Q1 (20 July 2021 Shift 1)

Let a be a positive real number such that $\int_0^a e^{x-[x]} dx = 10e - 9$ where $[x]$ is the greatest integer less than or equal to x. Then a is equal to :

(1) $10-\log_e(1+e)$

(2) $10+\log_e 2$

(3) $10+\log_e 3$

(4) $10+\log_e(1+e)$

Q2 (20 July 2021 Shift 1)

The value of the integral $\int_{-1}^{1} \log_e(\sqrt{1-x}+\sqrt{1+x}) dx$ is equal to :

(1) $\frac{1}{2}\log_e 2+\frac{\pi}{4}-\frac{3}{2}$

(2) $2\log_e 2+\frac{\pi}{4}-1$

(3) $\log_e 2+\frac{\pi}{2}-1$

(4) $2\log_e 2+\frac{\pi}{2}-\frac{1}{2}$

Q3 (20 July 2021 Shift 2)

If [x] denotes the greatest integer less than or equal to x, then the value of the integral $\int_{-\pi/2}^{\pi/2}[[x]-\sin x] dx$ is equal to :

(1) $-\pi$

(2) π

(3) 0

(4) 1

Q4 (20 July 2021 Shift 2)

Let $g(t)=\int_{-\pi/2}^{\pi/2} \cos\left(\frac{\pi}{4}t+f(x)\right) dx$, where $f(x)=\log_e\left(x+\sqrt{x^2+1}\right), x \in \mathbf{R}$. Then which one of the following is correct?

(1) $\mathrm{g}(1)=\mathrm{g}(0)$

(2) $\sqrt{2}\ \mathrm{g}(1)=\mathrm{g}(0)$

(3) $g(1)=\sqrt{2} g(0)$

(4) $g(1)+g(0)=0$

Q5 (22 July 2021 Shift 1)

If $\int_0^{100\pi} \frac{\sin^2 \mathrm{x}}{\mathrm{e}^{\left(\frac{\mathrm{x}}{\pi}-\left[\frac{x}{\pi}\right]\right)}} \mathrm{dx}=\frac{\alpha \pi^3}{1+4\pi^2}, \alpha \in \mathbf{R}$ where $[\mathrm{x}]$ is the

greatest integer less than or equal to x, then the value of α is:

(1) $200\left(1-\mathrm{e}^{-1}\right)$

(2) $100(1-\mathrm{e})$

(3) $50(\mathrm{e}-1)$

(4) $150\left(\mathrm{e}^{-1}-1\right)$

Q6 (25 July 2021 Shift 1)

The value of the definite integral

$\int_{x/24}^{5x/24} \frac{dx}{1+\sqrt[3]{\tan 2x}}$ is

(1) $\frac{\pi}{3}$

(2) $\frac{\pi}{6}$

(3) $\frac{\pi}{12}$

(4) $\frac{\pi}{18}$

Q7 (25 July 2021 Shift 1)

Let $f:[0, \infty) \rightarrow[0, \infty)$ be defined as

$\mathrm{f}(\mathrm{x})=\int_0^{\mathrm{x}}[\mathrm{y}] \mathrm{dy}$

where $[\mathrm{x}]$ is the greatest integer less than or equal to x. Which of the following is true?

(1) f is continuous at every point in $[0, \infty)$ and differentiable except at the integer points.

(2) f is both continuous and differentiable except at

the integer points in $[0, \infty)$.

(3) f is continuous everywhere except at the integer points in $[0, \infty)$.

