Strategic Quantization of a Noisy Source

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***Abstract*— This paper is concerned with strategic quantization of a noisy source where the encoder, which observes the remote source through a noisy channel, and the decoder, with a distortion defined over the remote source, have misaligned objectives. Such scenarios constitute a special class of games**

I. INTRODUCTION

In this paper, we study the quantizer design problem for the setting where an encoder that observes the source through a noisy channel, and a decoder with misaligned objectives communicate over a noiseless channel.

This problem in its conventional setting of identical objectives dates back to the seminal work of Dobrushin and Lyba [1], has been well studied in the literature since, see e.g., [2]–[4]. The main result of these prior works is that one can transform the problem of indirect source coding to a direct source coding problem with a modified distortion measure defined as the original distortion conditioned over the sensing channel output.

Our problem here is closely related to a class of communication games known as "information design," also known as "Bayesian Persuasion," where agents with diverging objectives communicate as detailed in Section II.B.

The problem setting has several applications in engineering as well as Economics. For an engineering application, consider the Internet of Things, where agents with misaligned objectives communicate over channels with delay constraints. For a more concrete, real-life application, consider two smart cars by competing manufacturers, e.g., Tesla and Honda, where the Tesla (decoder) car asks for a piece of specific information, such as traffic congestion, from the Honda (encoder) to decide on changing its route or not. Say Honda's objective is to make Tesla take a specific action, e.g., to change its route, while Tesla's objective is to estimate the congestion to make the right decision accurately. Honda's objective is obviously different from that of Tesla, hence has no incentive to convey a truthful congestion estimate. However, Tesla is aware of Honda's motives while still would like to use Honda's information (if possible). With the realistic assumption that Honda would observe this information through a noisy sensing channel (i.e., a sensor), how would these cars communicate over a fixed-rate zero-delay channel? Such problems can be handled using our model. Note that here Honda has three different behavioral choices: it can choose not to communication (non-revealing strategy), can communicate exactly what the Tesla wants (fully-revealing strategy), or it can craft a message that would make Tesla to change its route. Note that Tesla can choose not to use Honda's message, if it is statistically too far from the truth. Hence, crafting an optimal message for Honda that would serve its own objective, knowing that Tesla's objective differs from it, is a formidable research challenge.

II. PRELIMINARIES

A. Notation

In this paper, random variables are denoted by capital letters, their sample values are denoted by the respective lower case letters, and their alphabets are denoted by the respective calligraphic letters. This alphabet may be finite, countably infinite, or a continuum, like an interval $[a, b] \subset \mathbb{R}$. The expectation operator is denoted by $\mathbb{E}\{\cdot\}$. The scalar Gaussian with mean m, variance σ^2 is denoted by $\mathbb{N}(m, \sigma^2)$. All logarithms are base 2.

B. Strategic Quantization Prior Work

The strategic quantization problem can be described as follows: the encoder observes a signal $X \in \mathcal{X}$, and sends a message $Z \in \mathcal{Z}$ to the decoder, upon receiving which the decoder takes the action $Y \in \mathcal{Y}$. The encoder designs the quantizer decision levels Q to minimize its objective D_E, while the decoder designs the quantizer representative levels $\mathbf{y}$ to minimize its objective D_D. Note that the objectives of the encoder and the decoder are misaligned ($D_E \neq D_D$). The strategic quantizer is a mapping $Q : \mathcal{X} \to \mathcal{Z}$, with $|\mathcal{Z}| \leq M$ for a given quantization resolution $M \in \mathbb{Z}^+$, and given distortion measures D_E, D_D.

As mentioned earlier, our problem is a variation of the Bayesian Persuasion (or information design) class of problems where an encoder and decoder with misaligned objectives communicate [5]. This class of the problems have been an active research area in Economics due to their modeling abilities of real-life scenarios, see e.g., [6]–[9].

This problem was previously studied in Economics as well as Computer Science. In [10], authors showed the existence of optimal strategic quantizers in abstract spaces. Moreover, authors provide a low-complexity method to obtain the optimal strategic quantizer. In [11], [12], authors characterize sufficient conditions for the monotonicity of the optimal strategic quantizer, and as a byproduct of their analysis, characterize its behavior (non-revealing, fully revealing, or partially revealing) for some special settings. In Computer Science, in []

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In [13], we showed that a strategic variation of the Lloyd-Max algorithm does not converge to a locally optimal solution. As a remedy we develop a gradient descent based solution for this problem. We also demonstrated that even for well-behaving sources, such as scalar Uniform, there are multiple local optima, depending on the distortion measures chosen, in sharp contrast with the classical quantization for which the local optima is unique for the case of log-concave sources (which includes Uniform sources). We also analyzed the behavior of the optimal strategic quantizer for some typical settings. The behavior can be one of the following three: i) Non-revealing: the encoder does not send any information, i.e., $Q(X) = \text{constant}$. ii) Fully revealing: the encoder effectively sends the information the decoder asks, which simplifies the problem into classical quantizer design with the decoder's objective. iii) Partially revealing: the encoder sends some information but not exactly what the decoder wants.

In [14], [15], we carried out our analysis of strategic quantization to the scenario where there is a noisy communication channel between the encoder and the decoder, using random index mapping in conjunction with gradient descent based and dynamics programming solutions respectively. In [16], we derived the globally optimal strategic quantizer via a dynamic programming based solution to resolve the poor local minima issues with gradient descent based solutions.

