer design is difficult in general, we concede here to a solution[4] that achieves only the lower convex hull of the rate-distortion region, as common practice in classical quantization literature, see, e.g., [31]. We take our objective function as the Lagrangian $J(\lambda, R) = D_E(R) + \lambda R$ where $\lambda \in \mathbb{R}^+$ is the slope of the line supporting the convex hull of achievable D_E at rate R, i.e., each R corresponds to a specific λ. We refer to the quantizer that minimizes this Lagrangian cost as the Lagrangian-optimal strategic quantizer.

[4]If the objective, here $D_E(R)$, is convex, the Lagrangian relaxation does not introduce any loss. In general, however, $D_E(R)$ is not convex; see, e.g., [30], where authors show that for a uniform scalar source and the squared error distortion, the optimal variable rate quantizer, in the classical (nonstrategic) setting, coincides with the Lagrangian-optimal only at specific rates.

*Problem 2 (***Variable-rate***):* For a given Lagrangian λ (corresponding to the total entropy constraint R), scalar source X with a probability distribution function μ find the decision boundaries $\mathbf{q} = [x_0, x_1, \dots, x_M]$ that satisfy:

$$\mathbf{q}^* = \arg\min_{\mathbf{q}} \sum_{m=1}^{M} \mathbb{E}\{\eta_E(x, y_m(\mathbf{q}))|x \in [x_{m-1}, x_m)\} + \lambda \left(-\sum_{m=1}^{M} p_m \log p_m \right),$$

where actions $\mathbf{y}(\mathbf{q})$ satisfy

$$y_m^*(\mathbf{q}) = \arg\min_{y_m \in \mathcal{Y}} \mathbb{E}\left\{\eta_D(x, y_m)|x \in [x_{m-1}, x_m)\right\} \quad \forall m \in [1:M].$$

For the Channel-optimized extension, we use an index assignment chosen uniformly at random $\pi : [1 : M] \to [1 : M]$ and applied to the message Z. The message $\pi(Z)$ is transmitted over a noisy channel with transition probability matrix $p(z_j|z_i)$. After receiving the message Z, the decoder applies a mapping $\phi : \mathcal{Z} \to \mathcal{Y}$ on the message Z and takes an action $Y = \phi(Z)$. The encoder and the decoder minimize their respective objectives $D_E = \mathbb{E}_\pi\{\mathbb{E}\{\eta_E(X, Y)|\pi\}\}$ and $D_D = \mathbb{E}_\pi\{\mathbb{E}\{\eta_D(X, Y)|\pi\}\}$, which are misaligned ($\eta_E \neq \eta_D$). The encoder designs Q *ex-ante*, i.e., without the knowledge of the realization of X, using only the objectives η_E and η_D, the statistics of the source $\mu(\cdot)$, and the channel parameters (transition probability matrix $p(z_j|z_i)$). The objectives (η_E and η_D), the shared prior (μ), the index assignment (π), the channel transition probability matrix ($p(z_j|z_i)$), and the mapping (Q) are known to the encoder and the decoder. The problem is to design Q for the equilibrium, i.e., the encoder minimizes its distortion if used with a corresponding decoder that minimizes its own distortion.

*Problem 3 (***Noisy channel***):* Using random index assignment for a given noisy channel with rate R and bit error rate p_{err}, scalar source X with a probability density function μ, find the decision boundaries $\mathbf{q} = [x_0, x_1, \dots, x_M]$ and actions $\mathbf{y}(\mathbf{q}) = [y_1, \dots, y_M]$ as a function of boundaries that satisfy:

$$\mathbf{q}^* = \arg\min_{\mathbf{q}} \sum_{m=1}^{M} \mathbb{E}_\pi\{\mathbb{E}\{\eta_E(x, y_m(\mathbf{q}))|\pi, x \in [x_{m-1}, x_m)\}\},$$

where actions $\mathbf{y}(\mathbf{q})$ are $y_m^*(\mathbf{q}) = \arg\min_{y_m \in \mathcal{Y}} \mathbb{E}_\pi\{\mathbb{E}\{\eta_D(x, y_m)|\pi, x \in [x_{m-1}, x_m)\}\} \quad \forall m \in [1:M]$, and the rate satisfies $\log M \leq R$.

C. Numerical Examples

In this section, we provide simple numerical examples to demonstrate few intricacies of the strategic quantization problem that differentiate it from its classical analogue.

1) To reveal or not to reveal... or partially reveal?: As mentioned, the classical quantization problem is a team problem, i.e., the encoder and the decoder share identical objectives. In strategic quantization, this is no longer the case. Hence, depending on the misalignment between η_E and η_D, the encoder might choose not to send any information to the decoder (e.g., if they are too misaligned), which we refer to as "non-revealing" policy following the convention in the Economics literature. Alternatively, the strategic quantizer problem might simplify to classical quantization with a common distortion measure η_D, i.e., the encoder cannot utilize its information design advantage to persuade the decoder to take a specific action (referred to here as "fully-revealing" encoder policy). Finally, the encoder might choose to employ a quantizer that is not identical to the classical one, i.e., "partially-revealing" policy. It is relatively straightforward to provide an example for the first case; consider, e.g., $\eta_E(x, y) = -\eta_D(x, y)$, $\forall x, y$ which makes the problem a zero-sum game, hence, the optimal strategy for the encoder is not to send any information (see also the analysis below). We demonstrate the latter cases via a simple numerical example.

Consider the same continuous source $X \sim U[-1, 1]$ but with the distortion functions $\eta_D(x, y) = (x - y)^2$ and $\eta_E(x, y)$ is defined as follows:

$$\eta_E(x,y) = \begin{cases} (x^3-y)^2, & xy \geq 0 \\ \to \infty, & \text{otherwise.} \end{cases} \tag{6}$$

A fully non-revealing policy, i.e., the case of $R=0$ ($M=1$) yields $D_E(0) = \int_{-1}^{1} (t^3)^2 \frac{1}{2}\, dt = 1/7$.

We next consider $M=2$. From symmetry, the optimal encoding policy is simply setting the boundary at $q_1 = 0$. However, this yields $y_1 = -y_2 = -1/2$ and $D_E(1) = 1/7$, which is identical to $D_E(0)$. Hence, optimal strategic quantizer for $M=2$ does not send any information to the decoder, i.e., non-revealing.

We finally consider the case of $M=3$. Similar to the previous case, we parametrize $\mathcal{V}_m^*$ as $[-1,q], (q,0], (0,1]$ and express $\{y_m\}$ as a function of q, which yields

$$y_1 = \frac{-1+q}{2}, \quad y_2 = \frac{q+1}{2}, \quad y_3 = \frac{1}{2}. \tag{7}$$

Substituting again $\{y_m\}$ in $\eta_E(x,y)$:

$$J(q) = \frac{1}{2}\left(\int_{-1}^{q} (t^3-y_1)^2\, dt + \int_{q}^{0} (t^3-y_2)^2\, dt + \int_{0}^{1} (t^3-y_3)^2\, dt\right). \tag{8}$$

Enforcing the KKT conditions similar to (**??**), we obtain $q = 0, \pm\sqrt{1/2}$. Since $q \leq 0$, the possible solutions are $q = 0, -\frac{1}{\sqrt{2}}$. Of these two choices, $q = -1/\sqrt{2}$ yields a lower distortion $D_E = 25/224$, hence is optimal.

We note here that $D_E(\log 3) = 25/224 > 1/7 = D_E(1) = D_E(0)$. Hence, at $M=2$, the strategic quantizer does not communicate any information while, at $M=3$, in sharp contrast to $M=2$, uses the quantization channel fully to send three messages, demonstrating that the quantization constraint can change the nature of the optimum encoder policy. Moreover, it shows that the operational rate-distortion function $D_E(R)$ here is not a strictly decreasing function of rate R, since $D_E(1) = D_E(0)$, unlike its classical counterpart.

2) Quantizer resolution is binding: We next focus on the question: can the quantization constraint change the nature of optimal encoding policy in strategic communication? For example, is there a case where for $M = 2$ the encoder is non-revealing but for $M = 3$ the encoder prefers to send a message? The answer is, perhaps surprisingly, affirmative.

