Part a

```python
A = np.array([[3, 2, 0, -1], [-1, 3, 0, 2], [0, -4, -2, 7]])
b = np.array([[3], [1], [-4]])
```
[2] ✓ 0.0s

```python
def gradient_descent(x0, a, y, tol=1e-3, max_iters=10000):
    x = x0.copy()
    history = [(0, x.copy())]
    for k in range(1, max_iters+1):
        g = df(x, y)
        if np.linalg.norm(g) < tol:
            break
        x = x - a * g
        history.append((k, x.copy()))
    return history

def df(x, y):
    return A.T @ (A @ x - b) + y * x
```
[3] ✓ 0.0s

```python
a = 0.01
y = 2
x = np.array([[1], [1], [1], [1]])
```
[4] ✓ 0.0s

```python
h = gradient_descent(x, a, y)

# print first 5
print("First 5 iterates:")
for k, xk in h[:5]:
    print(f"k={k}, x={np.round(xk, 4)}")

# print last 5
print("\nLast 5 iterates:")
for k, xk in h[-5:]:
    print(f"k={k}, x={np.round(xk, 4)}")

print(f"\nConverged in {h[-1][0]} iterations; "
      f"‖∇f‖≈{np.linalg.norm(df(h[-1][1], y)):.4e}")
```
[5] ✓ 0.0s

```
First 5 iterates:
k=0, x=[[1]
 [1]
 [1]
 [1]]
k=1, x=[[0.98]
 [1.07]
 [1.08]
 [0.58]]
k=2, x=[[0.9393]
 [1.0117]
 [1.0908]
 [0.4222]]
k=3, x=[[0.8973]
 [0.934 ]
 [1.0835]
 [0.3383]]
k=4, x=[[0.8586]
 [0.862 ]
 [1.0712]
 [0.2795]]
```

```
Last 5 iterates:
k=383, x=[[ 0.4302]
 [ 0.5845]
 [ 0.0479]
 [-0.217 ]]
k=384, x=[[ 0.4302]
 [ 0.5845]
 [ 0.0479]
 [-0.217 ]]
k=385, x=[[ 0.4302]
 [ 0.5845]
 [ 0.0479]
 [-0.217 ]]
k=386, x=[[ 0.4302]
 [ 0.5845]
 [ 0.0479]
 [-0.217 ]]
k=387, x=[[ 0.4301]
 [ 0.5845]
 [ 0.0479]
 [-0.217 ]]

Converged in 387 iterations; ‖∇f‖≈9.8030e-04
```

Part b

(b.)

$$(y-\beta)^T(y-\beta) = \sum_{j=1}^{p}(y_j-\beta_j)^2$$

$$L(\beta) = \frac{1}{2p}\|y-\beta\|_2^2 + \lambda\|W\beta\|_2^2$$

$$\nabla L(\beta) = \frac{1}{p}(\beta - y) + 2\lambda W^TW\beta$$

Let $\nabla L(\beta) = 0$

$$0 = \frac{1}{p}(\beta - y) + 2\lambda W^TW\beta$$

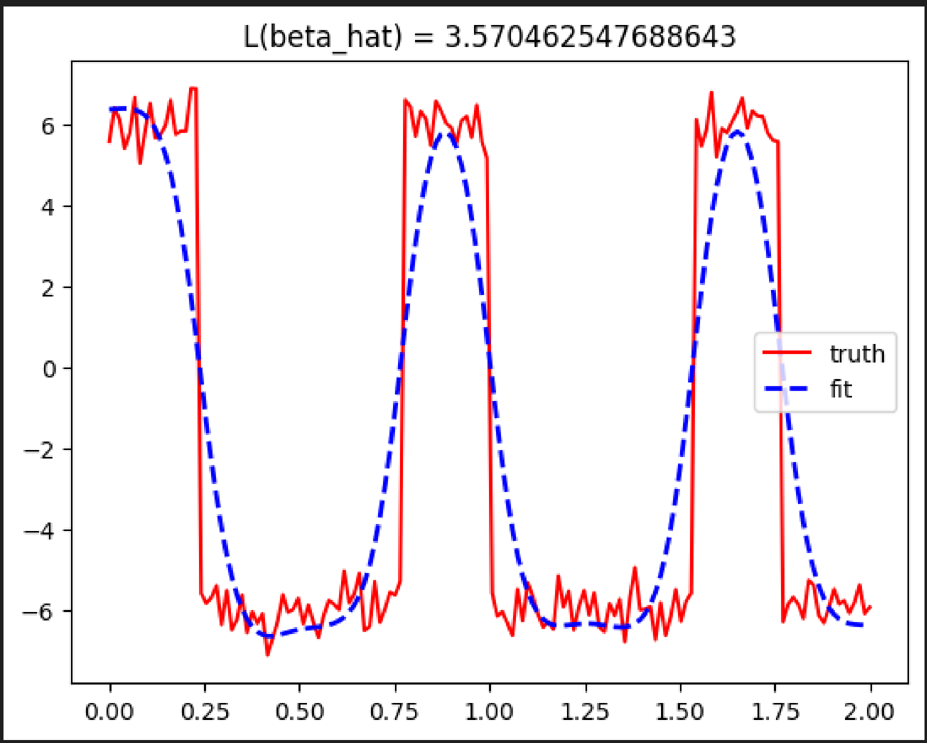
$$\frac{1}{p}y = \frac{1}{p}\beta + 2\lambda W^TW\beta$$