(4) f is differentiable at every point in $[0, \infty)$. Official

Q8 (25 July 2021 Shift 2)

If $f(x)=\begin{cases}\int_0^x(5+|1-t|)\,\mathrm{dt}, & x>2\\ 5x+1, & x\leq 2\end{cases}$, then

(1) $\mathrm{f(x)}$ is not continuous at $\mathrm{x}=2$

(2) $\mathrm{f(x)}$ is everywhere differentiable

(3) $\mathrm{f(x)}$ is continuous but not differentiable at $\mathrm{x}=2$

(4) $\mathrm{f(x)}$ is not differentiable at $\mathrm{x}=1$

Q9 (25 July 2021 Shift 2)

The value of the integral $\int_{-1}^{1}\log\left(x+\sqrt{x^2+1}\right)dx$ is:

(1) 2

(2) 0

(3) -1

(4) 1

Q10 (27 July 2021 Shift 1)

The value of $\lim_{n\to\infty}\frac{1}{n}\sum_{j=1}^{n}\frac{(2j-1)+8n}{(2j-1)+4n}$ is equal to :

(1) $5+\log_e\left(\frac{3}{2}\right)$

(2) $2-\log_e\left(\frac{2}{3}\right)$

(3) $3+2\log_e\left(\frac{2}{3}\right)$

(4) $1+2\log_e\left(\frac{3}{2}\right)$

Q11 (27 July 2021 Shift 1)

The value of the definite integral

$\int_{\frac{\pi}{4}}^{\frac{\pi}{4}} \frac{dx}{(1+e^{x\cos x})(\sin^4 x+\cos^4 x)}$

is equal to:

(1) $-\frac{\pi}{2}$

(2) $\frac{\pi}{2\sqrt{2}}$

(3) $-\frac{\pi}{4}$

(4) $\frac{\pi}{\sqrt{2}}$

Q12 (27 July 2021 Shift 1)

Let F : $[3,5] \rightarrow \mathbf{R}$ be a twice differentiable function on $(3,5)$ such that

$F(x)=e^{-x}\int_3^x\left(3t^2+2t+4F'(t)\right)dt$

If $F'(4)=\frac{\alpha e^{\beta}-224}{(e^{\beta}-4)^2}$, then $\alpha+\beta$ is equal to

Q13 (27 July 2021 Shift 2)

If $\int_0^{\pi}\left(\sin^3 x\right)e^{-\sin^2 x}dx=\alpha-\frac{\beta}{e}\int_0^1\sqrt{t}e^t dt$, then $\alpha+\beta$

is equal to

Answer Key

Q1 (2)	**Q2** (2)	**Q3** (1)	**Q4** (2)
Q5 (1)	**Q6** (3)	**Q7** (1)	**Q8** (3)
Q9 (2)	**Q10** (4)	**Q11** (2)	**Q12** (16)
Q13 (5)			

Q1

$a > 0$

Let $n \leq a < n+1, n \in W$

$$\therefore a = \underset{\text{G.I.F}}{\underset{\Downarrow}{[a]}} + \underset{\text{Fractional part}}{\underset{\Downarrow}{\{a\}}}$$

Here $[a] = n$

Now, $\int_0^a e^{x-[x]} dx = 10e - 9$

$\Rightarrow \int_0^n e^{(x)} dx + \int_n^a e^{x-[x]} dx = 10e - 9$

$\therefore \quad n \int_0^1 e^x dx + \int_n^a e^{x-n} dx = 10e - 9$

$\Rightarrow n(e-1) + (e^{a-n} - 1) = 10e - 9$

$\therefore \quad n = 0 \text{ and } \{a\} = \log_e 2$

So, $a = [a] + \{a\} = (10 + \log_e 2)$

$\Rightarrow$ Option (2) is correct.

Q2

Let $I = 2\int_0^1 \underbrace{\ln(\sqrt{1-x} + \sqrt{1+x})}_{(I)} \underbrace{1}_{(II)} dx$

(I.B.P.)