Consider the continuous source $X \sim U[0, 1]$ but with the distortion functions $\eta_D(x, y) = (x - y)^2$ and $\eta_E(x, y) = (x^2 - y)^2$

A fully non-revealing policy yields $D_E(0) = \int_0^1 (t^2 - 0.5)^2 \mathrm{d}t =$

$$\begin{aligned}\int_0^1 (t^2 - 0.5)^2 \mathrm{d}t &= \int_0^1 (t^4 - t^2 + 0.25)\mathrm{d}t \\ &= (\frac{t^5}{5} - \frac{t^3}{3} + 0.25t)\Big|_0^1 \\ &= 1/5 - 1/3 + 1/4 = -2/15 + 1/4 = (-8 + 15)/60 = 7/60\end{aligned} \tag{9}$$

$$J = \tag{10}$$

Consider the same continuous source $X \sim U[-1, 1]$ but with the distortion functions $\eta_D(x, y) = (x - y)^2$ and $\eta_E(x, y)$ is defined as follows:

$$\eta_E(x, y) = \begin{cases} (x^3 - y)^2, & xy \geq 0 \\ \to \infty, & \text{otherwise.} \end{cases} \tag{11}$$

A fully non-revealing policy, i.e., the case of $R = 0$ ($M = 1$) yields $D_E(0) = \int_{-1}^{1} (t^3)^2 \frac{1}{2}\, dt = 1/7$.

We next consider $M = 2$. From symmetry, the optimal encoding policy is simply setting the boundary at $q_1 = 0$. However, this yields $y_1 = -y_2 = -1/2$ and $D_E(1) = 1/7$, which is identical to $D_E(0)$. Hence, optimal strategic quantizer for $M = 2$ does not send any information to the decoder, i.e., non-revealing.

We finally consider the case of $M = 3$. Similar to the previous case, we parametrize $\mathcal{V}_m^*$ as $[-1, q], (q, 0], (0, 1]$ and express $\{y_m\}$ as a function of q, which yields

$$y_1 = \frac{-1+q}{2}, \quad y_2 = \frac{q+1}{2}, \quad y_3 = \frac{1}{2}. \tag{12}$$

Substituting again $\{y_m\}$ in $\eta_E(x, y)$:

$$J(q) = \frac{1}{2}\left(\int_{-1}^{q}(t^3 - y_1)^2\,dt + \int_{q}^{0}(t^3 - y_2)^2\,dt + \int_{0}^{1}(t^3 - y_3)^2\,dt\right). \tag{13}$$

Enforcing the KKT conditions similar to (**??**), we obtain $q = 0, \pm\sqrt{1/2}$. Since $q \leq 0$, the possible solutions are $q = 0, -\frac{1}{\sqrt{2}}$. Of these two choices, $q = -1/\sqrt{2}$ yields a lower distortion $D_E = 25/224$, hence is optimal.

We note here that $D_E(\log 3) = 25/224 > 1/7 = D_E(1) = D_E(0)$. Hence, at $M = 2$, the strategic quantizer does not communicate any information while, at $M = 3$, in sharp contrast to $M = 2$, uses the quantization channel fully to send three messages, demonstrating that the quantization constraint can change the nature of the optimum encoder policy. Moreover, it shows that the operational rate-distortion function $D_E(R)$ here is not a strictly decreasing function of rate R, since $D_E(1) = D_E(0)$, unlike its classical counterpart.

3) Failure of "Strategic Lloyd-Max": In this subsection, we investigate whether a simple strategic variation of Lloyd-Max approach would yield a locally optimal quantizer, as in the case of classical quantization. Consider a continuous source $X \sim U[-1, 1]$ quantized with $M = 3$ messages where $\eta_E(x, y) = (x^3 - y)^2$ and $\eta_D(x, y) = (x - y)^2$.

We initialize $\{y_m\}$ arbitrarily and find q_1 and q_2 that minimize $D_E = \mathbb{E}\{(x^3 - y)^2\}$, as

$$q_1 = \left(\frac{y_1 + y_2}{2}\right)^{\frac{1}{3}}, q_2 = \left(\frac{y_2 + y_3}{2}\right)^{\frac{1}{3}}. \tag{14}$$

We then find the decoder actions $\{y_m\}$ via (**??**), and iterate between (14) and (**??**) until convergence. We note that during these iterations, q_1 and q_2 move towards -1 and 1, respectively, i.e., the boundaries move towards the endpoints of the interval taken for quantization with each

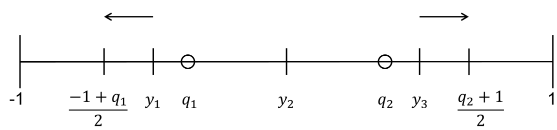


Fig. 1. Movement of the quantization boundaries through iterations in the running example.

iteration as demonstrated in Figure 1, hence the iterations converge to $M = 1$ solution which is non-revealing with $D_E = 1/7$. Now, let us examine whether this solution is a local optimum.

Any admissible perturbation of with some $1 > \epsilon > 0$ of $q_1 = -q_2 = -1$ would result in a $M = 3$ level quantizer with decision boundaries $q_1 = -1+\epsilon$, $q_2 = 1-\epsilon$ with the corresponding decoder actions $y_1 = -1 + \frac{\epsilon}{2}$, $y_2 = 0$, $y_3 = 1 - \frac{\epsilon}{2}$, yielding $D_E = 1/7 - \epsilon(1-\frac{\epsilon}{2})^2(1-\epsilon)^2$ which is smaller than than that of the non-revealing solution ($D_E = 1/7$), hence this is not a locally optimal solution. This observation indicates that the straightforward enforcement of optimality conditions may not yield a locally optimal solution, which contrasts sharply with the case in classical quantization. In other words, unlike its classical counterpart, a trivial extension of the Lloyd-Max algorithm adopted for strategic settings may not converge to a locally optimum solution.

D. Gradient descent issues:

Unlike classical quantization, strategic version can have multiple local minima even when used in conjunction with log-concave sources, as demonstrated in Fig 2. We show the number of locally optimal quantizers with rate for a non-strategic (increases with bit error rate) and a strategic quantizer for a Gaussian source.

III. Proposed Dynamic Programming Based Solution

DP methods work via connecting the solution of an M-stage problem to the solution of an $M-1$ stage problem via the DP (Bellman) equation. Then, for a given initial condition (the solution

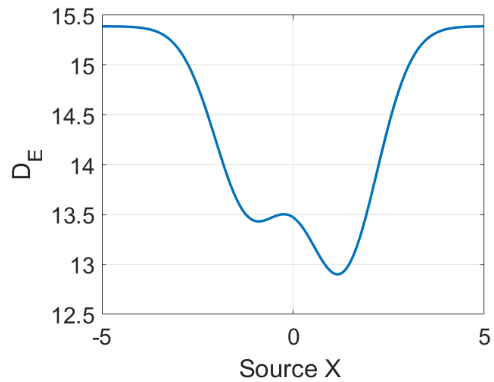
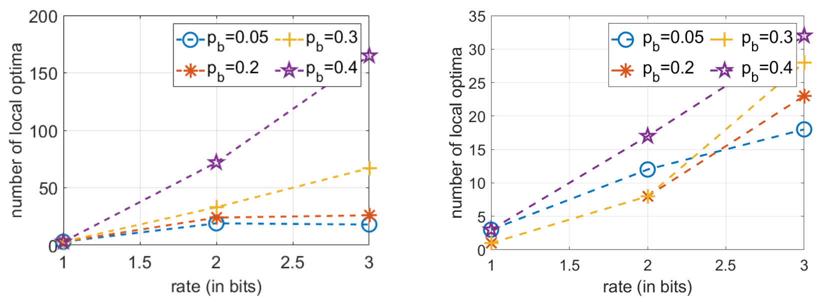


Fig. 2. Local optima in encoder distortion for 2-level quantization of $X \sim \mathcal{N}(0.1, 1)$ with $\eta_E = (x^3 - y)^2, \eta_D = (x - y)^2$



(a)$\eta_E = (x - y)^2$

(b)$\eta_E = (x^3 - y)^2$

Fig. 3. Number of locally optimal quantizers for a Gaussian source $X \sim \mathcal{N}(0, 1)$ with an MSE decoder $\eta_D(x, y) = (x - y)^2$.

of the 1-stage problem), the M-stage problem is solved recursively. Several seemingly unrelated problems admit a DP solution, including the classical quantization problem demonstrated by Bruce in his Ph.D. thesis [16]. Here, we extend this approach to the strategic setting.