$$\frac{1}{p}y = \left(\frac{1}{p}I + 2\lambda W^TW\right)\beta$$

$$\hat{\beta} = \frac{1}{p}y\left(\frac{1}{p}I + 2\lambda W^TW\right)^{-1}$$

```python
p = len(y_var)

def create_W(p):
    W = np.zeros((p-2, p))
    b = np.array([1,-2,1])
    for i in range(p-2):
        W[i,i:i+3] = b
    return W

def loss(beta, y, W, lam):
    term1 = (1/(2*p)) * np.linalg.norm(y - beta)**2
    term2 = lam * np.linalg.norm(W @ beta)**2
    loss_val = term1 + term2
    return loss_val

lam = 0.9
W = create_W(p)
I = np.eye(p)

beta_hat = np.linalg.solve(I + 2*lam*p*(W.T @ W), y_var)

Lval = loss(beta_hat, y_var, W, lam)

plt.plot(t_var, y_var, zorder=1, color='red', label='truth')
plt.plot(t_var, beta_hat, zorder=3, color='blue',
linewidth=2, linestyle='--', label='fit')
plt.legend(loc='best')
plt.title(f"L(beta_hat) = {loss(beta_hat, y_var, W, lam)}")
plt.show()
```

✓ 0.0s



Part c

(c.)

$$L(\beta) = \frac{1}{2p}\|y-\beta\|_2^2 + \lambda\|W\beta\|_2^2$$

$$\frac{1}{2p}\|y-\beta\|_2^2 = \frac{1}{2p}(y-\beta)^T(y-\beta) = \frac{1}{2p}\sum_{j=1}^{p}(y_j-\beta_j)^2$$

- mean squared error → makes sure that large deviations between β and y are penalised. This is minimised to make β follow the actual data accurately.

$$\lambda\|W\beta\|_2^2 = \lambda(W\beta)^T(W\beta) = \lambda\sum_{i=1}^{p-2}(\beta_i - 2\beta_{i+1} + \beta_{i+2})^2$$

- second difference → penalises quick or abrupt changes in the slope. minimising this increases regularity and smoothness.

Part d

(d.) $L_j(\beta) = \frac{1}{2}(y_j - \beta_j)^2 + \lambda \|W\beta\|_2^2$

$$\sum_{j=1}^{p} L_j(\beta) = \sum_{j=1}^{p} \frac{1}{2}(y_j - \beta_j)^2 + \sum_{j=1}^{p} \lambda \|W\beta\|_2^2$$

$$= \frac{1}{2} \sum_{j=1}^{p} (y_j - \beta_j)^2 + p\lambda \|W\beta\|_2^2$$

divide by p

$$\frac{1}{p} \sum_{j=1}^{p} L_j(\beta) = \frac{1}{2p} \sum_{j=1}^{p} (y_j - \beta_j)^2 + \lambda \|W\beta\|_2^2$$

$$= L(\beta)$$

as required.

