Strategic Quantization

Anonymous Authors
Please do NOT provide authors' names and affiliations
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***Abstract*—This paper is concerned with the quantization setting where the encoder and the decoder have misaligned objectives. While the unconstrained variation of this problem has been well-studied under the theme of information design problems in Economics, the problem becomes more appealing and relevant to engineering applications with a constraint on the cardinality of the message space. We first motivate the problem via a toy example demonstrating the strategic quantization problem's intricacies, explicitly showing that the quantization resolution can change the nature of optimal encoding policy and the iterative optimization of the decoder and the encoder mappings may not converge to a local optimum solution. As a remedy, we develop a gradient-descent based solution. We analyze the poor local optimal optima issues associated with the optimization method, and show that even for well-behaving sources, such as Gaussian, there are multiple local minima, depending on the distortion measures chosen, in sharp contrast with the classical quantization. We finally present numerical results obtained via the proposed algorithms that suggest the proposed algorithms' validity and demonstrate strategic quantization features that differentiate it from its classical counterpart.**

I. Introduction

Consider the following problem: an encoder observes a realization of source $X \in \mathcal{X}$ with a probability distribution μ and sends a message $Z \in \mathcal{Z}$ using an injective mapping $Q : \mathcal{X} \to \mathcal{Z}$, with $|\mathcal{Z}| \leq M$. After receiving the message Z, the decoder takes action $Y \in \mathcal{Y}$. The costs that the encoder and the decoder minimize are $D_E \triangleq \eta_E(X, Y)$ and $D_D \triangleq \mathbb{E}\{\eta_D(X, Y)\}$, with $\eta_E \neq \eta_D$ (misaligned objectives). The encoder designs Q *ex-ante*, i.e., without the knowledge of the realization of X, using only the functions η_E and η_D, and the statistics of the source, $\mu(\cdot)$. The functions (η_E and η_D), the shared prior (μ), and the mapping (Q) are known to the encoder and the decoder. The problem is to design Q. We call this setup *strategic quantization*, which is the focus of this paper.

The setting without the quantization aspect (in the practical sense, if M is asymptotically large) is known in the Economics literature as the information design, or the Bayesian persuasion problem [1], [2]. These problems analyze how a communication system designer (sender) can use the information to influence the action taken by a receiver. This framework has proven beneficial in analyzing a variety of real-life applications, such as the design of transcripts when schools compete to improve their students' job prospects [3] and voter mobilization and gerrymandering [4], as well as various engineering applications, including in modeling misinformation spread over social networks [5] and privacy-constrained information processing [6], and many more [7]. For an excellent survey of the related literature in Economics, see [8], [9].

The strategic quantization problem, as described above, was discussed in a few contemporary economics and computer science studies. In [10], authors analyze the problem via a computation lens and report approximate results on this problem, relating to another problem they solved conclusively. In one of their main results, the algorithmic complexity of finding the optimum strategic quantizer was shown to be NP-hard. In a recent working paper, Aybaş and Türkel [11] analyzed this problem using the methods in [2] and provided several theoretical properties of strategic quantization. A byproduct of their analysis yields a constructive method for deriving optimal quantizers based on a search over possible posterior distributions over their feasible set. Our objective here is to leverage the rich collection of results in quantization theory, e.g., the comprehensive survey of results by Gray and Neuhoff [12], to study the same problem via the engineering lens.

We note in passing that quantizers also arise as equilibrium strategies endegenously, i.e., without an external constraint, in a related but a distinctly different class of signaling games, namely the cheap talk [13]. In [13], the encoder chooses the mapping from the realization of the source X to message Z *after* observing it, *ex-post*, as different source realizations indicate optimality of different mappings for the encoder. The encoder's lack of commitment power in the cheap talk setting makes the notion of equilibrium a Nash equilibrium since both agents form a strategy that is the best response to each other's mapping. However, in our strategic quantization problem (and the information design problems in general as in [1], [2]), the encoder designs Q *ex-ante*, *before* seeing the source realization, and committed to the designed Q afterward. This commitment is known to the decoder and establishes a form of trust between the sender and the receiver, resulting in possibly higher payoffs for both agents. This difference also manifests itself in the notion of equilibrium we are seeking here since the encoder does not necessarily form the best response to the decoder due to its commitment to Q[1].

II. Model

The set $\mathcal{X}$ is divided into mutually exclusive and exhaustive sets $\mathcal{V}_1, \mathcal{V}_2, \ldots, \mathcal{V}_M$. The action of the decoder y_m can be

[1]These issues are well understood in the Economics literature, see, e.g., [14] for an excellent survey. However, we emphasize them here for a reader with an engineering background; see [7] for a detailed discussion through the engineering lens.

written as a quantization operator

$$y_m = Q(x) \quad \forall x \in \mathcal{V}_m \tag{1}$$

for all $m \in [1:M]$. Throughout this paper, we make the following "monotonicity" assumption on the sets $\{\mathcal{V}_m\}$.

Assumption 1: $\mathcal{V}_m$ is convex for all $m \in [1:M]$.

Remark 1: Assumption 1 is the first of the two regularity conditions commonly employed in the classical quantization literature, cf. [15]. Note that the second regularity condition, $y_m \in \mathcal{V}_m$, is not included in Assumption 1.

In the Economics parlance, Assumption 1 is referred as the "monotonicity" condition. In [16], [17] sufficient conditions on η_E and η_D for the monotonicity of optimal encoder strategies are characterized within the unconstrained (without quantization) variation of the same problem. We note that here we have the quantization constraint in the problem formulation as an exogenous constraint on the message set, hence it is not clear apriori whether the results [16], [17] would be applicable here. Under Assumption 1, $\mathcal{V}_m$ is an interval, i.e.,

$$\mathcal{V}_m = [x_{q_{m-1}}, x_{q_m}) \tag{2}$$

where $0 = q_0 < q_1 < \ldots < q_M = N$. The encoder chooses the boundary indices $\mathbf{q} = [q_0, q_1, \ldots, q_M]$. The decoder determines its actions $\mathbf{y} = [y_1, \ldots, y_M]$ as a best response to $\mathbf{q}$ to minimize its cost $D_D = \mathbb{E}\{\eta_D(x,y)\}$ as follows

$$y_m^* = \arg\min_{y_m \in \mathcal{Y}} \mathbb{E}\{\eta_D(x, y_m) | x \in \mathcal{V}_m\} \quad \forall m \in [1:M] \tag{3}$$