The following observation yields the DP equations needed to solve the strategic quantization problem for fixed and variable rate variations. Its proof follows from similar steps used in classical quantization and is presented in Appendix B for completeness.

Lemma 2: Let $\mathbf{q}^* = [x_0^*, \ldots, x_M^*]$ be the decision boundaries for the optimal M level strategic quantization of the source $\mathcal{X}$. Let the decision boundaries of the optimal $(M - 1)$ level strategic quantizer applied to $[x_0, \ldots, x^*_{M-1})$ be $\mathbf{p}^*$. Then, $\mathbf{p}^* = [x^*_0, x^*_1, \ldots, x^*_{M-1}]$.

Remark 3: We note that in Lemma 2, we do not impose any restrictions on the distortion measures $\eta_E(x, y)$ and $\eta_D(x, y)$. In the next section, we decrease the complexity of the proposed method by exploiting the structural properties of η_D if exists.

A. Fixed rate

We next specialize to the fixed-rate setting and define the following expressions used in the subsequent derivations.

1) The encoder and the decoder costs for source interval $[\alpha, \beta)$ for a given action t:
$$C_s(\alpha, \beta, t) \triangleq \int \eta_s(x, t)\mathrm{d}\mu \quad \text{for } s \in \{E, D\}. \tag{15}$$
2) The decoder's optimal action for the source interval $[\alpha, \beta)$:
$$\kappa(\alpha, \beta) \triangleq \underset{t \in \mathcal{Y}}{\arg\min}\, C_D(\alpha, \beta, t). \tag{16}$$
3) Costs for the source interval $[\alpha, \beta)$ in conjunction with the optimal action:
$$\epsilon_s(\alpha, \beta) \triangleq C_s(\alpha, \beta, \kappa(\alpha, \beta)) \quad \text{for } s \in \{E, D\}. \tag{17}$$
4) Equilibrium total costs associated with the m level optimal strategic quantizer for $[x_0, \beta)$:
$$D_m(x_0, \beta) \triangleq \min_{\mathbf{q}} \sum_{j=1}^{m} \epsilon_s(x_{j-1}, x_j) \quad \text{for } s \in \{E, D\}. \tag{18}$$
where $\mathbf{q} = [x_0, \ldots, x_m]$, $a = x_0 < \ldots < x_m = \beta$.
5) The set of all non-empty convex subsets of $\mathcal{X}$:
$$\mathcal{S} \triangleq \{[t_1, t_2) : t_1, t_2 \in \mathcal{X}, t_1 < t_2\}. \tag{19}$$

The encoder minimizes D_E with the choice of the quantizer decision levels $\mathbf{q}^* = [x^*_0, \ldots, x^*_M]$,
$$\mathbf{q}^* = \arg\min_{\mathbf{q}} \sum_{i=1}^{M} \int \eta_E(x, y_i)\mathrm{d}\mu, \tag{20}$$

where the representative levels y_m^* are chosen by the decoder to minimize its distortion D_D,

$$y_m^* = \arg\min_{y \in \mathcal{Y}} \int \eta_D(x,y) \mathrm{d}\mu. \tag{21}$$

The distortion in quantizing $[\alpha, \beta)$ with one representative level for the encoder, $\epsilon_E(\alpha, \beta)$, is computed for each $[\alpha, \beta) \in \mathcal{S}$. We set the $1-$level distortion $D_1(\alpha, \beta) = \epsilon_E(\alpha, \beta), \beta \in \mathcal{X} \backslash x_0$. The $m-$level distortion for an interval $[x_0, \beta) \in \mathcal{S}$ due to quantizing the interval with m representative levels can be written in terms of the $1-$level distortions,

$$D_m(x_0, \beta) = \min_{\substack{x_0, \ldots, x_m \in \mathcal{X} \\ a = x_0 < x_1 < \ldots < x_m = \beta}} \sum_{i=1}^{m} D_1(x_{i-1}, x_i).$$

The optimization for $m-$level quantization of $[x_0, \beta)$ can be written as the sum of $(m-1)$ level quantization of $[x_0, \alpha)$ and $1-$level quantization of $[\alpha, \beta)$ as the Bellman equations,

$$D_m(x_0, \beta) = D_{m-1}(x_0, r_{m-1}(x_0, \beta) + D_1(r_{m-1}(x_0, \beta), \beta) \tag{22}$$

$$r_{m-1}(x_0, \beta) \triangleq \arg\min_{\alpha \in \mathcal{X}} \{D_{m-1}(x_0, \alpha) + D_1(\alpha, \beta)\}. \tag{23}$$

Dynamic programming requires a forward and a backward pass. During the forward pass, we compute and store $r_{m-1}(x_0, \beta)$ and $D_m(x_0, \beta)$ for each pair $(m, \beta), m \in [2 : M], \beta \in \mathcal{X} \backslash x_0$ recursively starting from $m = 2$ using the pre-computed values of $D_1(\alpha, \beta)$. In the backward pass, we set $x_0^* = a, x_M^* = b$ and compute optimal decision (x_m^*) and representative levels (y_m^*) recursively as

$$x_{m-1}^* = r_{m-1}(x_0^*, x_m^*), \quad m = M, \ldots, 2, \tag{24}$$

$$y_m^* = \kappa(x_{m-1}^*, x_m^*), \quad m = [1 : M]. \tag{25}$$

Note that there is no computation involved in the backward pass, we only use the parameters that are computed and stored in the forward pass.

The derived steps above are presented in Algorithm A-A of Appendix A.

B. Variable rate

Let $p(\alpha, \beta)$ be the probability that x lies in the interval $[\alpha, \beta)$,

$$p(\alpha, \beta) = \int \mathrm{d}\mu. \tag{26}$$

We next define the following expressions for the variable rate derivation:

1) The entropy of quantizing an interval $[x_\alpha, x_\beta)$ to 1 level is given by

$$H_1(\alpha, \beta) \triangleq -p(\alpha, \beta) \log p(\alpha, \beta). \tag{27}$$

2) The entropy of quantizing $[x_0, \beta)$ to m levels with the quantizer $\mathbf{q}$ is given by,

$$H(\beta, m) \triangleq \sum_{i=1}^{m} H_1(x_{i-1}, x_i). \tag{28}$$

3) Analogous to $C_E(\alpha, \beta, t)$, the cost of quantizing the interval $[\alpha, \beta)$ for a given action t:

$$C_E^\lambda(\alpha, \beta, t) \triangleq C_E(\alpha, \beta, t) + \lambda H_1(\alpha, \beta). \tag{29}$$

4) Analogous to $\epsilon_s(\alpha, \beta)$, the cost of quantizing the interval $[\alpha, \beta)$,

$$\epsilon_{s,\lambda}(\alpha, \beta) \triangleq \epsilon_s(\alpha, \beta) + \lambda H_1(\alpha, \beta). \tag{30}$$

5) The Lagrangian cost:

$$D(\lambda, M) \triangleq D_M(x_0, b) + \lambda H(b, M). \tag{31}$$

6) The Lagrangian cost of quantizing $[x_0, \beta)$ to m levels for a given λ,

$$D_{m,\lambda}(x_0, \beta) \triangleq \min_{\mathbf{q}} \sum_{j=1}^{m} (\epsilon_E(q_{j-1}, q_j) + \lambda H_1(q_{j-1}, q_j)). \tag{32}$$

Similar to the fixed-rate setting, the encoder minimizes D_E with the choice of the quantizer decision levels $\mathbf{q}^* = [x_0^*, \ldots, x_M^*]$,

$$\mathbf{q}^* = \arg\min_{\mathbf{q}} \sum_{i=1}^{M} \bigg(\int \eta_E(x, y_i) \mathrm{d}\mu + \lambda H_1(x_{i-1}, x_i) \bigg), \tag{33}$$

where the representative levels y_m^* are chosen by the decoder to minimize its distortion D_D,

$$y_m^* = \arg\min_{y \in \mathcal{Y}} \int \eta_D(x, y) \mathrm{d}\mu. \tag{34}$$