Compute $\nabla L_j(\beta)$

$$\nabla L_j(\beta) = \frac{\partial}{\partial \beta_k} \frac{1}{2}(y_j - \beta_j)^2 + \nabla_\beta\left(\lambda \|W\beta\|_2^2\right)$$

if $k = j$

$$f_{\beta_k} = -(y_j - \beta_j)$$

if $k \neq j$
since we diff by β_j,

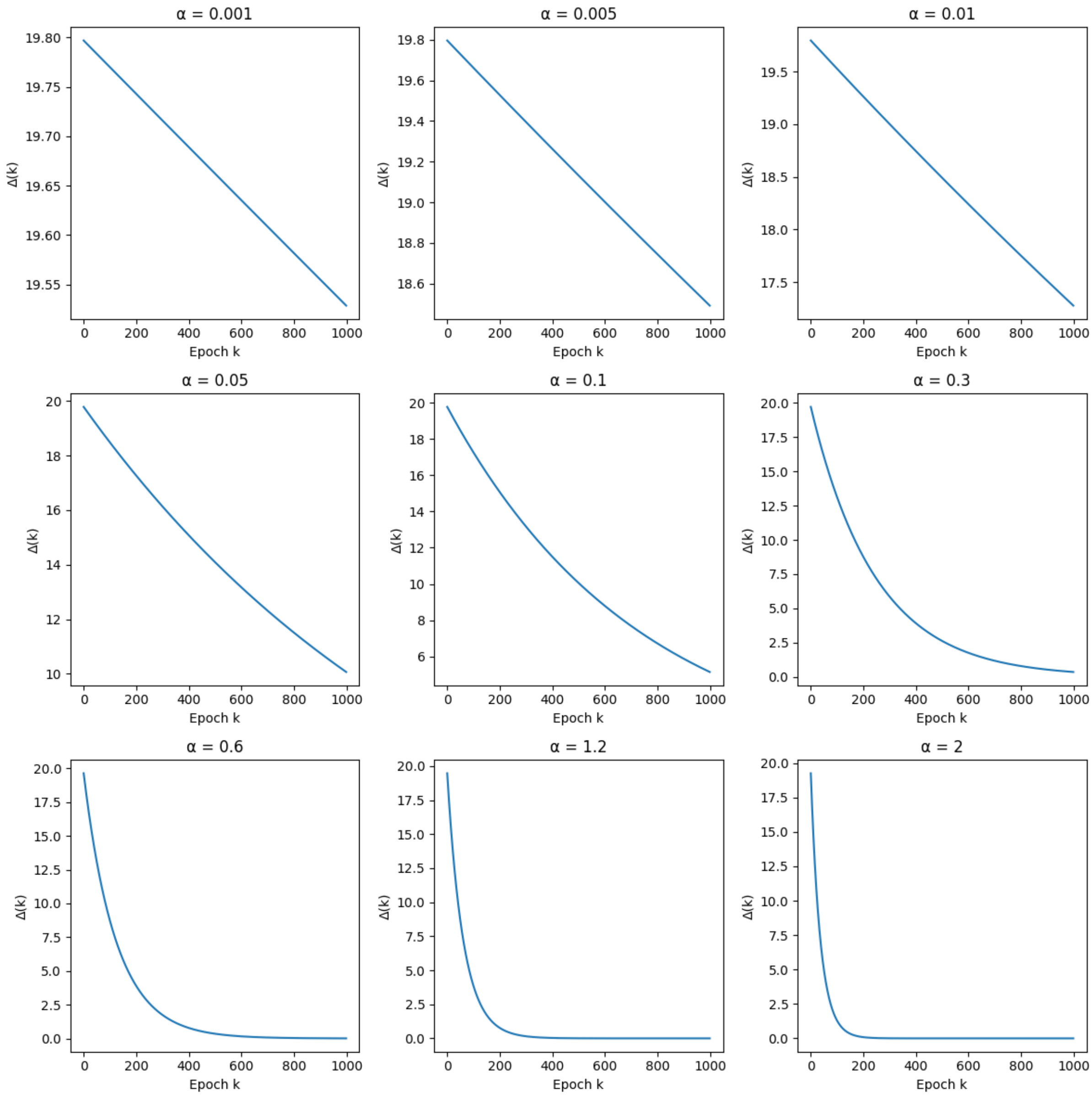
$$f_{\beta_k} = 0$$

$2\lambda W^T W \beta$

$$\therefore \nabla L_j(\beta) = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ -(y_j - \beta_j) \\ 0 \\ \vdots \\ 0 \end{pmatrix} + 2\lambda W^T W \beta$$

