# Optimal Strategic Quantizer Design via Dynamic Programming

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## Abstract

This paper is concerned with the quantization setting where the encoder and the decoder have misaligned objectives. We first motivate the problem via a toy example which demonstrates the intricacies of the strategic quantization problem, specifically shows that iterative optimization of the decoder and the encoder mappings may not converge to a local optimum. As a remedy, we propose a dynamic programming based optimal optimization method, inspired by the early works in the quantization theory. We then extend our approach to variable-rate (entropy-coded) quantization. We finally present numerical results obtained via the proposed algorithms.

## Introduction

Consider the following quantization problem: An encoder observes a realization of source $X \in \mathcal{X}$ generated from a probability distribution $\mathbb{P}_X$ and maps it into a message in a discrete set $Z \in \mathcal{Z}$, via a quantizer $Q : \mathcal{X} \to \mathcal{Z}$ subject to a cardinality constraint $|\mathcal{Z}| \leq M$. The decoder generates an action (reconstruction) $Y \in \mathcal{Y}$ based on the message $Z$ it receives. The objectives of the encoder and the decoder are to minimize $D_E \triangleq \mathbb{E}\{\eta_e(X,Y)\}$ and $D_D \triangleq \mathbb{E}\{\eta_d(X,Y)\}$ respectively, where distortion functions are misaligned, i.e., $\eta_e \neq \eta_d$. The encoder designs $Q$ *ex-ante* i.e., before seeing the realization of $X$, based only on the statistics and the objectives. The distortion functions $\eta_e$ and $\eta_d$, the shared prior $\mathbb{P}_X$, and the quantizer $Q$ are common knowledge. Then, what is $Q$ at the equilibrium?

This problem, which we call *strategic quantization*, is the main subject of this paper. This game setting, without any quantization constraints, i.e., if $M$ is arbitrarily large, is known as the Bayesian Persuasion [1].

Two most relevant studies to this work are by Dughmi and Xu [2] and by Aybas and Turkel [3]. In [2] authors address the algorithmic (complexity) aspect of the problem and show that it is NP-hard in general. In [3], authors present an exhaustive search based algorithm along with several complementary theoretical results.

Here, we consider the strategic quantization problem described above, via the lens of quantization theory. We first demonstrate, via a toy example, that the standard method of optimization by iteratively enforcing necessary conditions of optimality, known as the Lloyd-Max-I method[4], may not yield a locally optimal strategic quantizer. Inspired by the early work in quantization theory [5, 6], we then develop

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dynamic programming based algorithms that yield the globally optimal solutions of this problem.

We note, in passing, that the quantizers arise as equilibrium strategies in a related but distinctly different information transmission game: the cheap talk [7]. In this setting, the encoder determines its mapping after it sees the realization, i.e., the objective of the encoder is to minimize a functional of the form $\eta_e(X, Y)$ rather than $\mathbb{E}\{\eta_e(X, Y)\}$. This difference stems from the lack of the commitment assumption in the cheap talk setting.

## Preliminaries

*Notation*

In this paper, random variables are denoted by capital letters, their sample values are denoted by the respective lower case letters, and their alphabets are denoted by the respective calligraphic letters. This alphabet may be finite, countably infinite, or a continuum, like an interval $[a, b] \subset \mathbb{R}$. The expectation operator is denoted by $\mathbb{E}\{\cdot\}$. The uniform distribution over an interval $[a, b]$ and the scalar Gaussian with mean $\mu$, variance $\sigma^2$ are denoted by $U[a, b]$ and $\mathbb{N}(\mu, \sigma^2)$.