In Appendix I, we prove the following result, which is an extension of a result presented in [13], as well as in [11]:

Theorem 1. *For $\eta_E(u,y) = (\alpha u - y)^2$ and $\eta_D(u,y) = (u-y)^2$, the optimal strategic quantizer Q is:*

$$Q(x) = \begin{cases} \arg\min \mathbb{E}\{(X - Q(X))^2\}, & \textit{for } 0 \le \alpha \le 1/2 \\ \textit{constant}, & \textit{otherwise} \end{cases}$$

Note that the first case corresponds to the fully-revealing behavior, while the second one is non-revealing.

We refer to this theorem, later in the text, in order to demonstrate the use of our main result in this paper.

C. Remote Source Coding Prior Work

As mentioned earlier, this problem is well-studied in the classical, i.e., non-strategic, compression literature, under different names such as remote source coding, indirect rate-distortion, noisy quantization etc.

The main result, by Dobrushin and Tsybakov [1], adopted to the quantization setting as in [3] is presented as follows:

Theorem 2. *Consider the remote source coding problem where the source U is observed through a memoryless channel $P(X|U)$ by the encoder. The encoder quantizes the channel output, X, to minimize a common distortion measure $\mathbb{E}\{d(U, Q(X))\}$ subject to a rate constraint. Let Q_1 be the optimal quantizer, i.e.,*

$$Q_1 = \arg\min \mathbb{E}\{d(U, Q(X))\}$$

Let Q_2 be the optimal point-to-point quantizer for the distortion metric $d_2(C, b) = \mathbb{E}\{d(A, b)|C\}$ where $P_{U|X} = P_{A|C}$, i.e.,

$$Q_2 = \arg\min \mathbb{E}\{d_2(U, Q(U)\}$$

where d_2 is defined as above.

Then, $Q_1 = Q_2$.

D. Problem Definition

Consider the following quantization problem: an encoder observes a realization of a scalar source $U \in \mathcal{U}$ as $X \in \mathcal{X}, X = U + W$, where W is an additive noise independent of the source with a probability distribution μ_W. The joint probability distribution of the source and its noisy version is given by $\mu_{U,X}$. The encoder maps X to a message $Z \in \mathcal{Z}$, where $\mathcal{Z}$ is a set of discrete messages with a cardinality constraint $|\mathcal{Z}| \le M$ using an injective mapping, $Q : \mathcal{X} \to \mathcal{Z}$. After receiving the message Z, the decoder applies a mapping $\phi : \mathcal{Z} \to \mathcal{Y}$, where $|\mathcal{Y}| = |\mathcal{Z}|$, on the message Z and takes an action $Y = \phi(Z)$. The encoder and decoder minimize their respective objectives $D_E = \mathbb{E}\{\eta_E(U, Y)\}$ and $D_D = \mathbb{E}\{\eta_D(U, Y)\}$, which are misaligned ($\eta_E \ne \eta_D$). The encoder designs Q *ex-ante*, i.e., without the knowledge of the realization of X, using only the objectives η_E and η_D, and the statistics of the source $\mu_{U,X}(\cdot, \cdot)$. The objectives (η_E and η_D), the shared prior (μ), and the mapping (Q) are known to the encoder and the decoder. The problem is to design Q for the equilibrium, i.e., the encoder minimizes its distortion if used with a corresponding decoder that minimizes its own distortion. This communication setting is given in Figure 1.

III. MAIN RESULTS

A. Analysis

The objectives of the encoder and the decoder are given by $D_s = \mathbb{E}\{\eta_s(U, Q(X))\}, s \in \{E, D\}$. The set $\mathcal{X}$ is divided into mutually exclusive and exhaustive sets $\mathcal{V}_1, \mathcal{V}_2, \ldots, \mathcal{V}_M$. We make the following "monotonicity" assumption.

Assumption 3 (Convex code-cells)**.** *$\mathcal{V}_i$ is convex for all $i \in [1:M]$.*

Under assumption 1, $\mathcal{V}_i$ is an interval since X is a scalar, i.e.,

$$\mathcal{V}_i = [x_{i-1}, x_i).$$

The encoder chooses the quantizer Q with boundary levels $[x_0, \ldots, x_M]$. The decoder determines a set of actions $\mathbf{y} = [y_1, \ldots, y_M]$ as the best response to Q to minimize its cost D_D as

$$y_i^* = \arg\min_{y_i \in \mathcal{Y}} \sum_{i=1}^{M} \mathbb{E}\{\eta_D(u, \mathbf{y}) | x \in \mathcal{V}_i\}.$$

After observing x and θ, the encoder quantizes the source as

$$z_i = Q(x), \quad x \in \mathcal{V}_i.$$

The decoder receives the message z_i transmitted over a noiseless channel and takes the action

$$y_i = \phi(z_i).$$

Problem 1. *The source $U \in \mathcal{U}$ is corrupted by an independent additive noise $W \sim \mu_W$, and is observed by the encoder as $X = U + W$ with a joint probability distribution $\mu_{U,X}$. The encoder communicates to the decoder over a noiseless channel with rate R. The objectives of the encoder are misaligned and are given by η_E and η_D, respectively, with $\eta_E \neq \eta_D$. Find the quantizer decision levels Q, and the set of actions $\mathbf{y} = [y_1, \ldots, y_M]$ as a function of the quantizer decision levels that satisfy:*

$$q^* = \arg\min_{q} \sum_{i=1}^{M} \{\mathbb{E}\{\eta_E(x, \mathbf{y}) | x \in \mathcal{V}_i\},$$

where actions $\mathbf{y}$ are $y_i^* = \arg\min_{y_i \in \mathcal{Y}} \mathbb{E}\{\eta_D(x, \mathbf{y}) | x \in \mathcal{V}_i\} \forall i \in [1:M]$, and the rate satisfies $\log M \leq R$.

