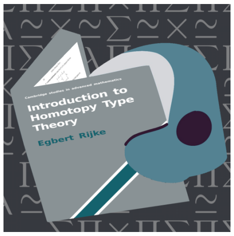

Worksheet 12

HoTTEST Summer School 2022

The HoTTEST TAs , and

19 August 2022

1 ($\star$)

For each of the following types, state how many unique[1] terms there are of that type, and list them. *Hint:* First figure out how many there are before defining them. Use Lemma 17.5.8 and think inductively!

(a)

$$\binom{\mathsf{Fin}\ 3}{\mathsf{Fin}\ 2}$$

(b)

$$\binom{\mathsf{Fin}\ n}{\mathbb{1}}$$

(c)

$$\binom{\mathsf{Fin}\ n}{\mathsf{Fin}\ n + \mathbb{1}}$$

2 ($\star\star$)

Consider a type A.

(a) We call a point $a : A$ *isolated* if the map $\mathsf{const}_a : \mathbb{1} \to A$ is a decidable embedding. Construct an equivalence

$$\binom{A}{\mathbb{1}} \simeq \sum_{a:A} \mathsf{is\text{-}isolated}(a).$$

(b) Show that if A is a set, then $\binom{A}{\mathbb{1}} \simeq A$

(c) Construct an equivalence

$$\binom{A}{\mathbb{1}} \simeq \left(\sum_{X:\mathcal{U}} (X + \mathbb{1}) \simeq A \right)$$

conclude that the map $X \mapsto X + \mathbb{1}$ on a univalent universe $\mathcal{U}$ is 0-truncated.

[1] Unique up to identity – assume univalence and function extensionality.

(d) More generally, construct an equivalence

$$\binom{A}{B} \simeq \sum_{X:\mathcal{U}_B} \sum_{Y:\mathcal{U}} X + Y \simeq A$$

3 (⋆⋆)

Given a type A, the type of **unordered pairs** in A is defined to be

$$\text{unordered-pairs}(A) := \sum_{X:BS_2} X \to A$$

(a) Construct an embedding

$$\binom{A}{\text{bool}} \hookrightarrow \text{unordered-pairs}(A)$$

Why does unordered-pairs(A) have, in general, more elements than $\binom{A}{\text{bool}}$? Which elements of unordered-pairs(A) are *not* in the image of this embedding?

(b) The type of homotopy commutative binary operations from A to B is defined as

$$\text{unordered-pairs}(A) \to B.$$

Show that if B is a set, then this type is equivalent to the type

$$\sum\nolimits_{m:A\to A\to B} \prod\nolimits_{x,y:A} m(x,y) = m(y,x).$$

(c) Show that the type of undirected graphs in $\mathcal{U}$ with at most one edge between any two vertices

$$\sum_{V:\mathcal{U}} (\text{unordered-pairs}(V) \to \text{Prop})$$

is equivalent to the type

$$\sum\nolimits_{V:\mathcal{U}} \sum\nolimits_{E:V\to V\to \text{Prop}} \prod\nolimits_{x,y:V} E(x,y) \to E(y,x).$$

4 (⋆⋆)

Consider the following claim.

$$\binom{\text{Fin}(n)}{\text{bool}} \simeq \sum_{k:\text{Fin}(n)} \text{Fin}(k) \qquad (*)$$

Prove (*) by induction on n, using the equivalences from Lemma 17.5.8 and the identities proved above. You should not need to unfold the definition of $\binom{A}{B}$ or explicitly construct any decidable embeddings.

5 $(\star\star)$

Given a finite type A, show that the following are equivalent:

(i) The type of all decidable equivalence relations on A

(ii) The type of all surjective maps $A \to X$ into a finite type X

(iii) The type of finite types X equipped with a family $Y : X \to \mathsf{Fin}$ of finite types, such that each $Y(x)$ is inhabited and equipped with an equivalence

$$e : \left(\sum_{x:X} Y(x) \right) \simeq A$$

1. The type of all decidable partitions of A, i.e. the type of all $P : (A \to \mathsf{dProp}) \to \mathsf{dProp}$ such that each Q in P is inhabited, and such that for each $x : A$ the type of $Q : A \to \mathsf{dProp}$ such that $Q(x)$ holds and $P(Q)$ holds is contractible.