*Strategic Quantization Problem*

We present the problem formulation in its most general form here however, we focus on the discrete distributions while designing the algorithm in the next section. The encoder and the decoder have continuous distortion functions $\eta_e(x, y)$ and $\eta_d(x, y)$ respectively that depend on the source realization $x \in \mathcal{X}$ and action taken by the decoder $y \in \mathcal{Y}$, where $\mathcal{X}$ is a compact metric space and $\mathcal{Y}$ is compact. The set of Borel probabilities over $\mathcal{X}$, a compact metric space in weak* topology, is denoted by $\Delta(\mathcal{X})$. The agents share a prior belief about $X$, $\mathbb{P}_X \in \Delta(\mathcal{X})$ which is common knowledge. A strategic (fixed-rate) quantizer is a measurable mapping $Q : \mathcal{X} \to \mathcal{Z}$ where $\mathcal{Z}$ denotes the compact metric space of messages that satisfies and $|\mathcal{Z}| \leq M$ for a given quantization resolution $M \in \mathbb{Z}^+$. Any quantizer induces a distribution $\tau$ over the messages given $\mathbb{P}_X$. A variable rate constraint is in the form of $-\int \log \tau d(\tau) \leq H_0$ instead of the fixed rate constraint $|\mathcal{Z}| \leq M$.

The timing of the game is as follows. First, the encoder designs a quantizer $Q$, based on the common knowledge, and announces to the decoder. Then, nature randomly selects a realization $x$ from $\mathcal{X}$ according to the common prior $\mu_0$. The encoder generates a message $z \in \mathcal{Z}$ through the announced quantizer and transmits to the decoder noiselessly. The decoder, upon observing $z \in \mathcal{Z}$, takes an action $r \in \mathcal{R}$. The solution concept sought after here is the encoder-preferred perfect Bayesian equilibrium.

This problem, without any constraints on the message space $|\mathcal{M}| \leq K$ is the well-known Bayesian Persuasion problem in the Economics literature [1]. In this constrained form, this problem is analyzed for discrete scalar sources in [3]. We make the following regularity assumption throughout this paper.

**Assumption:** Equilibrium quantizer consists of intervals (convex code-cells).



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This assumption is critical to the dynamic programming derivations presented in this paper. This is part of the regularity conditions in nonstrategic quantization literature (the other condition, which we do not assume here, states that the $i$-th reconstruction lies within the $i$-th interval). It is well understood that in most settings of engineering interest the optimal nonstrategic quantizer is regular, for example, for distortion measures of the form $\eta(x, y) = \rho(|x - y|)$ where $\rho$ is a nondecreasing convex function, for fixed rate and any source distribution, and for variable rate and any continuous sources, see e.g., [8]. However, due to the mismatch of distortion measures, the optimal strategic quantizer may not satisfy our assumption here. This "monotonicity" condition is studied in the recent Economics literature [9, 10] where conditions on distortion measures for monotonicity of the equilibrium mappings are characterized for the unconstrained Bayesian Persuasion setting where quantizers arise as optimal solutions without any exogenous constraint. We note that here we have the quantization in the problem formulation which imposes an exogenous constraint on the message set.

*A Toy Example*

Consider $X \sim U[-1, 1]$, $M = 3$, with $\eta_e(x, y) = (x^3 - y)^2$ and $\eta_d(x, y) = (x - y)^2$. The boundaries are parametrized as $[-1, r_1), [r_1, r_2)$, and $[r_2, 1]$ for some $r_1, r_2 \in [-1, 1]$ that satisfy $r_2 \geq r_1$. Then, the decoder reconstructions (actions) are:

$$y_1 = \frac{\int_{-1}^{r_1} \frac{1}{2} a da}{\int_{-1}^{r_1} \frac{1}{2} da} = \frac{1 + r_1}{2},\ y_2 = \frac{\int_{r_1}^{r_2} \frac{1}{2} a da}{\int_{r_1}^{r_2} \frac{1}{2} da} = \frac{r_2 + r_1}{2},\ y_3 = \frac{\int_{r_2}^{1} \frac{1}{2} a da}{\int_{r_2}^{1} \frac{1}{2} da} = \frac{r_2 + 1}{2}. \quad (1)$$

The cost function is then

$$J(r_1, r_2) = \int_{-1}^{r_1} (u^3 - \frac{1 + r_1}{2})^2 du + \int_{r_1}^{r_2} (u^3 - \frac{r_2 + r_1}{2})^2 du + \int_{r_2}^{1} (u^3 - \frac{1 + r_2}{2})^2 du \quad (2)$$