The distortions to the encoder and the decoder and the optimum decoder reconstruction for $i \in [1:M]$ are

$$D_s = \sum_{i=1}^{M} \int \int_{u \in \mathcal{U}} \eta_s(u, y_i) \mathrm{d}\mu_{U,X},$$

$$y_i = \arg\min_{y \in \mathcal{Y}} \int \int_{u \in \mathcal{U}} \eta_D(u, y_i) \mathrm{d}\mu_{U,X}.$$

The reconstruction levels $\mathbf{y}$ are found using KKT conditions,

$$\frac{\partial D_D}{\partial y_i} = \int \int_{u \in \mathcal{U}} \frac{\partial}{\partial y_i} \eta_D(u, y_i) \mathrm{d}\mu_{U,X}.$$

For $\eta_D(u, y) = (u - y)^2$, we have

$$\frac{\partial D_D}{\partial y_i} = -2 \int \int_{u \in \mathcal{U}} (u - y_i) \mathrm{d}\mu_{U,X}.$$

B. Main Result

Theorem 4. *The noisy strategic quantization problem described above, with distortions $\eta_E(u, y)$ and $\eta_D(u, y)$ is equivalent to the noiseless strategic quantization problem with a modified encoder distortion measure $\eta'_E(x, y) = \mathbb{E}\{\eta_E(u, y) | X = x\}$ for a given $P(X|U)$ observation channel.*

Proof: Consider

$$\mathbb{E}\{X\}$$

C. Example Application

As an example of Theorem 4, we consider the noisy variation of the setting presented in Theorem 1. Recall that for this setting, depending on the value of α, the behavior of strategic (noiseless) quantizer are different. We now analyze the noisy version with an additive Gaussian channel, with distortion measures $\eta_E(u, y) = (\alpha u - y)^2$ and $\eta_E(u, y) = (u - y)^2$. Our result pertaining to this example is presented in the following theorem.

Theorem 5. *Let Q_2 be the optimal strategic quantizer for the noisy setting with $\eta_E(u, y) = (\alpha u - y)^2$ and $\eta_E(u, y) = (u - y)^2$, and $X = U + N$ where $N \sim \mathbb{N}(0, \sigma^2)$, i.e,*

$$Q_2 = \arg\min_{Q} \mathbb{E}\{(\alpha U - \Phi(Q(X)))^2\}$$

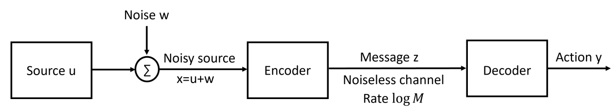


Fig. 1: Communication diagram

where the decoding function Φ minimi Let Q_1 be the optimal strategic quantizer for the noiseless setting with $\eta_E(u, y) = (\alpha u - y)^2$ and $\eta_E(u, y) = (u - y)^2$, i.e,

$$Q_1 = \arg\min \mathbb{E}\{\}$$

Then $Q_1 = Q_2$

D. Quadratic Gaussian Setting

Let $\eta_E(u, \theta, y) = (u + \theta - y)^2, \eta_D(u, y) = (u - y)^2$. All integrals are over $\mathcal{V}_{\theta,i}$, unless specified otherwise. The optimal representative levels y_i are computed by the decoder by minimizing its distortion $\mathbb{E}\{\eta_D(U, \theta, Q(X))\}$,

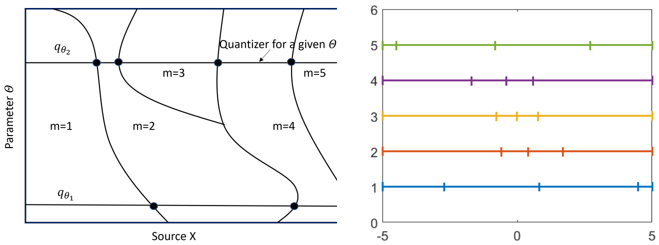
$$\mathbb{E}\{\eta_D(U, \theta, Q(X))\} = \sum_{i=1}^{M} \int_{\theta \in \mathcal{T}} \int \int_{u \in \mathcal{U}} (u - y_i)^2 \mathrm{d}\mu_{U,X,\theta},$$

$$y_i = \arg\min_{y \in \mathcal{Y}} \int_{\theta \in \mathcal{T}} \int \int_{u \in \mathcal{U}} (u - y)^2 \mathrm{d}\mu_{U,X,\theta}.$$

KKT optimality conditions imply

$$y_i = \frac{\int_{\theta \in \mathcal{T}} \int \int_{u \in \mathcal{U}} u \mathrm{d}\mu_{U,X,\theta}}{\int_{\theta \in \mathcal{T}} \int \int_{u \in \mathcal{U}} \mathrm{d}\mu_{U,\theta,X}} = \mathbb{E}\{U | X \in \mathcal{V}_{:,i}\}.$$

We show in Figure 3a that the nature of the quantizer may change with the value of θ. For the two quantizers shown, we see that the rate is $\log 5$ and $\log 3$. In Figure 3b, we show an example quantizer for θ discretized to 5 values. We rewrite the objective function D_E in terms of distortion due to the noisy source and distortion due to quantization. The term $Q(X)$ is written as Q for brevity. Note that $U - X - Q(X) - Y$, and $\theta - X - Q(X)$ forms a markov chain, with $Y = \mathbb{E}\{U|Q\}$.