$\therefore I = 2\left[(x \cdot \ln(\sqrt{1-x} + \sqrt{1-x}))_0^1\right.$

$\left.- \int_0^1 x \cdot \left(\frac{1}{\sqrt{1-x}+\sqrt{1+x}}\right) \cdot \left(\frac{1}{2\sqrt{1+x}} - \frac{1}{2\sqrt{1-x}}\right) dx\right]$

$= 2(\ln\sqrt{2} - 0) - \frac{2}{2}\int_0^1 \frac{x\sqrt{1-x}-\sqrt{1+x}dx}{(\sqrt{1-x}+\sqrt{1+x})\sqrt{1-x^2}}$

$= (\log_e 2) - \int_0^1 \frac{x\cdot\left(2-2\sqrt{1-x^2}\right)}{-2x\sqrt{1-x^2}} dx$

(After rationalisation)

$= (\log_e 2) + \int_0^1 \left(\frac{1-\sqrt{1-x^2}}{\sqrt{1-x^2}}\right) dx$

$= (\log_e 2) + \left(\sin^{-1} x\right)_0^1 - 1$

$= \log_e 2 + \left(\frac{\pi}{2} - 0\right) - 1$

$\therefore \mathrm{I}=(\log_e 2)+\frac{\pi}{2}-1$

$\Rightarrow$ Option (3) is correct.

Q3

$I=\int_{\frac{-\pi}{2}}^{\frac{\pi}{2}}([x]+[-\sin x])dx \ldots .(1)$

$I=\int_{\frac{-\pi}{2}}^{\frac{\pi}{2}}([-x]+[\sin x])dx \ldots \ldots(2)$

(King property) $2\mathrm{I}=\int_{\frac{-\pi}{2}}^{\frac{\pi}{2}}(\underbrace{[\mathrm{x}]+[-\mathrm{x}]}_{-1})+(\underbrace{[\sin \mathrm{x}]+[-\sin \mathrm{x}]}_{-1})\mathrm{dx}$

$2\mathrm{I}=\int_{\frac{-\pi}{2}}^{\frac{\pi}{2}}(-2)\mathrm{dx}=-2(\pi)$

$\mathrm{I}=-\pi$

Q4

$g(t)=\int_{-\pi/2}^{\pi/2}\left(\cos\frac{\pi}{4}t+f(x)\right)dx$

$g(t)=\pi\cos\frac{\pi}{4}t+\int_{-\pi/2}^{\pi/2}f(x)dx$

$g(t)=\pi\cos\frac{\pi}{4}t$

$g(1)=\frac{\pi}{\sqrt{2}}, g(0)=\pi$

Q5

$I=\int_0^{100\pi}\frac{\sin^2 x}{e^{(x/\pi)}}dx=100\int_0^{\pi}\frac{\sin^2 x}{e^{x/\pi}}dx$

$100\int_0^{\pi}e^{-x/\pi}\frac{(1-\cos 2x)}{2}dx$

$=50\left\{\int_0^{\pi}e^{-x/\pi}dx-\int_0^{\pi}e^{-x/\pi}\cos 2x dx\right\}$

$I_1=\int_0^{\pi}e^{-x/\pi}dx=\left[-\pi e^{-x/\pi}\right]_0^{\pi}=\pi\left(1-e^{-1}\right)$

$I_2=\int_0^{\pi}e^{-x/\pi}\cos 2x dx$

$=-\pi\mathrm{e}^{-x/\pi}\cos 2\mathrm{x}]_0^{\pi}-\int-\pi\mathrm{e}^{-x/\pi}(-2\sin 2\mathrm{x})\mathrm{dx}$

$=\pi\left(1-\mathrm{e}^{-1}\right)-2\pi\int_0^{\pi}\mathrm{e}^{-x/\pi}\sin 2\mathrm{xdx}$

$\pi\left(1-\mathrm{e}^{-1}\right)-2\pi\left\{-\pi\mathrm{e}^{-x/\pi}\sin 2\mathrm{x}]_0^{\pi}-\int_0^{\pi}-\pi\mathrm{e}^{-x/\pi}2\cos 2\mathrm{xdx}\right\}$