Applying the KKT optimality conditions $\frac{\partial J}{\partial r_1} = \frac{\partial J}{\partial r_2} = 0$ yields, after straightforward algebra, that the only non-degenerate solution is $r_1 = -0.7403$ and $r_2 = 0.7403$. We note that iteratively enforcing optimality conditions for the encoder and the decoder (as in Lloyd-Max algorithm) results in $r_1 \uparrow 0$ and $r_2 \downarrow 0$ since each iteration pushes the boundaries towards origin. Hence, straightforward enforcement of optimality conditions does not yield a locally optimal solution, since any perturbation of $r_1 = r_2 = 0$ would be preferred by the encoder to $r_1 = r_2 = 0$.

## Dynamic Programming Based Algorithms

Although we have formulated the problem in its most general form above, we next focus on discrete sources to develop algorithms. We note however that while the algorithms presented here are designed for discrete sources, they can be applied to continuous sources by first discretizing the source distribution at a suitable resolution.

Let $X$ be a discrete source taking values from $\mathcal{X} = \{x_1, \ldots, x_K\}, x_1 < x_2 < ... < x_K$, with a probability mass function $p_X(x_k) = p_k, k = 1, \ldots, k$. To simpify notation,



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we define an augmented set $\mathcal{O} = \{x_1, \ldots, x_K, x_{K+1}\}$, where $x_{K+1} = x_K + \epsilon$ for some $\epsilon > 0$. The set $\mathcal{X}$ is divided into non-overlapping subsets as $v_m = \{x_i | r_{m-1} \leq x_i < r_m\}$ for $m = 1, 2, \ldots, M$ where $r_m \in \mathcal{O}$ and $\cup_{m=1}^{m=M} v_m = \mathcal{X}$. The quantizer output is denoted by $\{y_m\}$:

$$y_m = Q(x) \quad \forall x \in [r_{m-1}, r_m) \quad m = 1 \ldots M. \tag{3}$$

To enable the computations in the proposed algorithm, we take $\mathcal{Y}$ (as defined in the previous section, the reconstruction space) as the discretized real line at a suitable resolution (in practice this depends on the distortion functions $\eta_e$ and $\eta_d$). The equilibrium decision levels (chosen by the encoder), the optimal representation levels (chosen by the decoder), the encoder and decoder distortions are denoted by $r_m^*$, $y_m^*$, $D_e^*$, and $D_d^*$ respectively. Similarly, we define $v_m^* = \{x_i | r_{m-1}^* \leq x_i < r_m^*\}$ for $m = 1, 2, \ldots, M$. We set $r_0^* = x_1$, and $r_M^* = x_{K+1}$. Finally, we define the set $\mathcal{S}$ as follows:

$$\mathcal{S} \triangleq \{(\alpha, \beta) : \alpha, \beta \in \mathcal{O}, x_1 \leq \alpha < \beta \leq x_{K+1}\}.$$

*Fixed Rate*

The encoder chooses $\{r_m\}$ for $m = 1, \ldots, M$ that minimize

$$D_e(r_0, r_1, \ldots, r_M) = \mathbb{E}\{\eta_e(X, Y)\} = \sum_{m=1}^{M} \sum_{k: x_k \in v_m} \eta_e(x_k, y_m) p_k, \tag{4}$$

where $y_m$ are chosen by the decoder to minimize $D_d = \mathbb{E}\{\eta_d(X, Y)\}$ as

$$y_m = \arg\min_{t \in \mathcal{Y}} \sum_{k: x_k \in v_m} \eta_d(x_k, t) p_k. \tag{5}$$

