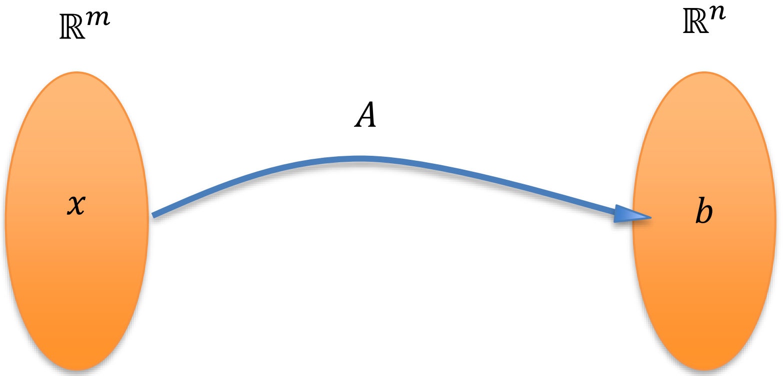
MATLAB Tutorial 09

ENME 303 Computational Methods for Engineers

Parham Oveissi

When Does a Solution Exist?!

$$\underset{\mathbb{R}^{n\times m}}{A}\,\underset{\mathbb{R}^{m}}{x} = \underset{\mathbb{R}^{n}}{b}$$

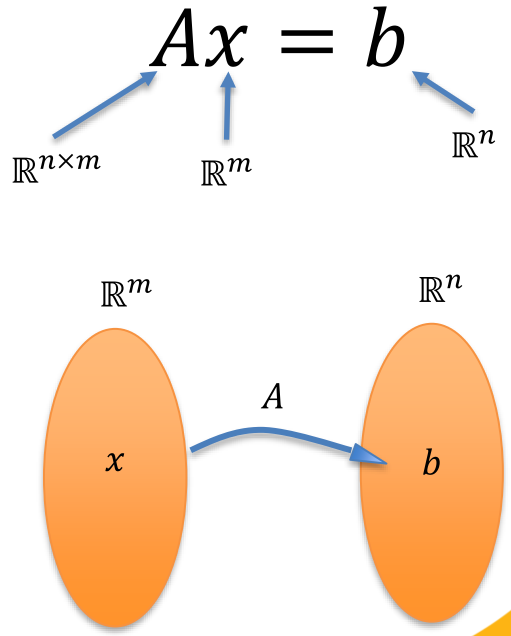


When Does a Solution Exist?!

- If b is not in $\mathcal{R}(A)$, then there is **no solution**.

- If $\mathcal{N}(A) \neq \{0\}$, then you can't have a unique solution (there are **infinite solutions**).
    1. $\mathrm{x_n} \in \mathcal{N}(\mathrm{A})$, then $\mathrm{Ax_n} = 0$
    2. Suppose there is a solution x_s s.t. $Ax_s = b$
    3. Then $A(x_s + x_n) = b$ is also a solution!
    4. Since there are infinite elements in $\mathcal{N}(\mathrm{A})$, there are infinite solutions!

- If $\mathcal{N}(A) = \{0\}$, then there exists a **unique solution**.



Existence and Uniqueness of a Solution

$$Ax = b$$

$\mathbb{R}^{n\times m}$ $\quad$ $\mathbb{R}^{m}$ $\quad$ $\mathbb{R}^{n}$

For $m = n$
(Square Matrices)

	$b \in \mathcal{R}(A)$	$b \notin \mathcal{R}(A)$
$\mathcal{N}(A) = \{0\}$	$x = A^{-1}b$	Can't happen $\mathcal{R}(A) = \mathbb{R}^n$
$\mathcal{N}(A) \neq \{0\}$	Infinitely many solutions	No solution exists