$=\pi\left(1-\mathrm{e}^{-1}\right)-4\pi^2\mathrm{I}_2$

$\Rightarrow \mathrm{I}_2=\frac{\pi\left(1-\mathrm{e}^{-1}\right)}{1+4\pi^2}$

$\therefore \mathrm{I}=50\left\{\pi\left(1-\mathrm{e}^{-1}\right)-\frac{\pi\left(1-\mathrm{e}^{-1}\right)}{1+4 \pi^2}\right\}$

$=\frac{200\left(1-\mathrm{e}^{-1}\right) \pi^3}{1+4 \pi^2}$

Q6

Let $I=\int_{\pi / 24}^{5 \pi / 24} \frac{(\cos 2 x)^{1 / 3}}{(\cos 2 x)^{1 / 3}+(\sin 2 x)^{1 / 3}} d x$ ....(i)

$\Rightarrow I=\int_{\pi / 2*}^{\operatorname{san} 4} \frac{\left(\cos \left\{2\left(\frac{\pi}{4}-x\right)\right\}\right)^{\frac{1}{3}}}{\left(\cos \left\{2\left(\frac{\pi}{4}-x\right)\right\}\right)^{\frac{1}{3}}+\left(\sin \left\{2\left(\frac{\pi}{4}-x\right)\right\}\right)^{\frac{1}{3}}} d x$

$\left\{\int_a^b f(x) d x=\int_a^b f(a+b-x) d x\right\}$

So $\mathrm{I}=\int_{\pi / 24}^{5 \pi / 24} \frac{(\sin 2 \mathrm{x})^{1 / 3}}{(\sin 2 \mathrm{x})^{1 / 3}+(\cos 2 \mathrm{x})^{1 / 3}} \mathrm{dx}$

Hence $2 \mathrm{I}=\int_{\pi / R 4}^{5 \pi / 24} \mathrm{dx} \quad[(\mathrm{i})+(\mathrm{ii})]$

$\Rightarrow 2 \mathrm{I}=\frac{4 \pi}{24} \Rightarrow \mathrm{I}=\frac{\pi}{12}$

Q7

$f:[0, \infty) \rightarrow[0, \infty), f(\mathrm{x})=\int_0^{\mathrm{x}}[\mathrm{y}] \mathrm{dy}$

Let $x=n+f, f \in(0,1)$

So $f(x)=0+1+2+\ldots+(n-1)+\int_n^{n+f} n d y$

$f(\mathrm{x})=\frac{\mathrm{n}(\mathrm{n}-1)}{2}+\mathrm{n} f$

$=\frac{[\mathrm{x}]([\mathrm{x}]-1)}{2}+[\mathrm{x}]\{\mathrm{x}\}$

Note $\lim _{x \rightarrow a^{+}} f(x)=\frac{n(n-1)}{2}, \lim _{x \rightarrow \infty} f(x)=\frac{(n-1)(n-2)}{2}+(n-1)$

$=\frac{\mathrm{n}(\mathrm{n}-1)}{2}$

$f(\mathrm{x})=\frac{\mathrm{n}(\mathrm{n}-1)}{2}\left(\mathrm{n} \in \mathrm{N}_0\right)$

so $f(\mathrm{x})$ is cont. $\forall \mathrm{x} \geq 0$ and diff. except at integer