We next define

$$D_m(\alpha, \beta) \triangleq \min_{\substack{r_1, r_2, \ldots, r_{m-1} \in \mathcal{O} \\ \alpha = r_0 < r_1 < \ldots < r_{m-1} < r_m = \beta}} \sum_{i=1}^{m} \sum_{k: x_k \in v_m} \eta_e(x_k, y_i) p_k, \tag{6}$$

where $y_i$ is determined via (5). Note that $D_m$ can be written in terms of $D_1$:

$$D_m(\alpha, \beta) = \min_{\substack{r_1, r_2, \ldots, r_{m-1} \in \mathcal{O} \\ \alpha = r_0 < r_1 < \ldots < r_{m-1} < r_m = \beta}} \sum_{i=1}^{m} D_1(r_{i-1}, r_i). \tag{7}$$

The key observation here is that $D_m$ can be written as a function of $D_1$ and $D_{m-1}$ since the equilibrium $m$ level quantizer can be decomposed as $m-1$ level equilibrium (optimal for the encoder) quantizer followed by a one level quantizer. This backward induction reasoning yields the following Bellman equations:

$$D_m(r_0, \alpha) = \min_{\substack{t \in \mathcal{O} \\ r_0 < t < \alpha}} \left(D_{m-1}(r_0, t) + D_1(t, \alpha)\right) \tag{8}$$



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$$r_{m-1}(r_0,\alpha) = \arg\min_{\substack{t\in\mathcal{O}\\ r_0<t<\alpha}}[D_{m-1}(r_0,t) + D_1(t,\alpha)]. \tag{9}$$

for $m = 2,\ldots,M$. The algorithm proceeds in two passes, a forward and a backward pass. During the forward pass, for each element of $\mathcal{S}$ in the form of $(r_0^*,\alpha), \alpha \in \mathcal{O}, r_0^* < \alpha \leq r_M^*$, the set of $x_i \in [r_0^*,\alpha)$ is quantized for $m = 1,2,\ldots,M$ levels, and the $(m-1)^{th}$ decision level is given by (9), for each $m$. During the backward pass, $r_{M-1}^*$ is found by applying (9) on the interval $[r_0^*, r_M^*)$:

$$r_{M-1}^* = r_{M-1}(r_0^*, r_M^*). \tag{10}$$

This process is repeated iteratively for $[r_0^*, r_m^*), m = M,\ldots,2$ to get the decision levels $\{r_m^*\}$ as

$$r_{m-1}^* = r_{m-1}(r_0^*, r_m^*). \tag{11}$$

The optimal representative levels, $\{y_m^*\}$ are found using (5).

We present the steps above in Algorithm 1, where we define $g(\cdot)$ as the indexing function, i.e., $g(x_k) = k,\ \forall x_k \in \mathcal{O}$ and three auxiliary variables as follows:

$$\xi([\alpha,\beta),\gamma) \triangleq \sum_{k=g(\alpha)}^{g(\beta)-1} \eta_e(x_k,\gamma)p_k, \quad \zeta(\alpha,\beta) \triangleq \arg\min_{t\in\mathcal{Y}} \sum_{k=g(\alpha)}^{g(\beta)-1} \eta_d(x_k,t)p_k. \tag{12}$$

$$\phi_t(\alpha,m) \triangleq D_{m-1}(r_0,t) + D_1(t,\alpha). \tag{13}$$

*Variable Rate*

Let $p(\alpha,\beta)$ be the probability that $x$ lies in the interval $[\alpha,\beta)$,

$$p(\alpha,\beta) = \sum_{k=g(\alpha)}^{g(\beta)-1} p_k. \tag{14}$$

We define $H(\alpha,\beta)$ and the total entropy $H_T(r_0,\ldots,r_m)$ as follows:

$$H(\alpha,\beta) = -p(\alpha,\beta)\log_2(p(\alpha,\beta)), \quad H_T(r_0,r_1,\ldots,r_m) = \sum_{i=1}^{m} H(r_{i-1},r_i). \tag{15}$$