(a) M=5 level quantizer (b) Example quantizer

Fig. 2: Quantization of X parameterized by θ.

$$
\begin{aligned}
D_E &= \mathbb{E}\{(U+\theta-Y)^2\} \\
&= \mathbb{E}_X\{\mathbb{E}\{(U+\theta-Y)^2|X\}\} \\
&= \mathbb{E}_X\{\mathbb{E}\{(U-\mathbb{E}\{U|X\}+\mathbb{E}\{U|X\}-\theta-Y)^2|X\}\} \\
&= \mathbb{E}_X\{\mathbb{E}\{(U-\mathbb{E}\{U|X\})^2+(\mathbb{E}\{U|X\}-\theta-Y)^2 \\
&\quad +2(U-\mathbb{E}\{U|X\})(\mathbb{E}\{U|X\}-\theta-Y)|X\}\} \\
&= \mathbb{E}\{(U-\mathbb{E}\{U|X\})^2\}+\mathbb{E}\{(\mathbb{E}\{U|X\}-\theta-Y)^2\} \\
&\quad +2\mathbb{E}_X\{\mathbb{E}\{(U-\mathbb{E}\{U|X\})(\mathbb{E}\{U|X\}-\theta-Y)|X\}\}
\end{aligned} \tag{1}
$$

The terms $\mathbb{E}\{(U-\mathbb{E}\{U|X\})\mathbb{E}\{U|X\}|X\}$ and $\mathbb{E}\{(U-\mathbb{E}\{U|X\})Y|X\}$ both vanish due to the orthogonality principle in optimal estimation. The third term,

$$\mathbb{E}\{(U-\mathbb{E}\{U|X\})\theta|X\}=\mathbb{E}\{U\theta|X\}-\mathbb{E}\{U|X\}\mathbb{E}\{\theta|X\}. \tag{2}$$

Assuming U and θ independent, we have

$$D_E=\mathbb{E}\{(U-\mathbb{E}\{U|X\})^2\}+\mathbb{E}\{(\mathbb{E}\{U|X\}-\theta-Y)^2\}. \tag{3}$$

Minimizing $D_E=\mathbb{E}\{d_1(X,\theta,Q(X))\}$ is equivalent to minimizing $D'_E=\mathbb{E}\{d_1^*(X,\theta,Y)\}$, where

$$d_1^*(X,\theta,Y)=\mathbb{E}\{(\mathbb{E}\{U|X\}+\theta-Y)^2\}, \tag{4}$$

since the other term $\mathbb{E}\{(U-\mathbb{E}\{U|X\})^2\}$ is distortion due to the noise at the source, which cannot be optimized. The encoder minimizes its equivalent distortion

$$
\begin{aligned}
D'_E &= \mathbb{E}\left\{\sum_{i=1}^{M}(\mathbb{E}\{U|X\}+\theta-y_i)^2|\mathcal{V}_i\right\} \\
&= \sum_{i=1}^{M}\int_{u\in\mathcal{U}}\int_{\theta}\int(\mathbb{E}\{U|X\}+\theta-y_i)^2 \mathrm{d}\mu_{U,X,\theta},
\end{aligned}
$$

where

$$y_i=\frac{\int_{\theta}\int\int_{u\in\mathcal{U}} u\mathrm{d}\mu_{U,X,\theta}}{\int_{\theta}\int\int_{u\in\mathcal{U}}\mathrm{d}\mu_{U,X,\theta}}=\mathbb{E}\{U|X\in\mathcal{V}_{:,i}\}.$$

E. Algorithm

The problem setting requires the encoder to choose decision levels first, following which the decoder chooses its reconstruction points. This allows a gradient descent based solution where the optimization parameter is the encoder's decision levels $\{x_{\theta,i}\}$, where at each step of the gradient computation, the encoder computes the representative levels y_i since it knows the decoder's distortion.

The gradient of the encoder's distortion with respect to the decision levels are

$$
\begin{aligned}
\frac{\partial D_E}{\partial x_{\theta',i}} &= \int_{u\in\mathcal{U}}(U+\theta'-y_i)^2\frac{\mathrm{d}\mu_{U,X,\theta}}{\mathrm{d}x}(x_{\theta',i},\theta') \\
&\quad -\int_{u\in\mathcal{U}}(U+\theta-y_{i+1})^2\frac{\mathrm{d}\mu_{U,X,\theta}}{\mathrm{d}x}(x_{\theta,i},\theta) \\
&\quad -2\int_{u\in\mathcal{U}}\int_{\theta}\int_{x_{\theta,i-1}}^{x_{\theta,i}}(U+\theta-y_i)\frac{\mathrm{d}y_i}{\mathrm{d}x_{\theta,i}}\mathrm{d}\mu_{U,X,\theta} \\
&\quad -2\int_{u\in\mathcal{U}}\int_{\theta}\int_{x_{\theta,i}}^{x_{\theta,i+1}}(U+\theta-y_{i+1})\frac{\mathrm{d}y_{i+1}}{\mathrm{d}x_{\theta,i}}\mathrm{d}\mu_{U,X,\theta},
\end{aligned}
$$