Hence, the decoder chooses the actions $\{y_m\}$ knowing the set of decision sets $\{\mathcal{V}_m\}$. The encoder computes what the decoder would choose as $\mathbf{y}$ given $\{\mathcal{V}_m\}$, and hence optimizes its own cost $\mathbb{E}\{\eta_E(x,y)\}$ over the choice of $\{\mathcal{V}_m\}$ accordingly:

$$\{\mathcal{V}_m^*\} = \arg\min_{\{\mathcal{V}_m\}} \sum_{m=1}^{M} \mathbb{E}\{\eta_E(x, y_m(\{\mathcal{V}_m\}) | x \in \mathcal{V}_m\} \tag{4}$$

or due to Assumption 1 equivalently over the choice of $\mathbf{q}$:

$$\mathbf{q}^* = \arg\min_{\mathbf{q}} \sum_{m=1}^{M} \mathbb{E}\{\eta_E(x, y_m(\mathbf{q})) | x \in [x_{q_{m-1}}, x_{q_m})\} \tag{5}$$

Remark 2: A key consideration here is that the encoder is committed to its choice of $\mathbf{q}$, it cannot determine $\mathbf{q}$ as the best response to $\mathbf{y}$. Hence, while the decoder can optimize its action $\mathbf{y}$ as the best response to $\mathbf{q}$, the encoder cannot choose $\mathbf{q}$ as the best response to $\mathbf{y}$, but that to a function of itself, i.e., the best response to $\mathbf{y}(\mathbf{q})$. This aspect of the problem introduces a hierarchy in the game play (the encoder plays first, and the decoder responds, which is referred to as the "Stackelberg equilibrium" in the computer science and control literature, and more formally constitutes an instance of subgame perfect Bayesian Nash equilibrium) and naturally is not a Nash equilibrium since $\mathbf{q}$ may not be the best response to $\mathbf{y}$. In cheap talk [13], Nash equilibria are sought after and the equilibria achieving strategies happen to be injective mappings, i.e., quantizers, without an exogenous rate constraint. It is essential to note the substantial difference between the problem formulation in this paper and the cheap talk literature [13].

III. Interesting Aspects of Strategic Quantization

In this subsection, we provide simple numerical examples to demonstrate few intricacies of the strategic quantization problem that differentiate it from its classical analogue.

A. To reveal or not to reveal... or partially reveal?

As mentioned earlier, the classical quantization problem is a team problem, i.e., the encoder and the decoder share identical objectives. In strategic quantization, this is no longer the case. Hence, depending on the misalignment between η_E and η_D, the encoder might choose not to send any information to the decoder (e.g., if they are too misaligned), which we refer to as "non-revealing" policy following the convention in the Economics literature. Alternatively, the strategic quantizer problem might simplify to classical quantization with a common distortion measure η_D, i.e., the encoder cannot utilize its information design advantage to persuade the decoder to take a specific action (referred to here as "fully-revealing" encoder policy). Finally, the encoder might choose to employ a quantizer that is not identical to the classical one, i.e., "partially-revealing" policy. It is relatively straightforward to provide an example for the first case; consider, e.g., $\eta_E(x,y) = -\eta_D(x,y)$, $\forall x, y$ which makes the problem a zero-sum game, hence, the optimal strategy for the encoder is not to send any information (see also the analysis below). We demonstrate the latter cases via a simple numerical example.

Consider a continuous $X \sim U(-1,1)$, $M = 3$, $\eta_E(x,y) = (x - \alpha y)^2$, $\eta_D(x,y) = (x-y)^2$ for a given $\alpha \in \mathbb{R}^+$. In other words, the decoder wants to reconstruct X as closely as possible, while the encoder wants the decoder's construction to be as close as possible to $\frac{1}{\alpha}X$, both in the MSE sense. Can the encoder "persuade" the decoder by carefully designing quantizer intervals $\mathcal{V}_m^*$?

Let us parametrize $\mathcal{V}_m^*$ as $[-1, q_1], (q_1, q_2], (q_2, 1]$, where $q_1, q_2 \in [-1,1]$, $q_2 \geq q_1$. For a given q_1, q_2, the decoder determines y_1, y_2, y_3 as follows:

$$y_1 = \frac{\int_{-1}^{q_1} \frac{1}{2} t \; dt}{\int_{-1}^{q_1} \frac{1}{2} \; dt} = \frac{-1+q_1}{2}, y_2 = \frac{\int_{q_1}^{q_2} \frac{1}{2} t \; dt}{\int_{q_1}^{q_2} \frac{1}{2} \; dt} = \frac{q_1+q_2}{2},$$
$$y_3 = \frac{\int_{q_2}^{1} \frac{1}{2} t \; dt}{\int_{q_2}^{1} \frac{1}{2} \; dt} = \frac{q_2+1}{2}. \tag{6}$$

Substituting $\{y_m\}$ in $\eta_E(x,y)$ expressed as a function q_1 and q_2, we obtain

$$J(q_1,q_2) = \frac{1}{2}\bigg(\int_{-1}^{q_1} (t - \alpha\frac{-1+q_1}{2})^2 dt + \int_{q_1}^{q_2} (t - \alpha\frac{q_1+q_2}{2})^2 dt + \int_{q_2}^{1} (t - \alpha\frac{q_2+1}{2})^2 dt \bigg). \tag{7}$$

Enforcing the first order necessary conditions for optimality

$$\frac{\partial J}{\partial q_1} = \frac{\partial J}{\partial q_2} = 0, \tag{8}$$

we obtain, after some straightforward algebra, that for $0 < \alpha < 2$ the only feasible solution that satisfies (8) is $q_1 = -q_2 = -1/3$ which is also the optimal quantizer for the nonstrategic case of $\eta_E(x, y) = \eta_D(x, y) = (x - y)^2$! Hence, for $2 > \alpha > 0$, optimal encoding policy is fully revealing. For $\alpha > 2$, optimum policy for the encoder is non-revealing, i.e., the strategic quantizer simplifies to its classical counterpart where both the encoder and the decoder have identical objectives.For $\alpha = 2$ and $\alpha = 0$, both fully revealing and non-revealing strategies are optimal.

1) Quantizer resolution is binding: We next focus on the question: can the quantization constraint change the nature of optimal encoding policy in strategic communication? For example, is there a case where for $M = 2$ the encoder is non-revealing but for $M = 3$ the encoder prefers to send a message? The answer is, perhaps surprisingly, affirmative.