For a given $M$, the encoder minimizes $D(\lambda,M)$ over $\{r_m\}$:

$$D(\lambda,M) = \mathbb{E}\{\eta_e(X,Y)\} + \lambda H_T(r_0,...,r_M) = \sum_{m=1}^{M}\sum_{k:x_k\in v_m} \eta_e(x_k,y_m)p_k + \lambda H_T(r_0,...,r_M),$$

for a given $\lambda > 0$ (that corresponds to the entropy constraint), where $y_m$ are chosen by the decoder, similar to the fixed-rate case, as in (5). For an element $(\alpha,\beta) \in \mathcal{S}$, we now redefine the $m$-level distortion, $D_m(\alpha,\beta,\lambda)$ for the set of $x_i \in [\alpha,\beta)$ that gives



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**Algorithm 1** Fixed-Rate Strategic Quantization

Input: $\mathcal{X}, \mathcal{O}, p_X(\cdot), M, \eta_e(\cdot,\cdot), \eta_d(\cdot,\cdot)$
Output: $\{r_m^*\}$, $\{y_m^*\}$, $D_e^*$, $D_d^*$
Initialization: $r_0^* = x_1$, $r_M^* = x_{K+1}$
$\mathcal{S} \leftarrow \{(\alpha,\beta) : \alpha,\beta \in \mathcal{O}, x_1 \leq \alpha < \beta \leq x_{K+1}\}$
**for** $(\alpha,\beta) \in \mathcal{S}$ **do**
  $y \leftarrow \zeta(\alpha,\beta)$
  $D_1(\alpha,\beta) \leftarrow \xi([\alpha,\beta), y)$
**end for**
**for** $m \leftarrow 2, 3, ..., M$ **do**
  **for** $\alpha \in \mathcal{O}\backslash\{x_1\}$ **do**
    $r_{M-1}(r_0^*, \alpha) \leftarrow \arg\min_{t\in\mathcal{Y}}(\phi_t(\alpha, m))$
    $D_m(r_0^*, \alpha) \leftarrow \phi_{r_{M-1}(r_0^*,\alpha)}(\alpha, m)$
  **end for**
**end for**
**for** $m \leftarrow M, M-1, \ldots, 2$ **do**
  $r_{m-1}^* = r_{m-1}(r_0^*, r_m^*)$
**end for**
**for** $m \leftarrow 1, 2, \ldots, M$ **do**
  $y_m^* \leftarrow \zeta(r_{m-1}^*, r_m^*)$
**end for**
$D_e^* \leftarrow \sum_{m=1}^{M} \sum_{k:x_k\in v_m^*} \eta_e(x_k, y_m^*) p_k$
$D_d^* \leftarrow \sum_{m=1}^{M} \sum_{k:x_k\in v_m^*} \eta_d(x_k, y_m^*) p_k$

the distortion due to quantizing the interval $[\alpha,\beta)$ with $m$ representation level for the given $\lambda$ parameter as:

$$D_m(\alpha,\beta,\lambda) = \min_{\substack{r_1,r_2,\ldots,r_{m-1}\in\mathcal{O} \\ \alpha=r_0<r_1<\ldots<r_{m-1}<r_m=\beta}} \sum_{i=1}^{m} \sum_{k:x_k\in v_m} \eta_e(x_k, y_i) p(\alpha,\beta) + \lambda H_T(r_0, r_1, ..., r_m),$$

where $y_i$ is determined via (5). By a the same reasoning in the fixed, rate, we obtain the following Bellman equations:

$$D_m(r_0,\alpha,\lambda) = \min_{\substack{t\in\mathcal{O} \\ r_0<t<\alpha}} \left(D_{m-1}(r_0,t,\lambda) + D_1(t,\alpha,\lambda)\right) \tag{16}$$

$$r_{m-1}(r_0,\alpha,\lambda) = \arg\min_{\substack{t\in\mathcal{O} \\ r_0<t<\alpha}} [D_{m-1}(r_0,t,\lambda) + D_1(t,\alpha,\lambda)]. \tag{17}$$

for $m = 2, \ldots, M$. The algorithm again proceeds in two passes. During the forward pass, similar to the fixed-rate case, for each element of $\mathcal{S}$ in the form of $(r_0^*, \alpha), \alpha \in \mathcal{O}, r_0^* < \alpha \leq r_M^*$, the set of $x_i \in [r_0^*, \alpha)$ is quantized for $m = 1, 2, \ldots, M$ levels, and the $(m-1)^{th}$ decision level is given by (17), for each $m$. During the backward pass,