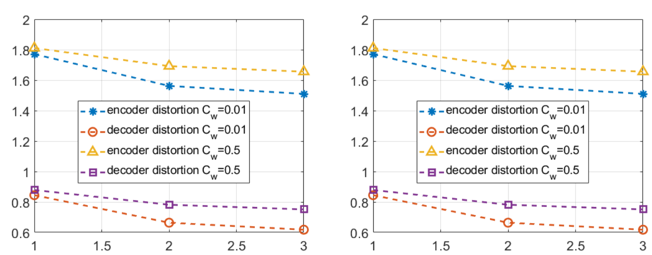
where

$$\frac{\mathrm{d}y_i}{\mathrm{d}x_{\theta,i}}=\frac{\int_{\theta}\int_{u\in\mathcal{U}}u\mathrm{d}\mu_{U,X,\theta}(u,x_{\theta,i},\theta)-y_i\int_{\theta}\int_{u\in\mathcal{U}}\mathrm{d}\mu_{U,X,\theta}}{\int_{\theta}\int_{x_{\theta,i-1}}^{x_{\theta,i}}\int_{u\in\mathcal{U}}\mathrm{d}\mu_{U,X,\theta}}$$

$$\frac{\mathrm{d}y_{i+1}}{\mathrm{d}x_{\theta,i}}=-\frac{\int_{\theta}\int_{u\in\mathcal{U}}u\mathrm{d}\mu_{U,X,\theta}(u,x_{\theta,i},\theta)-y_i\int_{\theta}\int_{u\in\mathcal{U}}\mathrm{d}\mu_{U,X,\theta}}{\int_{\theta}\int_{x_{\theta,i}}^{x_{\theta,i+1}}\int_{u\in\mathcal{U}}\mathrm{d}\mu_{U,X,\theta}}.$$

IV. NUMERICAL RESULTS

In this section, we present numerical results associated with the quadratic-Gaussian setting. Consider a 2-dimensional Gaussian source $(U,\theta), U\sim\mathcal{N}(0,1), \theta\sim\mathcal{N}(m_\theta,C_\theta)$ discretized, U and θ independent, and an independent additive noise $W\sim\mathcal{N}(0,0.01)$ and $W\sim\mathcal{N}(0,0.5)$. The encoder and decoder distortions for this setting is shown in Figure **??**.



(a) (b)

Fig. 3: Encoder and decoder distortions with varying noise variance.

V. CONCLUSIONS

Appendix I

The objective function is

$$J = \int_{\mathcal{U}} \sum_{m=1}^{M} \int_{x_{m-1}}^{x_m} (x + \alpha - \beta y_m)^2 \mathrm{d}\mu_{U,X},$$

where y_m is given by

$$y_m = \frac{\int_{\mathcal{U}} \int_{x_{m-1}}^{x_m} x \mathrm{d}\mu_X}{\int_{\mathcal{U}} \int_{x_{m-1}}^{x_m} \mathrm{d}\mu_X}$$

The derivative of the objective function with respect to quantizer decision level x_m,

$$\begin{aligned}\frac{\partial J}{\partial x_m} &= \int_{\mathcal{U}} (x_m + \alpha - \beta y_m)^2 \frac{\mathrm{d}\mu_{U,X}(u, x_m)}{\mathrm{d}x} \\ &- \int_{\mathcal{U}} (x_m + \alpha - \beta y_{m+1})^2 \frac{\mathrm{d}\mu_{U,X}(u, x_m)}{\mathrm{d}x} \\ &- 2\beta \frac{\mathrm{d}y_m}{\mathrm{d}x_m} \int_{\mathcal{U}} \int_{x_{m-1}}^{x_m} (x + \alpha - \beta y_m) \mathrm{d}\mu_{U,X} \\ &- 2\beta \frac{\mathrm{d}y_{m+1}}{\mathrm{d}x_m} \int_{\mathcal{U}} \int_{x_m}^{x_{m+1}} (x + \alpha - \beta y_{m+1}) \mathrm{d}\mu_{U,X}\end{aligned}$$

where $\frac{\mathrm{d}y_m}{\mathrm{d}x_m}$, $\frac{\mathrm{d}y_{m+1}}{\mathrm{d}x_m}$ are

$$\frac{\mathrm{d}y_m}{\mathrm{d}x_m} = \left(\int_{\mathcal{U}} x_m \frac{\mathrm{d}\mu_{U,X}(u, x_m)}{\mathrm{d}x} \int_{\mathcal{U}} \int_{x_{m-1}}^{x_m} \mathrm{d}\mu_{U,X} \right. \tag{5}$$

$$\left. - \int_{\mathcal{U}} \frac{\mathrm{d}\mu_{U,X}(u, x_m)}{\mathrm{d}x} \int_{\mathcal{U}} \int_{x_{m-1}}^{x_m} x \mathrm{d}\mu_{U,X} \right) \tag{6}$$

$$\Big/ \left(\int_{\mathcal{U}} \int_{x_{m-1}}^{x_m} \mathrm{d}\mu_{U,X} \right)^2$$

$$= \int_{\mathcal{U}} \frac{\mathrm{d}\mu_{U,X}(u, x_m)}{\mathrm{d}x} \frac{x_m - y_m}{\int_{\mathcal{U}} \int_{x_{m-1}}^{x_m} \mathrm{d}\mu_{U,X}}$$

$$\frac{\mathrm{d}y_{m+1}}{\mathrm{d}x_m} = -\left(\int_{\mathcal{U}} x_m \frac{\mathrm{d}\mu_{U,X}(u, x_m)}{\mathrm{d}x} \int_{\mathcal{U}} \int_{x_m}^{x_{m+1}} \mathrm{d}\mu_{U,X} \right. \tag{7}$$