Part e

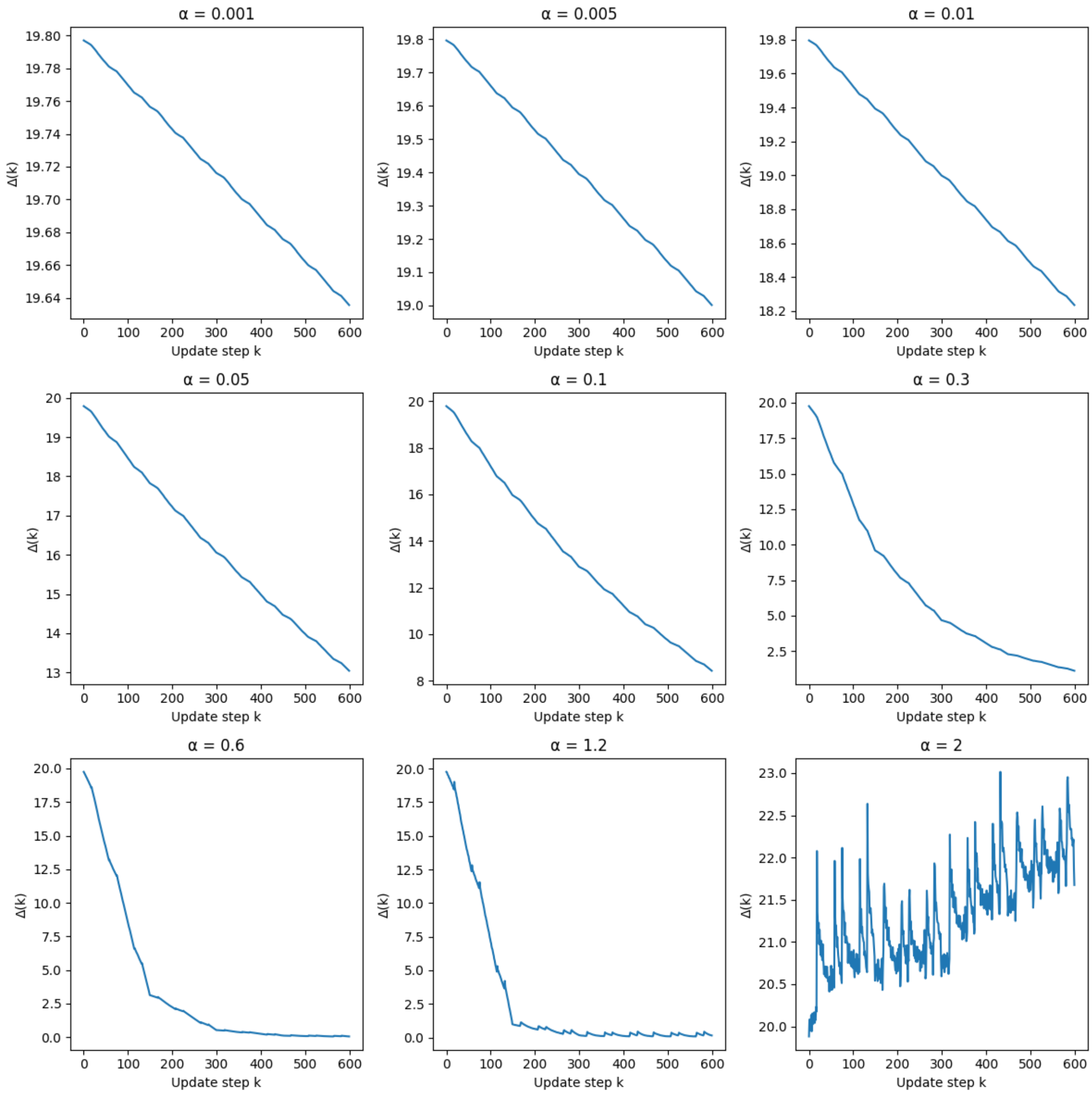
```python
lam = 0.001
beta_hat = np.linalg.solve(np.eye(p) + 2*lam*p*(W.T @ W), y_var)

def loss(beta):
    return 1/(2*p)*np.linalg.norm(y_var - beta)**2 + lam * np.linalg.norm(W @ beta)**2

def grad(beta):
    return (beta - y_var)/p + 2*lam*(W.T @ (W @ beta))

alphas = [0.001, 0.005, 0.01, 0.05, 0.1, 0.3, 0.6, 1.2, 2]

deltas = []
for alpha in alphas:
    beta = np.ones(p)
    delta_k = []
    for k in range(1, 1001):
        beta = beta - alpha * grad(beta)
        delta_k.append(loss(beta) - loss(beta_hat))
    deltas.append(delta_k)

fig, axes = plt.subplots(3, 3, figsize=(12, 12))
for idx, ax in enumerate(axes.flatten()):
    ax.plot(deltas[idx])
    ax.set_title(f'α = {alphas[idx]}')
    ax.set_xlabel('Epoch k')
    ax.set_ylabel('Δ(k)')
plt.tight_layout()
plt.show()
```



based on the plots, alpha = 2 is the best.