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**Algorithm 2** Variable-Rate Strategic Quantization

Input: $\mathcal{X}, \mathcal{O}, p_X(\cdot), \lambda, \eta_e(\cdot,\cdot), \eta_d(\cdot,\cdot)$
Output: $M^*$,$\{r_m^*\}$, $\{y_m^*\}$, $H_T(r_0^*, ..., r_M^*)$, $D_e^*, D_d^*$
Initialization: $r_0^* = x_1$, $r_M^* = x_{K+1}$, $th$, $M = 2$
$\mathcal{S} \leftarrow \{(\alpha, \beta) : \alpha, \beta \in \mathcal{O}, x_1 \leq \alpha < \beta \leq x_{K+1}\}$
**for** $(\alpha, \beta) \in \mathcal{S}$ **do**
  $y \leftarrow \zeta(\alpha, \beta)$
  $D_1(\alpha, \beta) \leftarrow \xi([\alpha, \beta), y) + \lambda H(\alpha, \beta)$
**end for**
**repeat**
  **for** $m \leftarrow 2, 3, ..., M$ **do**
    **for** $\alpha \in \mathcal{O} \backslash \{x_1\}$ **do**
      $r_{m-1}(r_0^*, \alpha) \leftarrow \arg\min_{t \in \mathcal{Y}}(\phi_t(\alpha, m))$
      $D_m(r_0^*, \alpha) \leftarrow \phi_{r_{m-1}(r_0^*, \alpha)}(\alpha, m)$
    **end for**
  **end for**
  **for** $m \leftarrow M, M-1, ..., 2$ **do**
    $r_{m-1}^* = r_{m-1}(r_0^*, r_m^*)$
  **end for**
  **for** $m \leftarrow 1, 2, ..., M$ **do**
    $y_m^* \leftarrow \zeta(r_{m-1}^*, r_m^*)$
  **end for**
  $H_T(r_0^*, r_1^*, ..., r_M^*) \leftarrow \sum_{m=1}^{M} H(r_{m-1}^*, r_m^*)$
  $D(\lambda, M) \leftarrow D_e(r_0^*, r_1^*, \ldots, r_M^*) + \lambda H(r_0^*, r_1^*, ..., r_M^*)$
  $M = M + 1$
**until** convergence in $D(\lambda, M)$: $\frac{D(\lambda,M) - D(\lambda,M-1)}{D(\lambda,M-1)} < th$
$M^* \leftarrow M$
$D_e^* \leftarrow \sum_{m=1}^{M^*} \sum_{k: x_k \in v_m^*} \eta_e(x_k, y_m^*) p_k$
$D_d^* \leftarrow \sum_{m=1}^{M^*} \sum_{k: x_k \in v_m^*} \eta_d(x_k, y_m^*) p_k$

$r_{M-1}^*$ is found by applying (17) on the interval $[r_0^*, r_M^*)$:

$$r_{M-1}^* = r_{M-1}(r_0^*, r_M^*). \tag{18}$$

which is performed for $[r_0^*, r_m^*), m = M, \ldots, 2$ to get the decision levels $\{r_m^*\}$ as

$$r_{m-1}^* = r_{m-1}(r_0^*, r_m^*). \tag{19}$$

Then, this process is repeated for $M = 2, 3 \ldots$ values until $D(\lambda, M)$ does not decrease (assuming $D(\lambda, M)$ is monotonic in $M$), and the corresponding $M$ is the optimal number of representative levels, $M^*$, for the given $\lambda$. The algorithm is presented in Algorithm 2.