Consider the same continuous source $X \sim U[-1, 1]$ but with the distortion functions $\eta_D(x, y) = (x - y)^2$ and

$$\eta_E(x, y) = \begin{cases} (x^3 - y)^2, & xy \geq 0 \\ \to \infty, & \text{otherwise.} \end{cases} \tag{9}$$

A fully non-revealing policy, i.e., the case of $R = 0$ ($M = 1$) yields $D_E(0) = \int_{-1}^{1} (t^3)^2 \frac{1}{2}\, dt = 1/7$.

We next consider $M = 2$. From symmetry, the optimal encoding policy is simply setting the boundary at $q_1 = 0$. However, this yields $y_1 = -y_2 = -1/2$ and $D_E(1) = 1/7$, which is identical to $D_E(0)$. Hence, optimal strategic quantizer for $M = 2$ does not send any information to the decoder, i.e., non-revealing.

We finally consider the case of $M = 3$. Similar to the previous case, we parametrize $\mathcal{V}_m^*$ as $[-1, q], (q, 0], (0, 1]$ and express $\{y_m\}$ as a function of q, which yields

$$y_1 = \frac{-1 + q}{2}, \quad y_2 = \frac{q + 1}{2}, \quad y_3 = \frac{1}{2}. \tag{10}$$

Substituting again $\{y_m\}$ in $\eta_E(x, y)$:

$$\begin{aligned} J(q) = \frac{1}{2}\bigg(&\int_{-1}^{q} (t^3 - y_1)^2\, dt + \int_{q}^{0} (t^3 - y_2)^2\, dt \\ &+ \int_{0}^{1} (t^3 - y_3)^2\, dt \bigg). \end{aligned} \tag{11}$$

Enforcing the KKT conditions similar to (8), we obtain $q = 0, \pm\sqrt{1/2}$. Since $q \leq 0$, the possible solutions are $q = 0, -\frac{1}{\sqrt{2}}$. Of these two choices, $q = -1/\sqrt{2}$ yields a lower distortion $D_E = 25/224$, hence is optimal.

We note here that $D_E(\log 3) = 25/224 > 1/7 = D_E(1) = D_E(0)$. Hence, at $M = 2$, the strategic quantizer does not communicate any information while, at $M = 3$, in sharp contrast to $M = 2$, uses the quantization channel fully to send three messages, demonstrating that the quantization constraint can change the nature of the optimum encoder policy. Moreover, it shows that the operational rate-distortion function $D_E(R)$ here is not a strictly decreasing function of rate R, since $D_E(1) = D_E(0)$, unlike its classical counterpart.

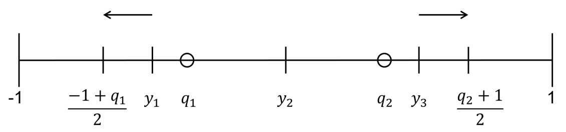


Fig. 1: Movement of the quantization boundaries through Lloyd-Max iterations in the running example.

2) Failure of "Strategic Lloyd-Max": In this subsection, we investigate whether a simple strategic variation of Lloyd-Max approach would yield a locally optimal quantizer, as in the case of classical quantization. Consider a continuous source $X \sim U[-1, 1]$ quantized with $M = 3$ messages where $\eta_E(x, y) = (x^3 - y)^2$ and $\eta_D(x, y) = (x - y)^2$.

We initialize $\{y_m\}$ arbitrarily and find q_1 and q_2 that minimize $D_E = \mathbb{E}\{(x^3 - y)^2\}$, as

$$q_1 = \left(\frac{y_1 + y_2}{2}\right)^{\frac{1}{3}}, q_2 = \left(\frac{y_2 + y_3}{2}\right)^{\frac{1}{3}}. \tag{12}$$

We then find the decoder actions $\{y_m\}$ via (6), and iterate between (12) and (6) until convergence. We note that during these iterations, q_1 and q_2 move towards -1 and 1, respectively, i.e., the boundaries move towards the endpoints of the interval taken for quantization with each iteration as demonstrated in Figure 1, hence the iterations converge to $M = 1$ solution which is non-revealing with $D_E = 1/7$. Now, let us examine whether this solution is a local optimum.

Any admissible perturbation of with some $1 > \epsilon > 0$ of $q_1 = -q_2 = -1$ would result in a $M = 3$ level quantizer with decision boundaries $q_1 = -1 + \epsilon$, $q_2 = 1 - \epsilon$ with the corresponding decoder actions $y_1 = -1 + \frac{\epsilon}{2}$, $y_2 = 0$, $y_3 = 1 - \frac{\epsilon}{2}$, yielding $D_E = 1/7 - \epsilon(1 - \frac{\epsilon}{2})^2(1 - \epsilon)^2$ which is smaller than than that of the non-revealing solution ($D_E = 1/7$), hence this is not a locally optimal solution. This observation indicates that the straightforward enforcement of optimality conditions may not yield a locally optimal solution, which contrasts sharply with the case in classical quantization. In other words, unlike its classical counterpart, a trivial extension of the Lloyd-Max algorithm adopted for strategic settings may not converge to a locally optimum solution.

B. Multiple Local Minima

Unlike classical quantization, strategic version can have multiple local minima even when used in conjunction with log-concave sources, as demonstrated in Fig 2.

IV. Proposed Solution

Having showed that a strategic variation of the Lloyd-Max algorithm, we propose a gradient descent based solution. Due to the Stackelberg equilibrium nature of the problem, the encoder (leader) decides the quantizer decision levels q first, and then the decoder (follower) decides the quantizer reconstruction levels as a function of Q, $\mathbf{y}(Q)$. So the gradient descent approach involves optimizing q along the direction of the gradients $\frac{\partial D_E}{\partial q_m}$.

Starting with an initial set of quantizers $Q = Q_0$, the reconstruction levels $\mathbf{y}(Q)$, and the associated encoder distortion D_E are computed. Then, the following steps are executed until convergence:

1) The gradients $\{\frac{\partial D_E}{\partial x_m}\}$ are computed.
2) The decision levels Q are updated: $x_m \triangleq x_m - \triangle \frac{\partial D_E}{\partial x_m}$, where $\triangle$ is the gradient descent parameter.
3) The reconstruction levels $\mathbf{y}(Q)$ are found.
4) The encoder distortion D_E is computed.