Part f

```python
def grad_j(beta, j):
    g = np.zeros_like(beta)
    g[j] = -(y_var[j] - beta[j])
    g += 2*lam*(W.T @ (W @ beta))
    return g

alphas = [0.001, 0.005, 0.01, 0.05, 0.1, 0.3, 0.6, 1.2, 2]
deltas_sgd = []

for alpha in alphas:
    beta = np.ones(p)
    delta_history = []
    total_updates = 4 * p
    for k in range(total_updates):
        j = k % p
        g = grad_j(beta, j)
        beta = beta - alpha * g
        delta_history.append(loss(beta) - loss(beta_hat))
    deltas_sgd.append(delta_history)

fig, axes = plt.subplots(3, 3, figsize=(12, 12))
for idx, ax in enumerate(axes.flatten()):
    ax.plot(deltas_sgd[idx])
    ax.set_title(f'α = {alphas[idx]}')
    ax.set_xlabel('Update step k')
    ax.set_ylabel('Δ(k)')
plt.tight_layout()
plt.show()
```



best is alpha = 1.2. SGD only uses one data point's gradient, so sometimes that estimate will make delta k go up.

Part g

(g.) $$L(\beta) = \frac{1}{2p}\sum_{h=1}^{p}(y_h - \beta_h)^2 + \lambda\sum_{i=1}^{p-2}(\beta_i - 2\beta_{i+1} + \beta_{i+2})^2$$

$$\frac{\partial L}{\partial \beta_j} = \frac{\beta_j - y_j}{p} + 2(W^T W\beta)_j$$

$$0 = \frac{\beta_j - y_j}{p} + 2(W^T W\beta)_j$$

$$\beta_j = \frac{\frac{y_j}{p} - 2\lambda\sum_{n\neq j}(W^T W)_{jn}\beta_n}{\frac{1}{p} + 2\lambda(W^T W)_{jj}}$$

$$(W^T W)_{jj} = \begin{cases} 1 & j=1 \text{ or } j=p \\ 5 & j=2 \text{ or } j=p-1 \\ 6 & 3\leq j\leq p-2 \end{cases}$$

$$(W^T W)_{j,j+1} = \begin{cases} -2 & \{j, j+1\} = \{1,2\} \text{ or } \{p-1, p\} \\ -4 & \text{otherwise.} \end{cases}$$

So for $j=1$

$$\hat{\beta}_1 = \frac{\frac{y_1}{p} + 4\lambda\beta_2}{\frac{1}{p} + 2\lambda}$$

$j = 2$

$$\hat{\beta}_2 = \frac{\frac{y_2}{p} + 4\lambda\beta_1 + 8\lambda\beta_3 - 2\lambda\beta_4}{\frac{1}{p} + 10\lambda}$$

$3 \leq j \leq p-2$

$$\hat{\beta}_j = \frac{\frac{y_j}{p} + 8\lambda(\beta_{j-1} + \beta_{j+1}) - 2\lambda(\beta_{j-2} + \beta_{j+2})}{\frac{1}{p} + 12\lambda}$$