Existence and Uniqueness of a Solution

$$Ax = b$$

$\mathbb{R}^{n\times m}$ $\mathbb{R}^{m}$ $\mathbb{R}^{n}$

For $m < n$
(Tall Matrices)

$$\begin{bmatrix}1 & 0\\ 0 & 1\\ 0 & 0\end{bmatrix}\begin{bmatrix}x_1\\ x_2\end{bmatrix} = \begin{bmatrix}b_1\\ b_2\\ b_3\end{bmatrix}$$

$\mathbb{R}^{3\times 2}$ $\mathbb{R}^{2}$ $\mathbb{R}^{3}$

	$b \in \mathcal{R}(A)$	$b \notin \mathcal{R}(A)$
$\mathcal{N}(A) = \{0\}$	$x = (A^T A)^{-1} A^T b$	No solution exists
$\mathcal{N}(A) \neq \{0\}$	Infinitely many solutions	No solution exists

Existence and Uniqueness of a Solution

$$Ax = b$$

$\mathbb{R}^{n\times m}$ $\mathbb{R}^m$ $\mathbb{R}^n$

For $m > n$
(Wide Matrices)

$$\underbrace{\begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}}_{\mathbb{R}^{2\times 3}} \underbrace{\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}}_{\mathbb{R}^3} = \underbrace{\begin{bmatrix} b_1 \\ b_2 \end{bmatrix}}_{\mathbb{R}^2}$$

	$b \in \mathcal{R}(A)$	$b \notin \mathcal{R}(A)$
$\mathcal{N}(A) = \{0\}$	Can't happen	Can't happen
$\mathcal{N}(A) \neq \{0\}$	Infinitely many solutions	No solution exists

Practice 01

- Consider the following system of linear equations.

$$\begin{cases} 9x + 5y + 8z = 3 \\ 7x + 8y + 5z = 8 \\ 9x + 4y + 0z = 4 \end{cases}$$

1. Write this system of equations in the $Ax = b$ form. Provide A and b.
2. Does a solution exist?
3. If a solutions exists, is it unique or not?
4. If the solution is unique find the unique solution and if not find a solution to the given system of equations.

Practice 02

- Consider the following system of linear equations.

$$\begin{cases} 9x + 5y + 4z = 52 \\ 7x + 8y - 1z = 61 \\ 9x + 4y + 5z = 47 \end{cases}$$

1. Write this system of equations in the $Ax = b$ form. Provide A and b.
2. Does a solution exist?
3. If a solutions exists, is it unique or not?
4. If the solution is unique find the unique solution and if not find a solution to the given system of equations.

Least Squares Solution

- Let $Ax = y$ be a system equations where $A \in \mathbb{R}^{m \times n}$ and $m > n$ (tall/skinny). A least squares solution of $Ax = y$ is a solution $\hat{x}$ in $\mathbb{R}^m$ such that:

$$\underbrace{\min}_{\hat{x}} ||A\hat{x} - b||_2^2$$

- One can find the least square solution by: (Proof)

$$x = (A^T A)^{-1} A^T b$$

- The term $(A^T A)^{-1} A^T$ is a left inverse (pseudo-inverse) of A.

Practice 03

- Consider the following system of linear equations.

$$\begin{cases} -5w + 3x - 5y - 8z = 1 \\ 10x + 6x + 11y + 2z = 24 \end{cases}$$

1. Write this system of equations in the $Ax = b$ form. Provide A and b.
2. Does a solution exist?
3. If a solutions exists, is it unique or not?
4. If the solution is unique find the unique solution and if not find a solution to the given system of equations.

UMBC

Least/Minimum Norm Solution

- Let $Ax = y$ be a system equations where $A \in \mathbb{R}^{m \times n}$ and $m < n$ (wide/fat). A least/minimum solution of $Ax = y$ is a solution x in $\mathbb{R}^m$ which minimizes $||x||$ among all the solutions (infinite number of solutions).
- One can find the least/minimum solution by: (Proof)

$$x = A^T(AA^T)^{-1}b$$

- The term $A^T(AA^T)^{-1}$ is a right inverse (pseudo-inverse) of A.

Thanks!