We present below the derivation of gradients for an MMSE decoder $\eta_D = (x-y)^2$ with an encoder with distortion measure $\eta_E(x,y)$. The gradients of the encoder's distortion with respect to the quantizer decision levels are given as:

$$\begin{aligned}\frac{\partial D_E}{\partial x_m} &= \eta_E(x_m, y_m)\frac{\mathrm{d}\mu_X(x_m)}{\mathrm{d}x} - \eta_E(x_m, y_{m+1})\frac{\mathrm{d}\mu_X(x_m)}{\mathrm{d}x} \\ &+ \frac{\partial y_m}{\partial x_m}\int_{x_{m-1}}^{x_m}\frac{\partial \eta_E(x,y_m)}{\partial y_m}\mathrm{d}\mu_X \\ &+ \frac{\partial y_{m+1}}{\partial x_m}\int_{x_m}^{x_{m+1}}\frac{\partial \eta_E(x,y_{m+1})}{\partial y_{m+1}}\mathrm{d}\mu_X,\end{aligned} \tag{13}$$

since the reconstruction levels y_m, y_{m+1} are the only ones which depend on x_m,

$$y_m = \frac{\int_{x_{m-1}}^{x_m} x\mathrm{d}\mu_X}{\int_{x_{m-1}}^{x_m}\mathrm{d}\mu_X}, \quad y_{m+1} = \frac{\int_{x_m}^{x_{m+1}} x\mathrm{d}\mu_X}{\int_{x_m}^{x_{m+1}}\mathrm{d}\mu_X}. \tag{14}$$

The gradients $\frac{\partial y_m}{\partial x_m}, \frac{\partial y_{m+1}}{\partial x_m}$ are

$$\begin{aligned}\frac{\partial y_m}{\partial x_m} &= \frac{x_m\frac{\mathrm{d}\mu_X(x_m)}{\mathrm{d}x}\int_{x_{m-1}}^{x_m}\mathrm{d}\mu_X - \frac{\mathrm{d}\mu_X(x_m)}{\mathrm{d}x}\int_{x_{m-1}}^{x_m}x\mathrm{d}\mu_X}{(\int_{x_{m-1}}^{x_m}\mathrm{d}\mu_X)^2} \\ &= \frac{\mathrm{d}\mu_X(x_m)}{\mathrm{d}x}\frac{x_m - y_m}{\int_{x_{m-1}}^{x_m}\mathrm{d}\mu_X},\end{aligned} \tag{15}$$

$$\begin{aligned}\frac{\partial y_{m+1}}{\partial x_m} &= -\frac{x_m\frac{\mathrm{d}\mu_X(x_m)}{\mathrm{d}x}\int_{x_m}^{x_{m+1}}\mathrm{d}\mu_X - \frac{\mathrm{d}\mu_X(x_m)}{\mathrm{d}x}\int_{x_m}^{x_{m+1}}x\mathrm{d}\mu_X}{(\int_{x_m}^{x_{m+1}}\mathrm{d}\mu_X)^2} \\ &= -\frac{\mathrm{d}\mu_X(x_m)}{\mathrm{d}x}\frac{x_m - y_{m+1}}{\int_{x_m}^{x_{m+1}}\mathrm{d}\mu_X}.\end{aligned} \tag{16}$$

V. Numerical Results

We consider the following settings:

A jointly Gaussian source $(X,\theta) \sim \mathcal{N}\left(\begin{bmatrix}0\\0\end{bmatrix}, \begin{bmatrix}\sigma_X^2 & \rho \\ \rho & \sigma_\theta^2\end{bmatrix}\right), 0 \leq \rho < 1$ and an encoder and decoder with distortion measures $\eta_E(x,\theta,y) = (x+\theta-y)^2, \eta_D(x,\theta,y) = (x-y)^2$.

We plot the encoder and decoder distortions associated with the above settings for correlation coefficient $\rho = [0, 0.2, 0.5, 0.9]$, in Figures 3 and 4 respectively. We observe that the encoder's distortion increases with correlation as expected due to the strategic nature of the problem.

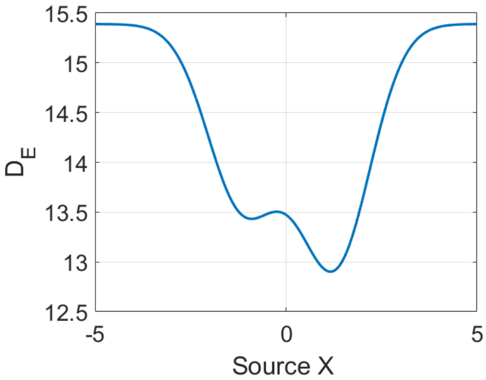
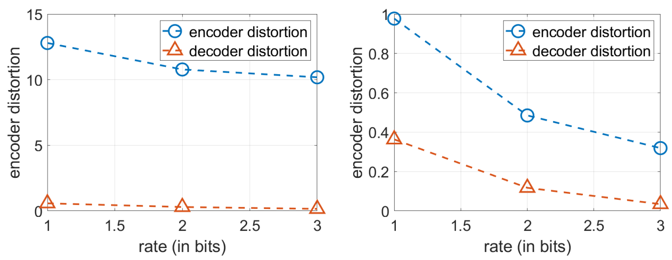


Fig. 2: Local optima in encoder distortion for 2-level quantization of $X \sim \mathcal{N}(0.1, 1)$ with $\eta_E = (x^3 - y)^2, \eta_D = (x-y)^2$

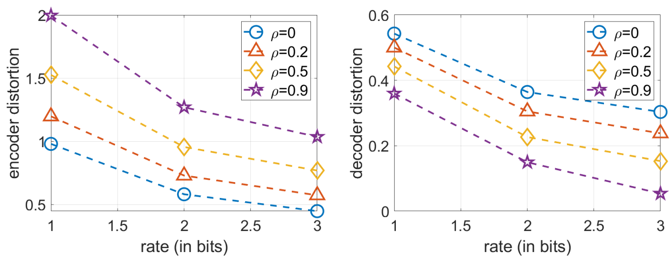


(a) $\eta_E = (x^3 - y)^2$ (b) $\eta_E = (1.5x - y)^2$

Fig. 3: $X \sim \mathcal{N}(0, 1), \eta_D = (x-y)^2$

VI. Conclusion

In this paper, we have first formulated the strategic quantization problem and shown its features that differentiate it from classical quantization. We have numerically demonstrated that strategic variation of Lloyd-Max may not converge to a local optimum. Motivated by this observation, we have developed gradient descent based solutions for the strategic quantization problem. Numerical results obtained via the proposed algorithm suggest several open theoretical questions about the behavior of the operational distortion-rate curve of the optimal strategic quantizers.