We first compute (and store) $\epsilon_{E,\lambda}(\alpha,\beta)$ for all $[\alpha,\beta)$ in $\mathcal{S}$. We set $D_{1,\lambda}(\alpha,\beta)=\epsilon_{E,\lambda}(\alpha,\beta)$. The $m-$level distortion for an interval $[x_0,\beta)$,

$$D_{m,\lambda}(x_0,\beta)=\min_{\substack{x_0,\ldots,x_m\in\mathcal{X}\\ a=x_0<x_1<\ldots<x_m=\beta}}\sum_{i=1}^{m}D_{1,\lambda}(x_{i-1},x_i), \tag{35}$$

can be written as the sum of $(m-1)-$level quantization of $[x_0,\alpha)$ and $1-$level quantization of $[\alpha,\beta)$ as the Bellman equations,

$$D_{m,\lambda}(x_0,\beta)=D_{m-1,\lambda}(x_0,r_{m,\lambda}(x_0,\beta))+D_{1,\lambda}(r_{m,\lambda}(x_0,\beta),\beta), m\in[2:n], \tag{36}$$

$$r_{m,\lambda}(x_0,\beta)\triangleq\arg\min_{\alpha\in\mathcal{X}}\left\{D_{m-1,\lambda}(x_0,\alpha)+D_{1,\lambda}(\alpha,\beta)\right\}. \tag{37}$$

Dynamic programming requires a forward and a backward pass. During the forward pass, we compute and store $r_{m-1,\lambda}(x_0,\beta)$ and $D_{m,\lambda}(x_0,\beta)$ for each pair $(m,\beta), m\in[2:M], \beta\in\mathcal{X}\backslash x_0$ recursively starting from $m=2$ using the pre-computed values of $D_1(\alpha,\beta)$. In the backward pass, we set $x_0^*=a, x_M^*=b$ and compute optimal decision (x_m^*) and representative levels (y_m^*) recursively as

$$x_{m-1}^*=r_{m-1,\lambda}(x_0^*,x_m^*),\quad m=M,\ldots,2,$$

$$y_m^*=\kappa(x_{m-1}^*,x_m^*),\quad m=[1:M].$$

We note that M is not specified in the problem formulation[5]. As a remedy, we go through several $M\leq|\mathcal{Y}|$ values and choose the one that yields minimum Lagrangian cost $D(\lambda,M)$. With the penalty of increased complexity, one can perform a full search over $[1:|\mathcal{Y}|]$.

The distortions are computed as $D_M(x_0^*,x_M^*)$ and $D_D(x_0^*,x_M^*)=\sum_{m=1}^{M}\epsilon_D(x_{m-1}^*,x_m^*)$, respectively for the encoder and the decoder. The rate is given by $H(b,M^*)$. We repeat the procedure over a range of λ values to sweep through the convex hull over the rates of interest.

We present the above steps in Algorithm A-B of Appendix A.

[5] In principle, M^* can be infinite for a continuous source, see [32] for a detailed analysis of this issue in the classical setting.

C. *Channel optimized strategic quantizer*

The encoder designs a quantizer q with random index mapping using only the objectives (η_E, η_D), the statistics of the source $(\mu(\cdot))$, and the channel transition probability matrix $(p(j|i))$, without the knowledge of the realization of X. After observing x, the encoder quantizes the source as

$$z_m = q(x), \quad x \in \mathcal{V}_m,$$

and uses an index mapping chosen uniformly at random to map z_m to z_i

$$z_i = \pi(z_m),$$

where $\pi : \{1, \ldots, M\} \to \{1, \ldots, M\}$. z_i is transmitted over a noisy channel and received as z_j with probability $p(j|i)$. The decoder receives the message and takes the action

$$y = \phi(z_j).$$

The average symbol error probability of the channel is

$$p_{err} = \frac{1}{M}\sum_{i=1}^{M}\sum_{i=1, j\neq i}^{M} p(j|i).$$

Let $c_1 = \frac{p_{err}}{M-1}, c_2 = 1 - Mc_1$. The probability that the receiver receives the noisy message $\hat{z} = z_j$ if z_i was transmitted using $\pi(q(x)) = z_i$ is given by $p(j|i)$, the channel transition probability.

The end-to-end distortion given an index assignment π is

$$\mathbb{E}\{\eta_s|\pi\} = \sum_{i=1}^{M}\sum_{j=1}^{M}\int \eta_s(x, y_j(q))p(j|i)\mathrm{d}\mu.$$

The average distortion over all possible index assignments is

$$\begin{aligned} D_s &= \sum_{i=1}^{M}\sum_{j=1}^{M}\int \eta_s(x, y_j(q))\mathbb{E}_\pi\{p(j|i)\}\mathrm{d}\mu \\ &= I_{j\neq i} + I_{j=i}, \end{aligned}$$

where $I_{j\neq i}$ and $I_{j=i}$ are defined as follows:

$$I_{j\neq i} = \sum_{i=1}^{M}\sum_{\substack{j=1\\ j\neq i}}^{M}\int \eta_s(x,y_j(q))\mathbb{E}_\pi\{p(j|i)\}\mathrm{d}\mu = \sum_{i=1}^{M}\sum_{\substack{j=1\\ j\neq i}}^{M}\int \eta_s(x,y_j(q))\frac{p_{err}}{M-1}\mathrm{d}\mu,$$

$$I_{j=i} = \sum_{i=1}^{M}\int \eta_s(x,y_i(q))\mathbb{E}_\pi\{p(j|i)\}\mathrm{d}\mu = \sum_{i=1}^{M}\int \eta_s(x,y_i(q))(1-p_{err})\mathrm{d}\mu.$$

$I_{j\neq i}$ can be further simplified as follows

$$\begin{aligned}
I_{j\neq i} &\overset{a}{=} c_1\Bigg(\sum_{i=1}^{M}\sum_{j=1}^{M}\int \eta_s(x,y_j(q))\mathrm{d}\mu(x) \quad -\sum_{i=1}^{M}\int \eta_s(x,y_i(q))\mathrm{d}\mu(x)\Bigg)\\
&\overset{b}{=} c_1\Bigg(\sum_{j=1}^{M}\sum_{i=1}^{M}\int \eta_s(x,y_j(q))\mathrm{d}\mu(x) - \sum_{i=1}^{M}\int \eta_s(x,y_i(q))\mathrm{d}\mu(x)\Bigg) \qquad (38)\\
&\overset{c}{=} c_1\Bigg(\sum_{j=1}^{M}\mathbb{E}\{\eta_s(x,y_j(q))\} - \sum_{i=1}^{M}\int \eta_s(x,y_i(q))\mathrm{d}\mu(x)\Bigg).
\end{aligned}$$

In Equation 38, equality a is given by adding and subtracting $\sum_{i=1}^{M}\int \eta_s(x,y_i(q))\mathrm{d}\mu(x)\Big)$, equality b is given by exchanging the summation over i and j, equality c is given by the definition of $\mathbb{E}\{\eta_s(x,y_j(q))\}$.

The average distortion and the optimum decoder reconstruction for $i\in[1:M]$ are

$$D_s = c_1\sum_{i=1}^{M}\mathbb{E}\{\eta_s(x,y_i(Q))\} + c_2\overline{D_s},$$

$$y_i = \arg\min_{y\in\mathcal{Y}} c_1\mathbb{E}\{\eta_s(x,y_i(q))\} + c_2\overline{D_s},$$

where $\overline{D_s}$ is the distortion in the noiseless setting

$$\overline{D_s} = \sum_{i=1}^{M}\int \eta_s(x,y_i)\mathrm{d}\mu. \qquad (39)$$

We assume that $0 < p_{err} < \frac{M-1}{M}$ so that $c_1, c_2 > 0$ since dynamic programming divides the optimization problem into sub problems that are individually optimal.