points

Q8

$f(x)=\int_0^1(5+(1-t)) d t+\int_1^x(5+(t-1)) d$

$=6-\frac{1}{2}+\left.\left(4 t+\frac{t^2}{2}\right)\right|_1^x$

$=\frac{11}{2}+4 \mathrm{x}+\frac{\mathrm{x}^2}{2}-4-\frac{1}{2}$

$= \frac{x^2}{2} + 4x + 1$

$f\left(2^{+}\right) = 2 + 8 + 1 = 11$

$f(2) = f(2) = 5 \times 2 + 1 = 11$

$\Rightarrow$ continuous at $x = 2$

Clearly differentiable at $x = 1$ $Lf'(2) = 5$

$Rf'(2) = 6$

$\Rightarrow$ not differentiable at $x = 2$

Q9

Let $I = \int_{-1}^{1} \log\left(x + \sqrt{x^2 + 1}\right) dx$

$\because \log\left(x + \sqrt{x^2 + 1}\right)$ is an odd function

$\therefore I = 0$

Q10

$\lim_{n \to \infty} \frac{1}{n} \sum_{j=1}^{n} \frac{\left(\frac{2j}{n} - \frac{1}{n} + 8\right)}{\left(\frac{2j}{n} - \frac{1}{n} + 4\right)}$

$\int_0^1 \frac{2x+8}{2x+4} dx = \int_0^1 \; dx + \int_0^1 \frac{4}{2x+4} dx$

$= 1 + 4\frac{1}{2}(\ln|2x + 4|)\Big|_0^1$

$= 1 + 2\ln\left(\frac{3}{2}\right)$

Q11

$I = \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \frac{dx}{\left(1+e^{x \cos x}\right)\left(\sin^4 x + \cos^4 x\right)} \quad \ldots(1)$

Using $\int_{=}^{b} f(x) dx = \int_{=}^{b} f(a + b - x) dx$

$I = \int_{-x/4}^{\pi/4} \frac{dx}{\left(1+e^{-x \cos x}\right)\left(\sin^4 x + \cos^4 x\right)}$

Add (1) and (2)

$2\mathrm{I} \; = \int_{-\frac{x}{4}}^{\frac{\pi}{4}} \frac{dx}{\sin^4 x+\cos^4 x}$

$2\mathrm{I} \; = 2\int_0^{\frac{\pi}{4}} \frac{dx}{\sin^4 x+\cos^4 x}$

$$I = \int_0^{\frac{\pi}{4}} \frac{\left(1+\tan^2 x\right)\sec^2 x}{\tan^4 x+1} dx$$

$$I = \int_0^{\frac{\pi}{4}} \frac{\left(1+\frac{1}{\tan^2 x}\right)\sec^2 x}{} \left(\tan x - \frac{1}{\tan x}\right)^2 + 2dx$$

$\tan x - \frac{1}{\tan x} = t$

$\left(1+\frac{1}{\tan^2 x}\right)\sec^2 x dx = dt$

$I = \int_{-\infty}^{0} \frac{dt}{t^2+2} = \left[\frac{1}{\sqrt{2}}\tan^{-1}\left(\frac{t}{\sqrt{2}}\right)\right]_{-\infty}^{0}$

Q12

$\mathrm{F}(3) = 0$

$\mathrm{e}^{\mathrm{x}}\mathrm{F}(\mathrm{x}) = \int_3^{\mathrm{x}} \left(3\mathrm{t}^2 + 2\mathrm{t} + 4\,\mathrm{F}'(\mathrm{t})\right)\mathrm{dt}$

$\mathrm{e}^{\mathrm{x}}\mathrm{F}(\mathrm{x}) + \mathrm{e}^{\mathrm{x}}\mathrm{F}'(\mathrm{x}) = 3\mathrm{x}^2 + 2\mathrm{x} + 4\,\mathrm{F}'(\mathrm{x})$

$(\mathrm{e}^{\mathrm{x}} - 4)\frac{\mathrm{dy}}{\mathrm{dx}} + \mathrm{e}^{\mathrm{x}}\mathrm{y} = \left(3\mathrm{x}^2 + 2\mathrm{x}\right)$

$\frac{\mathrm{dy}}{\mathrm{dx}} + \frac{\mathrm{e}^{\mathrm{x}}}{(\mathrm{e}^{\mathrm{x}}-4)}\mathrm{y} = \frac{(3\mathrm{x}^2+2\mathrm{x})}{(\mathrm{e}^{\mathrm{x}}-4)}$