(a) Encoder distortion (b) Decoder distortion

Fig. 4: $X \sim \theta \sim \mathcal{N}(0, 1), \eta_E = (x + \theta - y)^2, \eta_D = (x-y)^2$

REFERENCES

[1] L. Rayo and I. Segal, "Optimal information disclosure," *Journal of Political Economy*, vol. 118, no. 5, pp. 949–987, 2010.

[2] E. Kamenica and M. Gentzkow, "Bayesian persuasion," *American Economic Review*, vol. 101, no. 6, pp. 2590–2615, 2011.

[3] M. Ostrovsky and M. Schwarz, "Information disclosure and unraveling in matching markets," *American Economic Journal: Microeconomics*, vol. 2, no. 2, pp. 34–63, 2010.

[4] R. Alonso and O. Câmara, "Persuading voters," *American Economic Review*, vol. 106, no. 11, pp. 3590–3605, 2016.

[5] O. Candogan and K. Drakopoulos, "Optimal signaling of content accuracy: Engagement vs. misinformation," *Operations Research*, vol. 68, no. 2, pp. 497–515, 2020.

[6] E. Akyol, C. Langbort, and T. Başar, "Privacy constrained information processing," in *54th IEEE conference on decision and control (CDC)*. IEEE, 2015, pp. 4511–4516.

[7] ——, "Information-theoretic approach to strategic communication as a hierarchical game," *Proceedings of the IEEE*, vol. 105, no. 2, pp. 205–218, 2016.

[8] E. Kamenica, "Bayesian persuasion and information design," *Annual Review of Economics*, vol. 11, pp. 249–272, 2019.

[9] D. Bergemann and S. Morris, "Information design: A unified perspective," *Journal of Economic Literature*, vol. 57, no. 1, pp. 44–95, 2019.

[10] S. Dughmi and H. Xu, "Algorithmic Bayesian persuasion," *SIAM Journal on Computing*, no. 0, pp. STOC16–68, 2019.

[11] Y. C. Aybaş and E. Türkel, "Persuasion with coarse communication," *arXiv preprint arXiv:1910.13547*, 2019.

[12] R. M. Gray and D. L. Neuhoff, "Quantization," *IEEE Transactions on Information Theory*, vol. 44, no. 6, pp. 2325–2383, 1998.

[13] V. P. Crawford and J. Sobel, "Strategic information transmission," *Econometrica: Journal of the Econometric Society*, pp. 1431–1451, 1982.

[14] J. Sobel, "Giving and receiving advice," *Advances in Economics and Econometrics*, vol. 1, pp. 305–341, 2013.

[15] A. Gersho and R. M. Gray, *Vector quantization and signal compression*. Springer Sci. & Business Media, 2012, vol. 159.

[16] P. Dworczak and G. Martini, "The simple economics of optimal persuasion," *Journal of Political Economy*, vol. 127, no. 5, pp. 1993–2048, Oct. 2019.

[17] J. Mensch, "Monotone persuasion," *Games and Economic Behavior*, vol. 130, pp. 521–542, 2021. [Online]. Available: https://www.sciencedirect.com/science/article/pii/S0899825621001263

APPENDIX A

$$\eta_E(x,y)^2 = (x-ky)^2, \eta_D(x,y)^2 = (x-y)^2$$

A. $M = 1$

$$J(-1,1) = \int_{-1}^{1} (t)^2 \frac{1}{2}\, dt = \frac{t^3}{3}\frac{1}{2}\Big|_{-1}^{1} = \frac{1}{3} \tag{17}$$

B. $M = 2$

$$y_1 = \frac{-1+q}{2} \quad y_2 = \frac{q+1}{2}$$
$$y_1 - y_2 = -1$$
$$y_1^2 - y_2^2 = (y_1-y_2)(y_1+y_2) = -q$$
$$y_1^2 + y_2^2 = \frac{q^2+1}{2}$$
$$\int (t-h)^2\, dt = \int (t^2 - 2th + h^2)\, dt = \frac{t^3}{3} - t^2 h + h^2 t$$

$$\begin{aligned}
J(-1,q) + J(q,1) &= \int_{-1}^{q} (t-ky_1)^2 \frac{1}{2}\, dt + \int_{q}^{1} (t-ky_2)^2 \frac{1}{2}\, dt \\
&= \frac{1}{2}\Big[\frac{t^3}{3} - t^2 k y_1 + (ky_1)^2 t\Big]_{-1}^{q} \\
&\quad + \frac{1}{2}\Big[\frac{t^3}{3} - t^2 k y_2 + (ky_2)^2 t\Big]_{q}^{1} \\
&= \frac{1}{2}\big(\big[\frac{t^3}{3}\big]_{-1}^{1} - ky_1(q^2-1) + (ky_1)^2(q+1) \\
&\quad - ky_2(1-q^2) + (ky_2)^2(1-q)\big) \\
&= \frac{1}{2}\big(\frac{2}{3} - k(q^2-1)(y_1-y_2) \\
&\quad + qk^2(y_1^2-y_2^2) + k^2(y_1^2+y_2^2)\big) \\
&= \frac{1}{2}\big(\frac{2}{3} - k(q^2-1)(y_1-y_2) \\
&\quad + qk^2(y_1^2-y_2^2) + k^2(y_1^2+y_2^2)\big) \\
&= \frac{1}{2}\big(\frac{2}{3} + k(q^2-1) - q^2k^2 + k^2\frac{q^2+1}{2}\big) \\
&= \frac{1}{2}\big(\frac{2}{3} + k(q^2-1) - \frac{q^2k^2}{2} + \frac{k^2}{2}\big)
\end{aligned}$$

$$\frac{\partial J(-1,q) + J(q,1)}{\partial q} = 2kq - 2k^2 q + k^2 q = 2kq - k^2 q$$

Setting above equal to 0,

$$2kq - k^2 q = 0 \tag{18}$$

$q = 0, k = 0, k = 2$

$$\begin{aligned}
&\int_{-1}^{0} (x+0.5k)^2 + \int_{0}^{1} (x-0.5k)^2 \\
&\quad = \int_{-1}^{1} x^2 + \int_{-1}^{1} k^2 0.5^2 - \int_{0}^{1} 2xk0.5 + \int_{-1}^{0} 2xk0.5 \\
&\quad = \int_{-1}^{1} x^2 + 2k^2 0.5^2 - 2k0.5\frac{x^2}{2}\Big|_{0}^{1} + 2k0.5\frac{x^2}{2}\Big|_{-1}^{0} \\
&\quad = \int_{-1}^{1} x^2 + 2k^2 0.5^2 - 2k0.5
\end{aligned} \tag{19}$$

The term $2k^2 0.5^2 - 2k0.5$ is negative for $0 < k < 2$, 0 for $k = 2$, and positive for $k > 2$.