The reconstruction levels $\mathbf{y}$ are found using the first order derivative condition:

$$\frac{\partial D_D}{\partial y_i} = c_1\frac{\partial}{\partial y_i}\mathbb{E}\{\eta_D(x,y_i(q))\} + c_2\frac{\partial}{\partial y_i}D_D. \qquad (40)$$

When the decoder distortion is MMSE, $\eta_D(x,y)=(x-y)^2$,

$$\frac{\partial D_D}{\partial y_i} = -2c_1\sum_{m=1}^{M}\int (x-y_i)\mathrm{d}\mu - 2c_2\int (x-y_i)\mathrm{d}\mu(x), \tag{41}$$

$$y_i = \frac{c_1\mathbb{E}\{X\}+c_2\int x\mathrm{d}\mu}{c_1+c_2\int \mathrm{d}\mu(x)}. \tag{42}$$

We define the following expressions used in the subsequent derivations.

1) The encoder and the decoder costs for source interval $[\alpha,\beta)$ for a given action y:

$$C_s(\alpha,\beta,y) \triangleq \left(c_1\mathbb{E}\{\eta_s(x,y)\}+c_2\int \eta_s(x,y)\mathrm{d}\mu\right). \tag{43}$$

2) The decoder's optimal action for the source interval $[\alpha,\beta)$:

$$\kappa(\alpha,\beta)=\arg\min_{y\in\mathcal{Y}} C_D(\alpha,\beta,y). \tag{44}$$

3) Costs for the source interval $[\alpha,\beta)$ in conjunction with the optimal action:

$$\epsilon_{s,noisy}(\alpha,\beta)\triangleq C_s(\alpha,\beta,\kappa(\alpha,\beta)) \quad \text{for } s\in\{E,D\}. \tag{45}$$

4) Equilibrium total costs associated with the m level optimal strategic quantizer for $[x_0,\beta)$:

$$D_m(x_0,\beta)\triangleq\min_{\mathbf{q}}\sum_{j=1}^{m}\epsilon_{s,noisy}(x_{j-1},x_j) \quad \text{for } s\in\{E,D\}. \tag{46}$$

where $\mathbf{q}=[x_0,\ldots,x_m]$, $a=x_0<\ldots<x_m=\beta$.

5) The set of all non-empty convex subsets of $\mathcal{X}$:

$$\mathcal{S}\triangleq\{[t_1,t_2):t_1,t_2\in\mathcal{X},t_1<t_2\}. \tag{47}$$

The encoder minimizes D_E with the choice of the quantizer decision levels $\mathbf{q}^*=[x_0^*,\ldots,x_M^*]$,

$$\mathbf{q}^*=\arg\min_{\mathbf{q}}\sum_{i=1}^{M}\int\sum_{j=1}^{M}\eta_E(x,y_j)\mathbb{E}_\pi(p(z_j|z_i))\mathrm{d}\mu, \tag{48}$$

where the representative levels y_m^* are chosen by the decoder to minimize its distortion D_D,

$$y_m^*=\arg\min_{y}\sum_{i=1}^{M}\int \eta_D(x,y)\mathbb{E}_\pi(p(z_j|z_i))\mathrm{d}\mu.$$

The encoder's distortion in quantizing the interval $[\alpha, \beta)$ with one representation level, $\epsilon_E(\alpha, \beta)$, is computed for each $[\alpha, \beta) \in \mathcal{S}$. We set the $1-$level distortion $D_1(\alpha, \beta) = \epsilon(\alpha, \beta), \beta \in \mathcal{X} \backslash x_0$. The $m-$level distortion for an interval $[x_0, \beta) \in \mathcal{S}$ due to quantizing the interval with m representative levels can be written in terms of the $1-$level distortions,

$$D_m(x_0, \beta) = \min_{\substack{x_0, \ldots, x_m \in \mathcal{X} \\ a = x_0 < x_1 < \ldots < x_m = \beta}} \sum_{i=1}^{m} D_1(x_{i-1}, x_i).$$

The optimization for $m-$level quantization of $[x_0, \beta)$ can be written as the sum of $(m-1)$ level quantization of $[x_0, \alpha)$ and $1-$level quantization of $[\alpha, \beta)$ as the Bellman equations,

$$D_m(x_0, \beta) = D_{m-1}(x_0, r_{m-1}(x_0, \beta) + D_1(r_{m-1}(x_0, \beta), \beta)$$

$$r_{m-1}(x_0, \beta) \triangleq \arg\min_{\alpha \in \mathcal{X}} \{D_{m-1}(x_0, \alpha) + D_1(\alpha, \beta)\}.$$

Dynamic programming requires a forward and a backward pass. During the forward pass, we compute and store $r_{m-1}(x_0, \beta)$ and $D_m(x_0, \beta)$ for each pair $(m, \beta), m \in [2 : M], \beta \in \mathcal{X} \backslash x_0$ recursively starting from $m = 2$ using the pre-computed values of $D_1(\alpha, \beta)$. In the backward pass, we set $x_0^* = a, x_M^* = b$ and compute optimal decision (x_m^*) and representative levels (y_m^*) recursively as

$$x_{m-1}^* = r_{m-1}(x_0^*, x_m^*), \quad m = M, \ldots, 2,$$

$$y_m^* = \kappa(x_{m-1}^*, x_m^*), \quad m = [1 : M].$$

IV. Complexity Analysis

We first investigate the worst-case complexity of different design approaches in terms of three parameters[6]: N, M and $|\mathcal{Y}|$. We then focus on reducing the complexity of the proposed DP-based approach by incorporating the known methods in the literature.

[6]The extension of the big-O notation from one variable to multiple variables is not straightforward, and there is a lack of consistency in its treatment, see, e.g., [33]. Our focus here is to analyze how the overall complexity varies with one of these three variables while the other two are kept constant, and all three are asymptotically large.

A. Exhaustive search

A full search for $\mathbf{q}^*$ and $\mathbf{y}^*$ requires $\mathcal{O}(|\mathcal{Y}|M^N)$ complexity since there are M choices for each $x_i, i \in [1:M]$, and $|\mathcal{Y}|$ action choices for each quantization interval. Under Assumption 1, this complexity can be reduced to a full search for optimal partitions and actions, which is $\mathcal{O}(|\mathcal{Y}|N^M)$.

B. Posterior Search

In [11], the authors proposed a constructive algorithm based on the idea of searching through all Bayes-plausible[7] conditional distributions of the source given the message $p(x|z)$, that also characterizes the strategic quantizer in our formulation. However, there are $\binom{N}{(M-1)} = \mathcal{O}(N^M)$ ways of selecting such a Bayes-plausible posterior which is exponential in M, hence this approach might be infeasible in practice for large M values.

C. Basic Dynamic Programming

We note that the DP algorithm requires computing the following terms:

1) a) $\epsilon_E(\alpha,\beta) = \sum\limits_{v\in\mathcal{V}_i} \eta_E(v,y)P(v) \quad \forall\,[\alpha,\beta) \in \mathcal{S}$
   b) $y = \arg\min\limits_{t\in\mathcal{Y}} \sum\limits_{v\in\mathcal{V}_i} \eta_D(v,t)P(v)$
2) Forward pass:
   a) $D_E(n,m) = D_E(h(n,m),m-1) + \epsilon_E(h(n,m),n), m \in [2:M], n \in [m:N]$
   b) $h(n,m) = \arg\min\limits_{i\in[m-1:n-1]} \{D_E(i,m-1) + \epsilon_E(i,n)\}$
3) Backward pass: $q^*_{m-1} = h(q^*_m, m)$.

The computation of step 1a involves the term 1b. The term 1a has a complexity $\mathcal{O}(N)$ for a given $[\alpha,\beta)$, while the term 1b has a complexity $\mathcal{O}(N)$ for a given interval $[\alpha,\beta)$, for each

[7]Bayes plausibility of the posterior belief simply means that it satisfies the Bayes relationship given the source distribution $p(x)$, i.e., all $p(x|z)$ distributions such that $\sum_z p(x|z) = p(x)$.

$t \in \mathcal{Y}$. Since there are $\mathcal{O}(N^2)$ elements in $\mathcal{S}$, the complexity of computing $\epsilon_E(\alpha, \beta)$ in step 1a is $\mathcal{O}(N^3|\mathcal{Y}|)$.

The computation of step 2a involves the term in step 2b which has a complexity $\mathcal{O}(N^2M)$ because it is computed for different levels of quantization $m = [1 : M]$, for $\mathcal{O}(N)$ intervals with $\mathcal{O}(N)$ ways of dividing the interval. So, step 2 has a computational complexity $\mathcal{O}(N^2M)$.