$j = p-1$

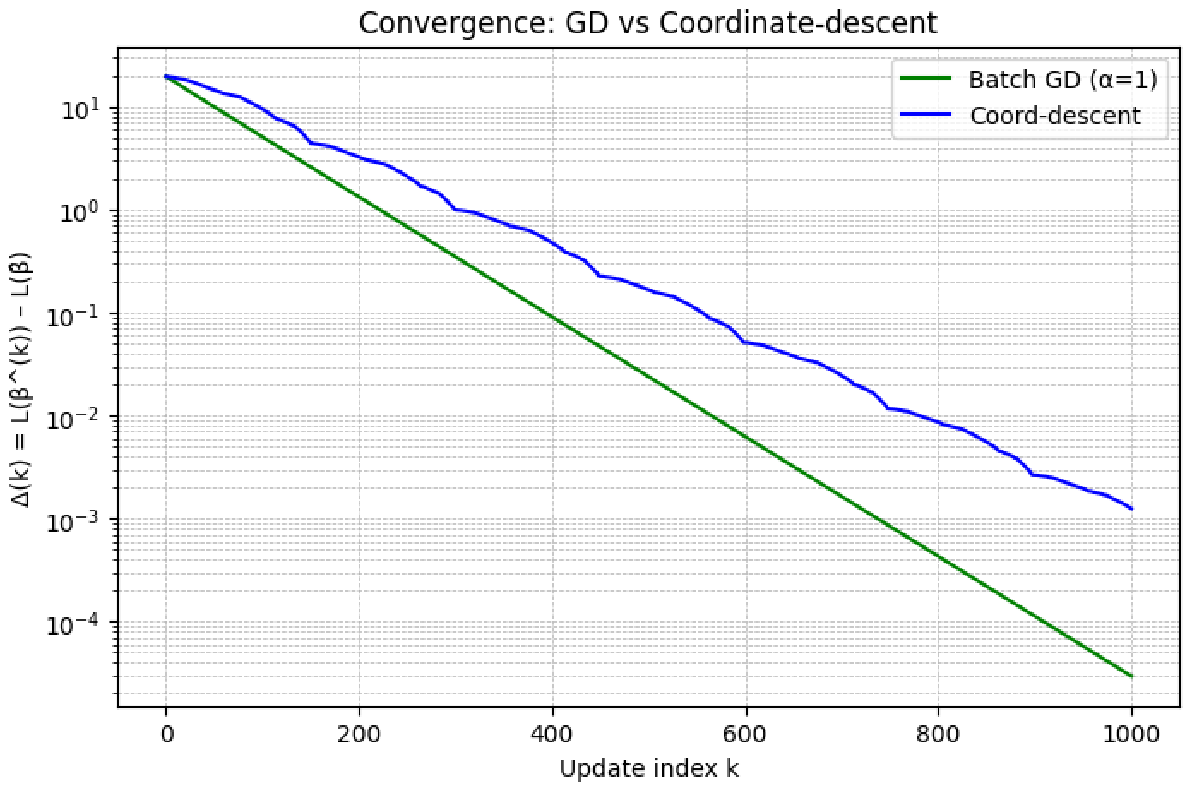
$$\hat{\beta}_{p-1} = \frac{\frac{y_{p-1}}{p} + 8\lambda\beta_{p-2} + 4\lambda\beta_p - 2\lambda\beta_{p-3}}{\frac{1}{p} + 10\lambda}$$

$j = p$

$$\hat{\beta}_p = \frac{\frac{y_p}{p} + 4\lambda\beta_{p-1}}{\frac{1}{p} + 2\lambda}$$

Part h

```python
M = W.T @ W

alpha = 1.0
beta_gd = np.ones(p)
delta_gd = [loss(beta_gd) - loss(beta_hat)]

for epoch in range(1, 1001):
    beta_gd = beta_gd - alpha * grad(beta_gd)
    delta_gd.append(loss(beta_gd) - loss(beta_hat))

beta_cd = np.ones(p)
delta_cd = [loss(beta_cd) - loss(beta_hat)]

for k in range(1, 1001):
    j = (k-1) % p
    row = M[j]
    numer = y_var[j]/p - 2*lam*(row @ beta_cd - row[j]*beta_cd[j])
    denom = 1/p + 2*lam*row[j]
    beta_cd[j] = numer/denom
    delta_cd.append(loss(beta_cd) - loss(beta_hat))

plt.figure(figsize=(8,5))
plt.plot(delta_gd, label="Batch GD (α=1)",  color="green")
plt.plot(delta_cd, label="Coord-descent", color="blue")
plt.xlabel("Update index k")
plt.ylabel("Δ(k) = L(β^(k)) − L(β̂)")
plt.title("Convergence: GD vs Coordinate-descent")
plt.legend()
plt.yscale("log")
plt.grid(True, which="both", ls="--", lw=0.5)
plt.show()
```



Part i

if you can afford to do GD (O(p^2) cost), then it will be faster. coordinate descent costs O(p) so with larger p, it would be better than GD.