C. $M = 3$

$$y_1 = \frac{-1+q_1}{2} \quad y_2 = \frac{q_1+q_2}{2} \quad y_3 = \frac{q_2+1}{2}$$

$$y_2 - y_1 = \frac{q_2+1}{2} \tag{20}$$

$$y_3 - y_2 = \frac{1-q_1}{2} \tag{21}$$

$$y_1 + y_2 = \frac{-1+2q_1+q_2}{2} \tag{22}$$

$$y_2 + y_3 = \frac{q_1+2q_2+1}{2} \tag{23}$$

$$\begin{aligned} H &= J(-1,q_1) + J(q_1,q_2) + J(q_2,1) \\ &= \frac{1}{2}(\int_{-1}^{q_1} (t-ky_1)^2\,dt + \int_{q_1}^{q_2} (t-ky_2)^2\,dt \\ &\quad + \int_{q_2}^{1} (t-ky_3)^2\,dt) \end{aligned} \tag{24}$$

$$\begin{aligned} \frac{\partial H}{\partial q_1} &= \frac{1}{2}((q_1-ky_1)^2 + \int_{-1}^{q_1} 2(t-ky_1)(\frac{-k}{2})\,dt \\ &\quad - (q_1-ky_2)^2 + \int_{q_1}^{q_2} 2(t-ky_2)(\frac{-k}{2})\,dt) \\ &= k(y_2-y_1)(2q_1 - k(y_1+y_2)) - k\frac{t^2}{2}\Big|_{-1}^{q_1} + k^2 y_1 t\Big|_{-1}^{q_1} \\ &\quad - k\frac{t^2}{2}\Big|_{q_1}^{q_2} + k^2 y_2 t\Big|_{q_1}^{q_2} \\ &= k(y_2-y_1)(2q_1 - k(y_1+y_2)) - k\frac{(q_1^2-1)}{2} \\ &\quad + k^2 y_1(q_1+1) - k\frac{q_2^2-q_1^2}{2} + k^2 y_2(q_2-q_1) \\ &= k\frac{q_2+1}{2}(2q_1 - k\frac{-1+2q_1+q_2}{2}) - k\frac{(q_1^2-1)}{2} \\ &\quad + k^2\frac{-1+q_1}{2}(q_1+1) - k\frac{q_2^2-q_1^2}{2} \\ &\quad + k^2\frac{q_1+q_2}{2}(q_2-q_1) \\ &= k\frac{q_2+1}{2}(2q_1 - k\frac{-1+2q_1+q_2}{2}) - k\frac{(q_2^2-1)}{2} \\ &\quad + k^2\frac{q_2^2-1}{2} \\ &= k\frac{q_2+1}{2}((2q_1 - k\frac{-1+2q_1+q_2}{2}) \\ &\quad - (q_2-1) + k(q_2-1)) \\ &= k\frac{q_2+1}{2}(2q_1 + \frac{k}{2} - kq_1 - \frac{kq_2}{2} - q_2 + 1 + kq_2 - k) \\ &= k\frac{q_2+1}{2}(q_1(2-k) + q_2(\frac{k}{2}-1) + k(\frac{-1}{2}) + 1) \end{aligned} \tag{25}$$

$$\begin{aligned} \frac{\partial H}{\partial q_2} &= (q_2-ky_2)^2 + \int_{q_1}^{q_2} 2(t-ky_2)(\frac{-k}{2})\,dt - (q_2-ky_3)^2 \\ &\quad + \int_{q_2}^{1} 2(t-ky_3)(\frac{-k}{2})\,dt \\ &= k(y_3-y_2)(2q_2 - k(y_2+y_3)) - k\frac{t^2}{2}\Big|_{q_1}^{q_2} + k^2 y_2 t\Big|_{q_1}^{q_2} \\ &\quad - k\frac{t^2}{2}\Big|_{q_2}^{1} + k^2 y_3 t\Big|_{q_2}^{1} \\ &= k\frac{1-q_1}{2}(2q_2 - k\frac{q_1+2q_2+1}{2}) - k\frac{q_2^2-q_1^2}{2} \\ &\quad + k^2\frac{q_1+q_2}{2}(q_2-q_1) - k\frac{1-q_2^2}{2} + k^2\frac{q_2+1}{2}(1-q_2) \\ &= k\frac{1-q_1}{2}(2q_2 - k\frac{q_1+2q_2+1}{2}) - k\frac{1-q_1^2}{2} + k^2\frac{1-q_1^2}{2} \\ &= k\frac{1-q_1}{2}(2q_2 - k\frac{q_1+2q_2+1}{2} - (1+q_1) + k(1+q_1)) \\ &= k\frac{1-q_1}{2}(q_1(-1+\frac{k}{2}) + q_2(2-k) + \frac{k}{2} - 1) \end{aligned} \tag{26}$$

For $k = 2$, setting above eqns to 0,

$$k\frac{q_2+1}{2}(q_1(2-k) + q_2(\frac{k}{2}-1) + k(\frac{-1}{2}) + 1) = 2\frac{q_2+1}{2}(0) = 0 \tag{27}$$

$$k\frac{1-q_1}{2}(q_1(-1+\frac{k}{2}) + q_2(2-k) + \frac{k}{2} - 1) = 2\frac{1-q_1}{2}(0) = 0 \tag{28}$$

$k = 2$, $k = 0$ solution?

$$J(-1,-\frac{1}{3}) + J(-\frac{1}{3},\frac{1}{3}) + J(\frac{1}{3},1) = 0.2778 \tag{29}$$

Appendix B

$\eta_E(x,y) = (x^3-y)^2, xy \geq 0; \to \infty$ OTHERWISE, $\eta_D(x,y) = (x-y)^2$