$$\left. - \int_{\mathcal{U}} \frac{\mathrm{d}\mu_{U,X}(u, x_m)}{\mathrm{d}x} \int_{\mathcal{U}} \int_{x_m}^{x_{m+1}} x \mathrm{d}\mu_{U,X} \right) \tag{8}$$

$$\Big/ \left(\int_{\mathcal{U}} \int_{x_m}^{x_{m+1}} \mathrm{d}\mu_{U,X} \right)^2$$

$$= -\int_{\mathcal{U}} \frac{\mathrm{d}\mu_{U,X}(u, x_m)}{\mathrm{d}x} \frac{x_m - y_m}{\int_{\mathcal{U}} \int_{x_m}^{x_{m+1}} \mathrm{d}\mu_{U,X}}$$

Let $\frac{\partial J}{\partial x_m} = J_1 + J_2 + I_1 + I_2$,

$$\begin{aligned}I_1 &= -2\beta \frac{\mathrm{d}y_m}{\mathrm{d}x_m} \int_{\mathcal{U}} \int_{x_{m-1}}^{x_m} (x + \alpha - \beta y_m) \mathrm{d}\mu_{U,X} \\ &= -2\beta \int_{\mathcal{U}} \frac{\mathrm{d}\mu_{U,X}(u, x_m)}{\mathrm{d}x} \frac{x_m - y_m}{\int_{\mathcal{U}} \int_{x_{m-1}}^{x_m} \mathrm{d}\mu_{U,X}} \\ &\quad \int_{\mathcal{U}} \int_{x_{m-1}}^{x_m} (x + \alpha - \beta y_m) \mathrm{d}\mu_X \\ &= -2\beta \int_{\mathcal{U}} \frac{\mathrm{d}\mu_{U,X}(u, x_m)}{\mathrm{d}x} (x_m - y_m)((1-\beta) y_m + \alpha)\end{aligned}$$

$$\begin{aligned}I_2 &= -2\beta \frac{\mathrm{d}y_{m+1}}{\mathrm{d}x_m} \int_{\mathcal{U}} \int_{x_m}^{x_{m+1}} (x + \alpha - \beta y_{m+1}) \mathrm{d}\mu_{U,X} \\ &= 2\beta \int_{\mathcal{U}} \frac{\mathrm{d}\mu_{U,X}(u, x_m)}{\mathrm{d}x} \frac{x_m - y_{m+1}}{\int_{\mathcal{U}} \int_{x_m}^{x_{m+1}} \mathrm{d}\mu_{U,X}} \\ &\quad \int_{\mathcal{U}} \int_{x_m}^{x_{m+1}} (x + \alpha - \beta y_{m+1}) \mathrm{d}\mu_{U,X} \\ &= 2\beta \int_{\mathcal{U}} \frac{\mathrm{d}\mu_{U,X}(u, x_m)}{\mathrm{d}x} (x_m - y_{m+1})((1-\beta) y_{m+1} + \alpha)\end{aligned}$$

$$\begin{aligned}J_1 + J_2 &= (\beta(y_{m+1} - y_m))(2x_m + 2\alpha - \beta(y_m + y_{m+1})) \\ &\quad \int_{\mathcal{U}} \frac{\mathrm{d}\mu_{U,X}(u, x_m)}{\mathrm{d}x}\end{aligned}$$

$$\begin{aligned}I_1 + I_2 &= 2\beta \int_{\mathcal{U}} \frac{\mathrm{d}\mu_{U,X}(u, x_m)}{\mathrm{d}x} \Big((x_m - y_{m+1})(y_{m+1}(1-\beta) + \alpha) \\ &\quad - (x_m - y_m)(y_m(1-\beta) + \alpha) \Big) \\ &= 2\beta \int_{\mathcal{U}} \frac{\mathrm{d}\mu_{U,X}(u, x_m)}{\mathrm{d}x} (y_{m+1} - y_m)((1-\beta) x_m \\ &\quad - (1-\beta)(y_{m+1} + y_m) - \alpha)\end{aligned}$$

$$\frac{\partial J}{\partial x_m} = J_1 + J_2 + I_1 + I_2 \tag{9}$$

$$= \beta \int_{\mathcal{U}} \frac{\mathrm{d}\mu_{U,X}(u, x_m)}{\mathrm{d}x} (y_{m+1} - y_m) \Big(2(1-\beta)(x_m \tag{10}$$

$$- (y_{m+1} + y_m)) - 2\alpha + (2\alpha + x_m - \beta y_m + x_m \tag{11}$$

$$- \beta y_{m+1}) \Big)$$

$$= \beta \int_{\mathcal{U}} \frac{\mathrm{d}\mu_{U,X}(u,x_m)}{\mathrm{d}x}(y_{m+1}-y_m)\Big(x_m(2(1-\beta)+2) \tag{12}$$

$$- (y_{m+1}+y_m)(2(1-\beta)+\beta)\Big)$$

$$= \beta \int_{\mathcal{U}} \frac{\mathrm{d}\mu_{U,X}(u,x_m)}{\mathrm{d}x}(y_{m+1}-y_m)\Big(x_m(4-2\beta) \tag{13}$$

$$- (y_{m+1}+y_m)(2-\beta)\Big)$$

$$= \beta \int_{\mathcal{U}} \frac{\mathrm{d}\mu_{U,X}(u,x_m)}{\mathrm{d}x}(y_{m+1}-y_m)\Big(2x_m(2-\beta) \tag{14}$$