In the backward pass, the algorithm only uses data computed in the forward pass. The total computational complexity is then $\mathcal{O}(N^3|\mathcal{Y}| + N^2M)$.

D. Complexity Reduction

Recursively solving a DP equation is often too complex because the same subproblems are solved repeatedly if the larger subproblems are solved before the smaller ones. Complexity can therefore be decreased by explicitly enumerating the distinct subproblems and solving them in the correct order. This idea is incorporated in the classical quantizer design problem in [18], and we carry out a similar analysis here. Moreover, if $\eta_D(\cdot, \cdot)$ is of some specific form, there are additional complexity reductions. In this section, inspired by [18], we discuss several complexity reduction strategies and analyze the overall complexity in terms of N, M and $|\mathcal{Y}|$ variables for strategic quantization.

1) Complexity Reduction with Pre-computation: We note that in step 1b, the summation $C_D(\alpha, \beta, t) = \sum_{i=\alpha}^{\beta-1} \eta_D(x_i, t) P(x_i)$ is required. We next introduce the matrices Γ_E and Γ_D:

$$\Gamma_s[\alpha, t] \triangleq C_s(x_0, x_\alpha, t), \quad \alpha \in [1 : N],\, t \in \mathcal{Y},\, s \in \{E, D\}. \tag{49}$$

The matrices Γ_s, $s \in \{E, D\}$ are of size $N \times |\mathcal{Y}|$. The t^{th} column of Γ_s, requires $\mathcal{O}(N)$ computations to build via $\Gamma_s[\alpha, t] = \Gamma_s[\alpha, t-1] + \eta_D(\alpha, t)P(x_\alpha)$, $\alpha \in [1 : N]$. Computing each matrix Γ_s has a total complexity of $\mathcal{O}(N|\mathcal{Y}|)$.

We note that the terms in steps 1a and 1b can be expressed as

$$C_s(\alpha, \beta, t) = \Gamma_s(\beta, t) - \Gamma_s(\alpha, t), \tag{50}$$

which implies that with precomputed Γ_s, for a given $[\alpha, \beta)$, the term $\kappa(\alpha, \beta)$ can be computed in $\mathcal{O}(|\mathcal{Y}|)$ complexity, and $\epsilon_E(\alpha, \beta)$ takes $\mathcal{O}(1)$ operations. The total complexity of the algorithm is then $\mathcal{O}(N^2(|\mathcal{Y}| + M))$.

2) Complexity Reduction via Matrix Search with a Decoder Cost Assumption: A well-known method of reducing complexity in the DP literature is matrix search [34], which is incorporated into classical quantization via DP method in [18]. We follow the same approach here.

Given a $y_m \in \mathcal{Y}$, we compute the distortion for $C_E(\alpha, \beta, t)$ for a given $[x_\alpha, x_\beta)$ region for each $t \in \mathcal{Y}$. In order to compute $\epsilon_E(\alpha, \beta)$ and $\epsilon_D(\alpha, \beta)$, we first need to compute $\kappa(\alpha, \beta)$. The following assumption on the decoder's distortion function, $\eta_D(x, y)$, enables a low-complexity approach for computing $\kappa(\alpha, \beta)$.

Assumption 3: The distortion measure of the decoder, $\eta_D(x, y)$ is non-negative and strictly increasing in $|x - y| \in [0, \infty)$ for a given x.

In most engineering applications, this assumption holds, for example, $\eta_D(x, y) = g(x)(x - y)^2$, where $g : \mathbb{R} \to \mathbb{R}^+$ is an arbitrary weight function, is widely used in compression, see e.g, [35]. We next present the following auxiliary lemma whose proof appears in Appendix C.

Lemma 4: Given $[t_1, t_2] \subset [x_u, x_v) \subset [x_u, x_w)$, $u, v, w \in [0 : N]$, if $C_D(u, v, t_1) \geq C_D(u, v, t_2)$, then $C_D(u, w, t_1) \geq C_D(u, w, t_2)$.

We use Lemma 4 to obtain the following result, whose proof is presented in Appendix D.

Lemma 5: $\kappa(\alpha, \beta) \in [x_\alpha, x_\beta)$ and is increasing in α, for a fixed β, and β for a fixed α.

Lemma 5 implies that the search for the representative level of an interval can be restricted to the interval itself. Hence, under Assumption 3, we use Lemma 5 to decrease the linear search interval for $\kappa(\alpha, \beta)$ from $\mathcal{Y}$ to $\mathcal{Y} \cap [x_\alpha, x_\beta), [\alpha, \beta) \in \mathcal{S}$, resulting in a reduction in complexity. Moreover, using the second part of Lemma 5, we reduce the interval of linear search for the following cases: Given $t = \kappa(\alpha, v), v < \beta$, the search interval for $\kappa(\alpha, \beta)$ becomes $\mathcal{Y} \cap [t, x_\beta)$ where $t \geq x_\alpha$, instead of $\mathcal{Y} \cap [x_\alpha, x_\beta)$. Given $t = \kappa(\alpha, \beta)$, the search interval for $\kappa(u, \beta), u > \alpha$ becomes $\mathcal{Y} \cap [\max(t, x_u), x_\beta)$, instead of $\mathcal{Y} \cap [x_u, x_\beta)$.

We next define the following expressions that will be used in the subsequent derivations.

*Definition 1 (***Submatrix of a matrix***):* For a matrix A, a submatrix is defined as the elements in the intersection of the rows and columns considered, i.e.,

$$A[[i_1,\dots,i_{t_1}],[j_1,\dots,j_{t_2}]] = \begin{bmatrix} A[i_1,j_1] & \dots & A[i_1,j_{t_2}] \\ & \vdots & \\ A[i_{t_1},j_1] & \dots & A[i_{t_1},j_{t_2}] \end{bmatrix}.$$

*Definition 2 (***Monotone matrix***):* A matrix A is monotone if $\arg\min_j A[i_1,j] \leq \arg\min_j A[i_2,j]$ for $i_1,i_2,j \in [1:n]$, $i_1 < i_2$.

*Definition 3 (***Totally monotone matrix***):* If $A[i,j] \leq A[i,j']$ implies $A[i',j] \geq A[i',j']$ for $i < i'$ and $j < j'$, then A is totally monotone.

We note that a matrix is totally monotone if every submatrix in it is monotone. Equivalently, if every 2×2 submatrix of A is monotone, then A is totally monotone. We then express the linear search for $\kappa(\alpha,\beta)$ as a matrix search problem using an upper triangular matrix defined as

$$S_\alpha[\beta,t] \triangleq C_D(\alpha,\beta,t),\ \alpha \in [0:N-1], \beta \in [\alpha+1:N],\ t \in \mathcal{Y}. \tag{51}$$

Finding $\kappa(\alpha,\beta)$ is equivalent to finding the minimum element of the row $S_\alpha[\beta,:]$. Under Assumption 3, we have from Lemma 4 that for an interval $[x_\alpha, x_{\beta_1})$, if $S_\alpha[\beta_1,t_1] \leq S_\alpha[\beta_1,t_2]$, then $S_\alpha[\beta_2,t_1] \geq S_\alpha[\beta_2,t_2], \beta_1 < \beta_2, t_1 < t_2$. Hence, we conclude that S_α is totally monotone. The complexity of finding $\kappa(\alpha,\beta)$ for all $[\alpha,\beta)$ in $\mathcal{S}$ via the matrix search algorithm is $\mathcal{O}(N^2)$. Hence, under Assumption 3, the total complexity is $\mathcal{O}(N^2 M + N|\mathcal{Y}|)$, i.e., increases linearly with M and $|\mathcal{Y}|$, and in a quadratic manner with N.

3) Complexity Reduction via Optimal Action Assumption: If $y_m = \mathbb{E}\{x|z_m\}$, which is the case for any Bregman loss function including (but not limited to, see [36]) the important case of $\eta_D(x,y) = (x-y)^2$, we do not need to search for $\kappa(\alpha,\beta)$ since we directly compute it:

$$\kappa(\alpha,\beta) = \frac{M_1(\beta) - M_1(\alpha)}{M_0(\beta) - M_0(\alpha)}, \tag{52}$$

where

$$M_k(\alpha) \triangleq \sum_{i=0}^{\alpha-1} x_i^k P(x_i),\ k = 0, 1 \tag{53}$$

are the k^{th} cumulative moments of $P(x_i)$.