$\mathrm{ye}^{\int \frac{\mathrm{e}^x}{(\mathrm{e}^x-4)\mathrm{dx}}} = \int \frac{(3\mathrm{x}^2+2\mathrm{x})}{(\mathrm{e}^{\mathrm{x}}-4)} \mathrm{e}^{\int \frac{\mathrm{e}^x}{\mathrm{e}^x-4}\mathrm{dx}}\mathrm{dx}$

$\mathrm{y}\cdot(\mathrm{e}^{\mathrm{x}} - 4) = \int \left(3\mathrm{x}^2 + 2\mathrm{x}\right)\mathrm{dx} + \mathrm{c}$

$\mathrm{y}\,(\mathrm{e}^{\mathrm{x}} - 4) = \mathrm{x}^3 + \mathrm{x}^2 + \mathrm{c}$

Put $\mathrm{x} = 3 \Rightarrow \mathrm{c} = -36$

$\mathrm{F}(\mathrm{x}) = \frac{(\mathrm{x}^3+\mathrm{x}^2-36)}{(\mathrm{e}^{\mathrm{x}}-4)}$

$\mathrm{F}'(\mathrm{x}) = \frac{(3\mathrm{x}^2+2\mathrm{x})(\mathrm{e}^{\mathrm{x}}-4)-(\mathrm{x}^3+\mathrm{x}^2-36)\mathrm{e}^{\mathrm{x}}}{(\mathrm{e}^{\mathrm{x}}-4)^2}$

$\mathrm{F}'(4) = \frac{56(\mathrm{e}^4-4)-4\mathrm{ne}^{\mathrm{n}}}{(\mathrm{e}^4-4)^2}$

$= \frac{12\mathrm{e}^4-22\mathrm{y}}{(\mathrm{e}^{\mathrm{y}}-4)^2} \Rightarrow \alpha = 12$

$\beta = 4$

$\alpha + \beta = 16$

Q13

$\mathrm{I}=2 \int_0^{\pi / 2} \sin ^3 \mathrm{x} \mathrm{e}^{-\sin ^2 \mathrm{x}} \mathrm{dx}$

$=2 \int_0^{\pi / 2} \sin x \mathrm{e}^{-\sin ^2 x} \mathrm{dx}+\int_0^{\pi / 2} \cos x \underbrace{\underbrace{\mathrm{e}^{-\sin ^2 x}(-\sin 2 \mathrm{x})}_{\text{in}} \mathrm{dx}}$

$=2 \int_0^{\pi / 2} \sin \mathrm{xe}^{-\sin ^2 \mathrm{x}} \mathrm{dx}+\left[\cos \mathrm{xe}^{-\sin ^2 \mathrm{x}}\right]_0^{\pi / 2}$

$+\int_0^{\pi / 2} \sin \mathrm{xe}^{-\sin ^2 x} \mathrm{dx}$

$=3 \int_0^{\pi / 2} \sin \mathrm{xe}^{-\sin ^2 \mathrm{x}} \mathrm{dx}-1$

$=\frac{3}{2} \int_{-1}^0 \frac{\mathrm{e}^\alpha \mathrm{d} \alpha}{\sqrt{1+\alpha}}-1 \quad\left(\text { Put }-\sin ^2 \mathrm{x}=\mathrm{t}\right)$

$=\frac{3}{2 \mathrm{e}} \int_0^1 \frac{\mathrm{e}^{\mathrm{x}}}{\sqrt{\mathrm{x}}} \mathrm{dx}-1(\text { put } 1+\alpha=\mathrm{x})$

$=\frac{3}{2 \mathrm{e}} \int_0^1 \mathrm{e}_{\mathrm{u}}^{\mathrm{x}} \frac{1}{\sqrt{\mathrm{x}}} \mathrm{dx}-1$

$=2-\frac{3}{\mathrm{e}} \int_0^1 \mathrm{e}^{\mathrm{x}} \sqrt{\mathrm{x}} \mathrm{dx}$

Hence, $\alpha+\beta=5$