A. $M = 3$

$$y_1 = \frac{-1+q}{2} \quad y_2 = \frac{q}{2} \quad y_3 = \frac{1}{2} \tag{30}$$

$$y_1^2 = \frac{q^2-2q+1}{4} \tag{31}$$

$$y_2 - y_1 = \frac{1}{2} \tag{32}$$

$$y_1 + y_2 = \frac{-1}{2} + q \tag{33}$$

$$\begin{aligned} J(-1,q_1) + J(q_1,q_2) + J(q_2,1) &= \frac{1}{2}(\int_{-1}^{q} (t^3-y_1)^2\,dt \\ &\quad + \int_{q}^{0} (t^3-y_2)^2\,dt + \int_{0}^{1} (t^3-y_3)^2\,dt) \end{aligned} \tag{34}$$

$$
\begin{aligned}
\frac{\partial J}{\partial q} &= (q^3-y_1)^2 + \int_{-1}^{q} 2(t^3-y_1)(\frac{-1}{2}) - (q^3-y_2)^2 \\
&\quad + \int_{q}^{0} 2(t^3-y_2)(\frac{-1}{2}) \\
&= (q^3-y_1)^2 - \frac{t^4}{4}\Big|_{-1}^{q} + y_1 t\Big|_{-1}^{q} - (q^3-y_2)^2 \\
&\quad - \frac{t^4}{4}\Big|_{q}^{0} + y_2 t\Big|_{q}^{0} \\
&= (y_2-y_1)(2q^3-y_1-y_2) - \frac{q^4-1}{4} + y_1(q+1) \\
&\quad + \frac{q^4}{4} - y_2 q \\
&= \frac{1}{2}(2q^3+\frac{1}{2}-q) + \frac{1}{4} + \frac{q^2-1}{2} - \frac{q^2}{2} \\
&= \frac{1}{2}(2q^3+\frac{1}{2}-q) + \frac{-1}{4} \\
&= (q^3-\frac{q}{2})
\end{aligned} \tag{35}
$$

Setting $\frac{\partial J}{\partial q} = 0$,

$$2q^3 - q = 0 \tag{36}$$

$$q(2q^2-1) = 0 \tag{37}$$

$$q = 0, \pm\frac{1}{\sqrt{2}}$$

Since we assume $q <= 0$,

$$q = 0, q = -\frac{1}{\sqrt{2}}$$

$$
\begin{aligned}
&J(-1,q) + J(q,0) + J(0,1) \\
&= \frac{1}{2}(\int_{-1}^{q}(t^3-y_1)^2\,dt + \int_{q}^{0}(t^3-y_2)^2\,dt + \int_{0}^{1}(t^3-y_3)^2\,dt) \\
&= \frac{1}{2}([\frac{t^7}{7} - \frac{t^4 y_1}{2} + y_1^2 t]_{-1}^{q} + [\frac{t^7}{7} - \frac{t^4 y_2}{2} + y_2^2 t]_{q}^{0} \\
&\quad + [\frac{t^7}{7} - \frac{t^4 y_3}{2} + y_3^2 t]_{0}^{1}) \\
&= \frac{1}{2}(\frac{t^7}{7}\Big|_{-1}^{1} - \frac{-1+q}{4}(q^4-1) + \frac{1+q^2-2q}{4}(q+1) \\
&\quad - (-q^4)\frac{q}{4} + \frac{q^2}{4}(-q) - \frac{1}{4} + \frac{1}{4}) \\
&= \frac{1}{2}(\frac{2}{7} + q^4(\frac{1}{4}) + q^2(-\frac{1}{4}))
\end{aligned} \tag{38}
$$

From integral function in matlab, $q = -\frac{1}{\sqrt{2}}$ is the solution (distortion = 0.1116), while $q = 0$ distortion 0.1429.

$$\frac{1}{2}(\frac{2}{7} + q^4(\frac{1}{4}) + q^2(-\frac{1}{4})) = \frac{1}{2}(\frac{2}{7} + \frac{1}{4}(\frac{1}{4}) + \frac{1}{2}(-\frac{1}{4})) = \frac{25}{224} \tag{39}$$

Appendix C

$$\eta_E(x,y)^2 = (x^3-y)^2, \quad \eta_D(x,y)^2 = (x-y)^2$$

A. $M = 1$

$$J = \int_{-1}^{1}(u^3)^2\frac{1}{2}du = \int_{-1}^{1} u^6 \frac{1}{2} du = (\frac{u^7}{14})|_{-1}^{1} = \frac{1}{7} = 0.1429 \tag{40}$$

1) ϵ perturbation:

$$
\begin{aligned}
&J([-1,-1+\epsilon,1-\epsilon,1]) \\
&= \frac{1}{2}\Big(\int_{-1}^{-1+\epsilon}(x^3-(-1+\frac{\epsilon}{2}))^2\,dx + \int_{-1+\epsilon}^{1-\epsilon} x^6\,dx \\
&\quad + \int_{1-\epsilon}^{1}(x^3-(1-\frac{\epsilon}{2}))^2\,dx\Big) \\
&= \frac{1}{2}\Big(\int_{-1}^{1} x^6 - 2\int_{-1}^{-1+\epsilon}(-1+\frac{\epsilon}{2})x^3 - 2\int_{1-\epsilon}^{1}(1-\frac{\epsilon}{2})x^3 \\
&\quad + \int_{-1}^{-1+\epsilon}(-1+\frac{\epsilon}{2})^2 + \int_{1-\epsilon}^{1}(1-\frac{\epsilon}{2})^2\Big) \\
&= \frac{1}{2}\Big(\frac{2}{7} + 2(1-\frac{\epsilon}{2})\frac{x^4}{4}\Big|_{-1}^{-1+\epsilon} - 2(1-\frac{\epsilon}{2})\frac{x^4}{4}\Big|_{1-\epsilon}^{1} \\
&\quad + (-1+\frac{\epsilon}{2})^2\epsilon + (1-\frac{\epsilon}{2})^2(\epsilon)\Big) \\
&= \frac{1}{7} + (1-\frac{\epsilon}{2})\frac{x^4}{4}\Big|_{-1}^{-1+\epsilon} - (1-\frac{\epsilon}{2})\frac{x^4}{4}\Big|_{1-\epsilon}^{1} + (1-\frac{\epsilon}{2})^2\epsilon \\
&= \frac{1}{7} - (1-\frac{\epsilon}{2})\frac{1-(-1+\epsilon)^4}{4} - (1-\frac{\epsilon}{2})\frac{1-(1-\epsilon)^4}{4} - (1-\frac{\epsilon}{2})^2\epsilon \\
&= \frac{1}{7} - 2(1-\frac{\epsilon}{2})\frac{1-(-1+\epsilon)^4}{4} - (1-\frac{\epsilon}{2})^2(\epsilon) \\
&= \frac{1}{7} - (1-\frac{\epsilon}{2})\frac{1-(-1+\epsilon)^4}{2} - (1-\frac{\epsilon}{2})^2\epsilon \\
&= \frac{1}{7} - (1-\frac{\epsilon}{2})\epsilon(1-\frac{\epsilon}{2})(2+\epsilon^2-2\epsilon) - (1-\frac{\epsilon}{2})^2\epsilon \\
&= \frac{1}{7} - (1-\frac{\epsilon}{2})^2(\epsilon(2+\epsilon^2-2\epsilon)-\epsilon) \\
&= \frac{1}{7} - (1-\frac{\epsilon}{2})^2(\epsilon^3-2\epsilon^2+\epsilon) \\
&= \frac{1}{7} - (1-\frac{\epsilon}{2})^2\epsilon(\epsilon^2-2\epsilon+1) \\
&= \frac{1}{7} - (1-\frac{\epsilon}{2})^2\epsilon(\epsilon-1)^2
\end{aligned} \tag{41}
$$