If $M_k(\alpha), \alpha \in [1 : N], k = 0, 1$ are precomputed, then $\kappa(\alpha, \beta)$ can be computed when required, on the fly, in $\mathcal{O}(1)$ complexity independent of length of the interval (α, β) followed by computation of $\epsilon_E(\alpha, \beta)$ with complexity $\mathcal{O}(N)$. The complexity of computation of $M_k(\alpha)$ for $\alpha \in [1 : N], k = 0, 1$ is of the order $\mathcal{O}(N)$. Since $\kappa(\alpha, \beta)$ can be found in $\mathcal{O}(1)$, it can be found on the fly for step 2b. The value of $\kappa(\alpha, \beta)$ is used to find $\epsilon_E(\alpha, \beta)$ in $\mathcal{O}(N)$ complexity. Step 2 requires $\mathcal{O}(NM)$ computations. The total computational complexity is then $\mathcal{O}(NM)$, i.e., linear in N and M. We note that the Bregman loss function assumption on η_D eliminates the search over $\mathcal{Y}$, hence this term does not appear in complexity computations.

Method	Time Complexity
Full Search	$\mathcal{O}(N^M\lvert\mathcal{Y}\rvert)$
Posterior Search [11]	$\mathcal{O}(N^M)$
Basic DP	$\mathcal{O}(N^3\lvert\mathcal{Y}\rvert + N^2M)$
DP with Γ_s	$\mathcal{O}(N^2\lvert\mathcal{Y}\rvert + N^2M)$
DP with Assumption 3 on η_D	$\mathcal{O}(N^2M + N\lvert\mathcal{Y}\rvert)$
DP with Bregman η_D	$\mathcal{O}(NM)$

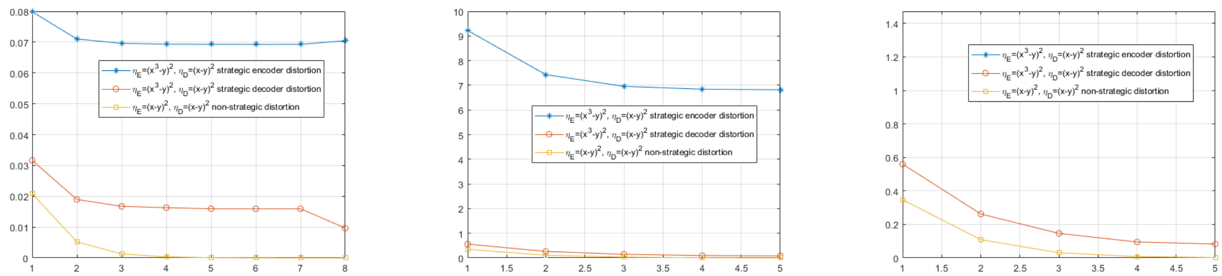
TABLE I

COMPLEXITY OF STRATEGIC QUANTIZATION WITH DIFFERENT METHODS AND ASSUMPTIONS.

The complexity of different approaches are summarized in Table I.

V. NUMERICAL RESULTS

We present the results for a zero-mean, unit variance Gaussian source, and a uniform source in $[0, 1]$. The distortion measures are taken as $\eta_E(x, y) = (x^3 - y)^2$ and $\eta_D(x, y) = (x - y)^2$. We



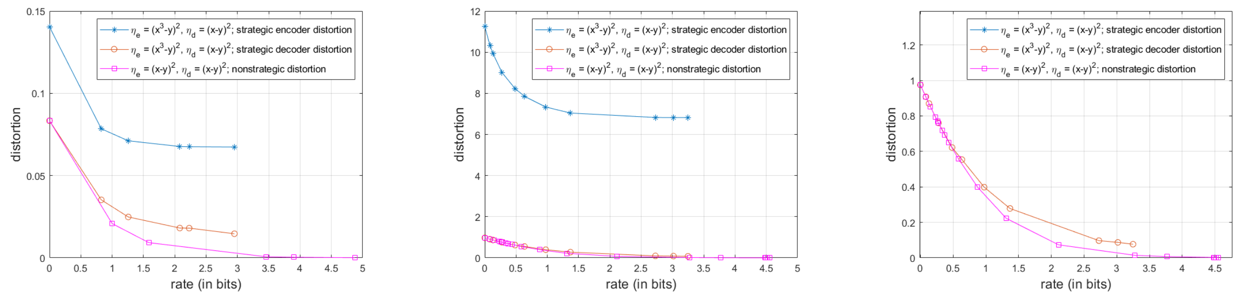
(a)$X \sim U[0,1]$ (b)$X \sim \mathcal{N}(0,1)$ (c)$X \sim \mathcal{N}(0,1)$ magnified

Fig. 4. Fixed-rate strategic quantization of a uniform and a Gaussian source for $\eta_E(x,y) = (x^3 - y)^2$ and $\eta_D(x,y) = (x-y)^2$.

take $\mathcal{Y} = \mathcal{X}$ for all scenarios. We set $N = 500$ for fixed-rate quantization of the uniform source (since we consider high rate regions), $N = 300$ for fixed rate Gaussian source, and $N = 100$ for variable rate settings (due to complexity). For the variable-rate case, we search for M^* in the set $[1:30]$. The distortion versus rate graphs are plotted in Figures 4 and 5 respectively for fixed and variable rate strategic quantization. We have made our codes that have generates the figures publicly available at [20].

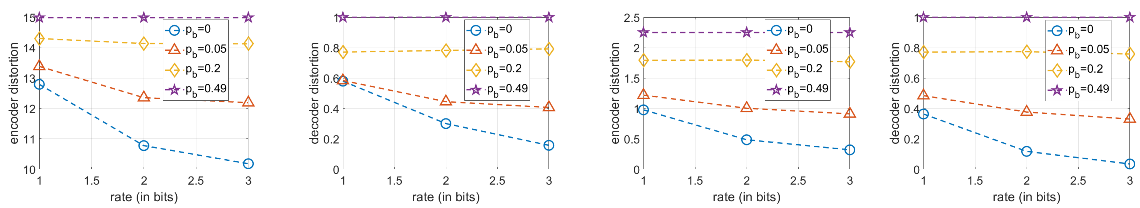
We next make a few observations on the numerical results. As expected, both distortions decrease monotonically in the low-rate regime, similar to the case for the classical quantization. However, unlike its classical counterpart, the distortions do not vanish in the high-rate region. This is due to the misaligned objectives between the encoder and the decoder, as also observed in [7] for the strategic rate-distortion function. The most interesting observation, however, is the fact that forcing the encoder to use all $M = 256$ quantization resolution increases the encoder's distortion as shown in Figure 4-a. This observation points out to an interesting phenomenon: the encoder might prefer to use smaller rate, even if there is no cost associated with rate. This sharply contrasts with the common engineering practice where increasing the communication rate can never "hurt" the communicating agents.

A natural question here pertains to the characterization of the settings where there is a cut-off



(a)$X \sim U[0,1]$ (b)$X \sim \mathcal{N}(0,1)$ (c)$X \sim \mathcal{N}(0,1)$ magnified

Fig. 5. Variable-rate strategic quantization of a uniform and a Gaussian source, for $\eta_E(x,y) = (x^3 - y)^2$ and $\eta_D(x,y) = (x-y)^2$.



(a)$\eta_E = (x^3 - y)^2$ (b)$\eta_E = (x^3 - y)^2$ (c)$\eta_E = (1.5x - y)^2$ (d)$\eta_E = (1.5x - y)^2$

Fig. 6. Channel optimized quantization of a Gaussian source with $\eta_D = (x-y)^2$.

rate R_0 for which in the region $R > R_0$, the encoder chooses not to use the channel fully, i.e., $D_E(R) = D_E(R_0)$. Recent analysis in the unconstrained variation of this problem indicates that a class of persuasion problems[8], admits an optimal "bi-pooling" policy over the induced posteriors[24], [37] hence in this class, for any strategic quantizer applied to purely discrete sources, there exists a cutoff rate R_0, unless the unconstrained policy is fully revealing. We leave analysis of R_0, and more generally the structure of operational rate-distortion function, as a future work.