We note that the Lagrangian approach assumes that the distortion-rate function of the optimal quantizer is convex. While this assumption does not hold most non-strategic settings (see e.g., [11] where the authors show this assumption is satisfied



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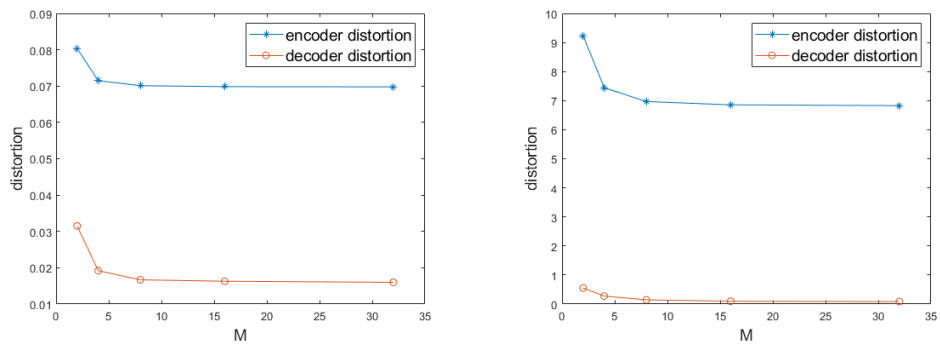
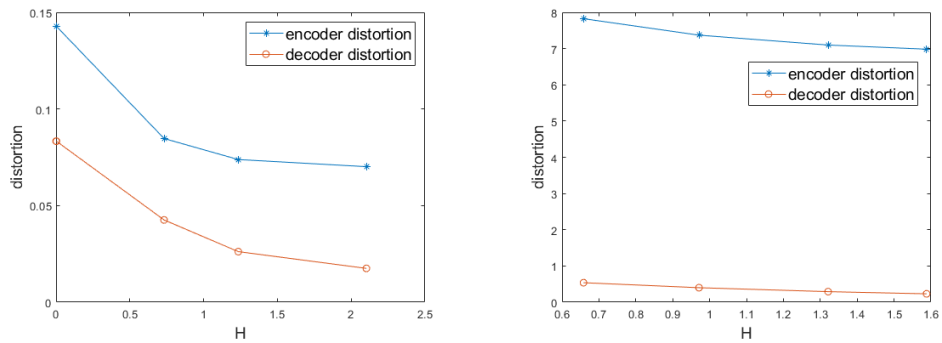



(a)$X \sim U[0,1]$ (b)$X \sim \mathbb{N}(0,1)$

Figure 1: Fixed rate quantization of a uniform and a Gaussian source, for $\eta_e(x,y) = (x^3 - y)^2$ and $\eta_d(x,y) = (x-y)^2$.




(a)$X \sim U[0,1]$ (b)$X \sim \mathbb{N}(0,1)$

Figure 2: Variable rate quantization of a uniform and a Gaussian source, for $\eta_e(x,y) = (x^3 - y)^2$ and $\eta_d(x,y) = (x-y)^2$.

only at integer rates for a uniform source), the difference is negligible in practice, see e.g., [12]. Alternatively, one can start with the Lagrangian formulation of the problem which obviously makes this "duality gap" issue vanish and makes the proposed solution optimal.

## Numerical Results

We consider Gaussian, $\mathbb{N}(0,1)$ and uniform $U[0,1]$ sources for $\eta_e(x,y) = (x^3 - y)^2$ and $\eta_d(x,y) = (x-y)^2$. The encoder and the decoder distortions for fixed and variable rate quantization are plotted in Figures 1 and 2 respectively. As expected, both distortions monotonically decrease with rate, however, unlike their nonstrategic counterparts, here they stay almost constant as rate increases in the high rate region. This is because of the mismatch between the objectives of the encoder and the decoder: even if there was no quantization at all, distortions would not vanish, see e.g., [13] for more details.



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## Discussion

In this paper, we have developed dynamic programming algorithms for the strategic quantization problem inspired by the early non-strategic quantization literature which employed dynamic programming to avoid the poor local minima issues in iterative optimization methods such as Lloyd-Max. Here, our purpose is beyond resolving the poor local optima issue as we have shown that the iterative solution may not even yield a locally optimal quantizer via simple examples. Numerical results obtained via the proposed algorithm suggest several open theoretical questions pertaining to the behavior of the operational distortion-rate curve of the optimal strategic quantizers.

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