$$
\begin{aligned}
1-(-1+\epsilon)^4 &= (1-(-1+\epsilon)^2)(1+(-1+\epsilon)^2) \\
&= (1-(1+\epsilon^2-2\epsilon))(1+(1+\epsilon^2-2\epsilon)) \\
&= (-\epsilon^2+2\epsilon)(2+\epsilon^2-2\epsilon) \\
&= \epsilon(-\epsilon+2)(2+\epsilon^2-2\epsilon)
\end{aligned} \tag{42}
$$

B. $M = 2$

$$J(q) = \int_{-1}^{q}(t^3-\frac{-1+q}{2})^2\frac{1}{2}dt + \int_{q}^{1}(t^3-\frac{q+1}{2})^2\frac{1}{2}dt \tag{43}$$

$$
\begin{aligned}
\frac{\partial J}{\partial q} &= (q^3 - \frac{-1+q}{2})^2\frac{1}{2} - (q^3 - \frac{q+1}{2})^2\frac{1}{2} \\
&\quad + \int_{-1}^{q} 2(t^3 - \frac{-1+q}{2})\frac{1}{2}(\frac{-1}{2})dt \\
&\quad + \int_{q}^{1} 2(t^3 - \frac{q+1}{2})\frac{1}{2}(\frac{-1}{2})dt \\
&= \frac{1}{2}(-\frac{-1+q}{2} + \frac{q+1}{2})(2q^3 - q) + \frac{-1}{2}(\frac{t^4}{4}\big|_{-1}^{q} \\
&\quad - \frac{-1+q}{2}t_{-1}^{q} + \frac{t^4}{4}\big|_{q}^{1} - \frac{q+1}{2}t\big|_{q}^{1}) \\
&= (-\frac{-1+q}{2} + \frac{q+1}{2})(2q^3 - q) - (\frac{t^4}{4}\big|_{-1}^{q} \\
&\quad - \frac{-1+q}{2}t\big|_{-1}^{q} + \frac{t^4}{4}\big|_{q}^{1} - \frac{q+1}{2}t\big|_{q}^{1}) \\
&= (-\frac{-1+q}{2} + \frac{q+1}{2})(2q^3 - q) - \frac{t^4}{4}\big|_{-1}^{1} + \frac{-1+q}{2}t\big|_{-1}^{q} \\
&\quad + \frac{q+1}{2}t\big|_{q}^{1} \\
&= (2q^3 - q) - (\frac{1}{4} - \frac{1}{4}) + \frac{-1+q}{2}(q+1) + \frac{q+1}{2}(1-q) \\
&= (2q^3 - q) + \frac{-1+q}{2}(q+1) + \frac{q+1}{2}(1-q) \\
&= (2q^3 - q)
\end{aligned}
\tag{44}
$$

Setting $\frac{\partial J}{\partial q} = 0$,

$$
\begin{aligned}
(2q^3 - q) &= 0 \\
q(2q^2 - 1) &= 0
\end{aligned}
\tag{45}
$$

$q = 0, \pm\frac{1}{\sqrt{2}}$

$$
J(0) = \int_{-1}^{0}(t^3 - \frac{-1}{2})^2\frac{1}{2}dt + \int_{0}^{1}(t^3 - \frac{1}{2})^2\frac{1}{2}dt = 0.1429 \tag{46}
$$

$$
\int (t^3 - k)^2\frac{1}{2}dt = \int (t^6 - 2t^3k + k^2)\frac{1}{2} = \frac{1}{2}(\frac{t^7}{7} - \frac{kt^4}{2} + k^2t) \tag{47}
$$

$$
\begin{aligned}
J(q) &= \int_{-1}^{q}(t^3 - \frac{-1+q}{2})^2\frac{1}{2}dt + \int_{q}^{1}(t^3 - \frac{q+1}{2})^2\frac{1}{2}dt \\
&= \frac{1}{2}((\frac{t^7}{7} - \frac{-1+q}{2}\frac{t^4}{2} + (\frac{-1+q}{2})^2t)\big|_{-1}^{q} \\
&\quad + (\frac{t^7}{7} - \frac{q+1}{2}\frac{t^4}{2} + (\frac{q+1}{2})^2t)\big|_{q}^{1}) \\
&= \frac{1}{2}(\frac{q^7+1}{7} - \frac{-1+q}{2}\frac{q^4-1}{2} + (\frac{-1+q}{2})^2(q+1) \\
&\quad + \frac{1-q^7}{7} - \frac{q+1}{2}\frac{1-q^4}{2} + (\frac{q+1}{2})^2(1-q)) \\
&= \frac{1}{2}(\frac{2}{7} + \frac{q^4-1}{2} - \frac{q^2-1}{2})
\end{aligned}
\tag{48}
$$

$$
J(\pm\frac{1}{\sqrt{2}}) = \frac{1}{2}(\frac{2}{7} - \frac{3}{8} + \frac{1}{4}) = 0.0804 \tag{49}
$$

$$
\begin{aligned}
J(-1,0) + J(0,1) &= \frac{1}{2}(\int_{-1}^{0}(t^3 - (\frac{-1}{2}))^2\,dt + \int_{0}^{1}(t^3 - \frac{1}{2})^2\,dt) \\
&= \frac{1}{2}(\int_{-1}^{0}(t^6 + t^3 + \frac{1}{4}))^2\,dt \\
&\quad + \int_{0}^{1}(t^6 - t^3 + \frac{1}{4})^2\,dt) \\
&= \frac{1}{2}(\frac{t^7}{7}\big|_{-1}^{1} + \frac{1}{4}t\big|_{-1}^{1} + \frac{t^4}{4}\big|_{-1}^{0} - \frac{t^4}{4}\big|_{0}^{1}) \\
&= \frac{1}{2}(\frac{2}{7} + \frac{2}{4} - \frac{1}{4} - \frac{1}{4}) \\
&= \frac{1}{7} = 0.1429
\end{aligned}
\tag{50}
$$