$$- (y_{m+1}+y_m)(2-\beta)\Big) \tag{15}$$

$$= \beta(y_{m+1}-y_m)(2-\beta)(2x_m-(y_{m+1}+y_m)) \tag{16}$$

$$\int_{\mathcal{U}} \frac{\mathrm{d}\mu_{U,X}(u,x_m)}{\mathrm{d}x}$$

If $\beta = 0, 2$, $\frac{\partial J}{\partial x_m}$ will be 0 regardless of x_m, y_m, y_{m+1} values. For $\beta \neq 0, 2$, since $(y_{m+1} - y_m) \neq 0$ (otherwise regions m and $m+1$ would not be unique),

$$(2x_m - (y_{m+1}+y_m)) = 0$$

$$x_m = \frac{y_m + y_{m+1}}{2}$$

i.e., the quantizer is the same as the non-strategic quantizer, if the encoder decides to send something.

$$J = \int_{\mathcal{U}} \sum_{m=1}^{M} \int_{x_{m-1}}^{x_m} (x+\alpha-\beta y_m)^2 \mathrm{d}\mu_{U,X}$$

$$= \sum_{m=1}^{M} \int_{x_{m-1}}^{x_m} ((x+\alpha)^2 - 2\beta(x+\alpha)y_m + \beta^2 y_m^2)\mathrm{d}\mu_X$$

$$= \sum_{m=1}^{M} \int_{x_{m-1}}^{x_m} \Big(x^2+\alpha^2+2\alpha x-2\beta x y_m-2\alpha\beta y_m+\beta^2 y_m^2\Big)\mathrm{d}\mu_X$$

$$= \sum_{m=1}^{M} \int_{x_{m-1}}^{x_m} x^2 \mathrm{d}\mu_X + \alpha^2 \sum_{m=1}^{M} \int_{x_{m-1}}^{x_m} \mathrm{d}\mu_X$$

$$+ 2\alpha \sum_{m=1}^{M} \int_{x_{m-1}}^{x_m} x\mathrm{d}\mu_X - 2\beta \sum_{m=1}^{M} y_m \int_{x_{m-1}}^{x_m} x\mathrm{d}\mu_X$$

$$- 2\alpha\beta \sum_{m=1}^{M} y_m \int_{x_{m-1}}^{x_m} \mathrm{d}\mu_X + \beta^2 \sum_{m=1}^{M} y_m^2 \int_{x_{m-1}}^{x_m} \mathrm{d}\mu_X$$

$$= \int_{a_X}^{b_X} x^2 \mathrm{d}\mu_X + \alpha^2 + 2\alpha \int_{a_X}^{b_X} x\mathrm{d}\mu_X$$

$$- 2\beta \sum_{m=1}^{M} y_m \int_{x_{m-1}}^{x_m} x\mathrm{d}\mu_X - 2\alpha\beta \sum_{m=1}^{M} \frac{\int_{x_{m-1}}^{x_m} x\mathrm{d}\mu_X}{\int_{x_{m-1}}^{x_m} \mathrm{d}\mu_X} \int_{x_{m-1}}^{x_m} \mathrm{d}\mu_X$$

$$+ \beta^2 \sum_{m=1}^{M} y_m \frac{\int_{x_{m-1}}^{x_m} x\mathrm{d}\mu_X}{\int_{x_{m-1}}^{x_m} \mathrm{d}\mu_X} \int_{x_{m-1}}^{x_m} \mathrm{d}\mu_X$$

$$= \int_{a_X}^{b_X} x^2 \mathrm{d}\mu_X + \alpha^2 + 2\alpha \int_{a_X}^{b_X} x\mathrm{d}\mu_X$$

$$- 2\beta \sum_{m=1}^{M} y_m \int_{x_{m-1}}^{x_m} x\mathrm{d}\mu_X - 2\alpha\beta \sum_{m=1}^{M} \int_{x_{m-1}}^{x_m} x\mathrm{d}\mu_X$$

$$+ \beta^2 \sum_{m=1}^{M} y_m \int_{x_{m-1}}^{x_m} x\mathrm{d}\mu_X$$

$$= \int_{a_X}^{b_X} x^2 \mathrm{d}\mu_X + \alpha^2 + 2\alpha(1-\beta) \int_{a_X}^{b_X} x\mathrm{d}\mu_X$$

$$+ \beta(\beta-2) \sum_{m=1}^{M} y_m \int_{x_{m-1}}^{x_m} x\mathrm{d}\mu_X$$

Distortion for a non-informative quantizer D_n:

$$D_n = \int_{a_X}^{b_X} (x+\alpha-\beta y)^2 \mathrm{d}\mu_X$$

$$= \int_{a_X}^{b_X} \Big(x^2+\alpha^2+2\alpha x-2\beta x y-2\alpha\beta y+\beta^2 y^2\Big)\mathrm{d}\mu_X$$

$$= \int_{a_X}^{b_X} x^2 \mathrm{d}\mu_X + \alpha^2 + 2\alpha \int_{a_X}^{b_X} x\mathrm{d}\mu_X$$

$$- 2\beta y \int_{a_X}^{b_X} x\mathrm{d}\mu_X - 2\alpha\beta y \int_{a_X}^{b_X} \mathrm{d}\mu_X + \beta^2 y^2 \int_{a_X}^{b_X} \mathrm{d}\mu_X$$