[8]Ones that can be solved via linear programming as mentioned in the Introduction.

VI. Conclusion

In this paper, we have first formulated the strategic quantization problem and shown its features that differentiate it from classical quantization. We have numerically demonstrated that strategic variation of Lloyd-Max may not converge to a local optimum. Motivated by this observation, we have developed dynamic programming algorithms for the strategic quantization problem inspired by the early classical quantization literature. We then developed fixed and variable rate strategic quantizer design algorithms based on dynamic programming. The proposed method yields the optimal strategic quantizer under the monotonicity assumption. Numerical results obtained via the proposed algorithm suggest several open theoretical questions about the behavior of the operational distortion-rate curve of the optimal strategic quantizers.

While the DP method inherently requires a discrete source, any continuous source, under mild smoothness assumptions, can be discretized via sampling, and our methods can be applied. The required accuracy can be achieved by increasing the sampling rate and, hence, the computational complexity. We have analyzed the complexity of the proposed and possible competitor methods in conjunction with complexity reduction methods that can be incorporated into our DP-based approach. We have shown that the overall complexity can be reduced to linear in the quantization resolution, M, and source alphabet cardinality N in the particular case of a Bregman loss decoder cost function η_D which includes the important special case of MSE, $\eta_D(x, y) = (x - y)^2$.

Appendix A
Algorithms

A. *Fixed Rate Strategic Quantization*

Input: $\mu(\cdot), \mathcal{X}, M, \eta_E(\cdot,\cdot), \eta_D(\cdot,\cdot)$
Output: $\mathbf{q}^*$, $\{y_m^*\}$, $D_E(N,M)$, $D_D(N,M)$
Initialization: $q_0^* = 0$, $q_M^* = N$
Parameters: N
$\mathcal{S} \leftarrow \{[\alpha,\beta) : \alpha,\beta \in [0:N], \alpha < \beta\}$
for $[\alpha,\beta) \in \mathcal{S}$ **do**
    $t \leftarrow \kappa(\alpha,\beta)$
    $\epsilon_E(\alpha,\beta) \leftarrow C_E(\alpha,\beta,t)$
end for
for $n \leftarrow 2,3,\ldots,N$ **do**
    $D_E(n,1) \leftarrow \epsilon_E(0,n)$
end for
for $m \leftarrow 2,3,\ldots,M$ **do**
    for $n \leftarrow m,\ldots,N$ **do**
        $h(n,m) \leftarrow \underset{i\in[m-1:n-1]}{\arg\min} \{D_E(i,m-1) + \epsilon_E(i,n)\}$
        $D_E(n,m) \leftarrow D_E(h(n,m),m-1) + \epsilon_E(h(m,n),n)$
    end for
end for
for $m \leftarrow M, M-1,\ldots,2$ **do**
    $q_{m-1}^* = h(q_m^*, m)$
end for
for $m \leftarrow 1,2,\ldots,M$ **do**
    $y_m^* \leftarrow \kappa(q_{m-1}^*, q_m^*)$
end for
$D_E^* \leftarrow D_E(N,M)$
$D_D^* \leftarrow D_D(N,M)$

B. Variable Rate Strategic Quantization

Input: $\mu(\cdot), \mathcal{X}, \lambda, \eta_E(\cdot,\cdot), \eta_D(\cdot,\cdot)$
Output: M^*, $\mathbf{q}^*$, $\{y_m^*\}$, D_E^*, D_D^*, H^*
Initialization: $q_0^* = 0$, $q_M^* = N$
Parameters: N, $\mathcal{P}$
$\mathcal{P} \leftarrow [1:30]$
$\mathcal{S} \leftarrow \{[\alpha,\beta) : \alpha,\beta \in [0:N], \alpha < \beta\}$
for $[\alpha,\beta) \in \mathcal{S}$ **do**
  $t \leftarrow \kappa(\alpha,\beta)$
  $\epsilon_{E,\lambda}(\alpha,\beta) \leftarrow C_E^\lambda(\alpha,\beta,t)$
end for
for $n \leftarrow 2,3,\ldots,N$ **do**
  $D_{E,\lambda}(n,1) \leftarrow \epsilon_{E,\lambda}(0,n)$
end for
for $M \in \mathcal{P}$ **do**
  for $m \leftarrow 2,3,\ldots,M$ **do**
    for $n \leftarrow m,\ldots,N$ **do**
      $h_\lambda(n,m) \leftarrow \underset{i\in[m-1:n-1]}{\arg\min}\{D_{E,\lambda}(i,m-1)+\epsilon_{E,\lambda}(i,n)\}$
      $D_{E,\lambda}(n,m) \leftarrow D_{E,\lambda}(h_\lambda(n,m),m-1)+\epsilon_{E,\lambda}(h_\lambda(n,m),n)$
    end for
  end for
  for $m \leftarrow M, M-1,\ldots,2$ **do**
    $q_{m-1}^* = h_\lambda(q_m^*,m)$
  end for
  for $m \leftarrow 1,2,\ldots,M$ **do**
    $y_m \leftarrow \kappa(q_{m-1}^*, q_m^*)$
  end for
  $d_E(M) \leftarrow \sum_{m=1}^{M} \epsilon_E(q_{m-1}^*, q_m^*)$
  $d_D(M) \leftarrow \sum_{m=1}^{M} \epsilon_D(q_{m-1}^*, q_m^*)$

$H_M \leftarrow H(N, M)$

$D(\lambda, M) \leftarrow D_{E,\lambda}(N, M)$

$M = M + 1$

end for

$M^* \leftarrow \arg\min_{M \in \mathcal{P}} D(\lambda, M)$

$H^* \leftarrow H_{M^*}$

$D_E^* \leftarrow d_E(M^*)$

$D_D^* \leftarrow d_D(M^*)$

Appendix B
Proof of Lemma 2

Let $\mathbf{p}^*$ be the optimal $M-1$-level strategic quantizer applied to $[x_0, x_{q^*_{M-1}})$. If $\mathbf{p}^* \neq [q_0^*, \ldots, q_{M-1}^*]$, then we can use $\mathbf{p}^*$ to construct $\mathbf{q}^*$ as $[\mathbf{p}^*, q_M^*]$ having a smaller D_E, which is a contradiction.

Appendix C
Proof of Lemma 4

We have $x_i - t_2 < x_i - t_1$ for $i \in [v : w-1]$ and since $\eta_D(x, y)$ is monotonic in $|x - y|$,

$$C_D(v, w, t_1) \geq C_D(v, w, t_2). \tag{54}$$

If the following inequality holds:

$$C_D(u, v, t_1) \geq C_D(u, v, t_2), \tag{55}$$

then,

$$\begin{aligned} C_D(u, w, t_2) &= C_D(u, v, t_2) + C_D(v, w, t_2) && \text{(56a)} \\ &\leq C_D(u, v, t_1) + C_D(v, w, t_1) && \text{(56b)} \\ &= C_D(u, w, t_1). && \text{(56c)} \end{aligned}$$

Lemma 4 yields $C_D(u, v, t_2) \leq C_D(u, v, t_1)$, and (54) implies $C_D(v, w, t_2) \leq C_D(v, w, t_1)$. Using both inequalities in (56a), we obtain (56b). Combining terms in (56b) leads to (56c).

APPENDIX D
PROOF OF LEMMA 5

For $i \in [\alpha : \beta - 1]$, if $t \geq x_\beta$, we obtain from Assumption 3 that $\eta_D(x_i, t) \geq \eta_D(x_i, x_\beta)$, which implies that $\kappa(\alpha, \beta) < x_\beta$. We also have that for $t < x_\alpha$, $\eta_D(x_i, t) \geq \eta_D(x_i, x_\alpha)$, hence $\kappa(\alpha, \beta) \geq x_\alpha$. It follows from Lemma 4 that $\kappa(\alpha, \beta)$ is monotone increasing in β. Using an argument analogous to the proof of Lemma 4 with the right end of interval fixed, we show that $\kappa(\alpha, \beta)$ is increasing in α.

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