$$= \int_{a_X}^{b_X} x^2 \mathrm{d}\mu_X + \alpha^2 + 2\alpha \int_{a_X}^{b_X} x\mathrm{d}\mu_X - 2\beta y \int_{a_X}^{b_X} x\mathrm{d}\mu_X$$

$$- 2\alpha\beta \frac{\int_{a_X}^{b_X} x\mathrm{d}\mu_X}{\int_{a_X}^{b_X} \mathrm{d}\mu_X} \int_{a_X}^{b_X} \mathrm{d}\mu_X + \beta^2 y \frac{\int_{a_X}^{b_X} x\mathrm{d}\mu_X}{\int_{a_X}^{b_X} \mathrm{d}\mu_X} \int_{a_X}^{b_X} \mathrm{d}\mu_X$$

$$= \int_{a_X}^{b_X} x^2 \mathrm{d}\mu_X + \alpha^2 + 2\alpha \int_{a_X}^{b_X} x\mathrm{d}\mu_X - 2\beta y \int_{a_X}^{b_X} x\mathrm{d}\mu_X$$

$$- 2\alpha\beta \int_{a_X}^{b_X} x\mathrm{d}\mu_X + \beta^2 y \int_{a_X}^{b_X} x\mathrm{d}\mu_X$$

$$= \int_{a_X}^{b_X} x^2 \mathrm{d}\mu_X + \alpha^2 + 2\alpha(1-\beta) \int_{a_X}^{b_X} x\mathrm{d}\mu_X$$

$$+ \beta(\beta-2) y \int_{a_X}^{b_X} x\mathrm{d}\mu_X$$

$$J = \int_{a_X}^{b_X} x^2 \mathrm{d}\mu_X + \alpha^2 + 2\alpha(1-\beta) \int_{a_X}^{b_X} x\mathrm{d}\mu_X$$

$$+ \beta(\beta-2) \sum_{m=1}^{M} y_m \int_{x_{m-1}}^{x_m} x\mathrm{d}\mu_X$$

$$= D_n + \beta(\beta-2) \Bigg(\sum_{m=1}^{M} y_m \int_{x_{m-1}}^{x_m} x\mathrm{d}\mu_X - y \int_{a_X}^{b_X} x\mathrm{d}\mu_X \Bigg)$$

$$= D_n + \beta(\beta-2) \Bigg(\sum_{m=1}^{M} \frac{\int_{x_{m-1}}^{x_m} x\mathrm{d}\mu_X}{\int_{x_{m-1}}^{x_m} \mathrm{d}\mu_X} \int_{x_{m-1}}^{x_m} x\mathrm{d}\mu_X$$

$$- \frac{\sum_{m=1}^{M} \int_{x_{m-1}}^{x_m} x\mathrm{d}\mu_X}{\sum_{m=1}^{M} \int_{x_{m-1}}^{x_m} \mathrm{d}\mu_X} \sum_{m=1}^{M} \int_{x_{m-1}}^{x_m} x\mathrm{d}\mu_X \Bigg)$$

$$= D_n + \beta(\beta-2) \Bigg(\sum_{m=1}^{M} \frac{\left(\int_{x_{m-1}}^{x_m} x\mathrm{d}\mu_X \right)^2}{\int_{x_{m-1}}^{x_m} \mathrm{d}\mu_X}$$

$$- \frac{\left(\sum_{m=1}^{M} \int_{x_{m-1}}^{x_m} x\mathrm{d}\mu_X \right)^2}{\sum_{m=1}^{M} \int_{x_{m-1}}^{x_m} \mathrm{d}\mu_X} \Bigg)$$

The first term is the distortion for a non-informative quantizer ($M = 1$). In order for the quantizer to be informative, the second term has to be less than 0. This happens in three cases

1) $k < 0$ and $\xi < 0$,
2) $0 < k < 2$ and $\xi > 0$,
3) $k > 2$ and $\xi < 0$,

where $\xi = \sum_{m=1}^{M} \frac{\left(\int_{x_{m-1}}^{x_m} x\mathrm{d}\mu_X \right)^2}{\int_{x_{m-1}}^{x_m} \mathrm{d}\mu_X} - \frac{\left(\sum_{m=1}^{M} \int_{x_{m-1}}^{x_m} x\mathrm{d}\mu_x^2 \right)^2}{\sum_{m=1}^{M} \int_{x_{m-1}}^{x_m} x\mathrm{d}\mu_x}$.

From Cauchy-Shwarz inequality $| < \mathbf{u}, \mathbf{v} > | \leq ||\mathbf{u}||||\mathbf{v}||$, substituting $u_i' = u_i/\sqrt{v_i}$ and $v_i' = \sqrt{v_i}$), we have that for real numbers $u_1, u_2, \ldots, u_n$ and positive real numbers $v_1, v_2, \ldots, v_n$:

$$\frac{(\sum_{i=1}^{n} u_i)^2}{\sum_{i=1}^{n} v_i} \leq \sum_{i=1}^{n} \frac{u_i^2}{v_i}.$$

In our case, $u_i = \int_{x_{m-1}}^{x_m} x\mathrm{d}\mu_X, v_i = \int_{x_{m-1}}^{x_m} \mathrm{d}\mu_X$ with real u_i and positive real v_i.

Applying this result to our problem, we obtain $\xi \leq 0$ which implies that the only possible case is case 2 with $0 < k < 2$, and the encoder is a non-strategic quantizer and non-revealing for $k \notin [0, 2]$.

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