MATLAB Tutorial 06

ENME 303 Computational Methods for Engineers

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Gauss Elimination Method

The Gauss elimination method can be used to solve system of linear equations. In this procedure, a system of equations is given in a general form and is manipulated to be in **Upper Triangular form**, which is then solved by using **Back Substitution**.

To perform row reduction on a matrix, one uses a sequence of elementary row operations to modify the matrix until the lower left-hand corner of the matrix is filled with zeros, as much as possible. There are three types of elementary row operations:

- Swapping two rows
- Multiplying a row by a nonzero number
- Adding a multiple of one row to another row

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Gauss Elimination Method

- System of linear equations:

$$\begin{cases} a_{11}x_1 + a_{12}x_2 + a_{31}x_3 = b_1 \\ a_{21}x_1 + a_{22}x_2 + a_{32}x_3 = b_2 \\ a_{31}x_1 + a_{32}x_2 + a_{33}x_3 = b_3 \end{cases}$$

- Matrix form $(AX = B)$:

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$$

- Augmented Matrix:

$$\left[\begin{array}{ccc|c} a_{11} & a_{12} & a_{13} & b_1 \\ a_{21} & a_{22} & a_{23} & b_2 \\ a_{31} & a_{32} & a_{33} & b_3 \end{array}\right]$$

Row Operations

- Upper Triangular form

$$\left[\begin{array}{ccc|c} a_{11} & a_{12} & a_{13} & b_1 \\ 0 & a_{22}' & a_{23}' & b_2' \\ 0 & 0 & a_{33}'' & b_3'' \end{array}\right]$$

Upper Triangular form

- Step 1: Keep the 1st equation (1st row) unchanged and eliminate the terms that include the first variable x_1 in all the other equations using row operations.
- Step 2: Keep the 1st and 2nd equations (1st and 2nd rows) unchanged and eliminate the terms that include the first two variable x_1, x_2 in all the other equations using row operations.

$$\vdots$$

- Step m: Keep the first m equations unchanged and eliminate the terms that include the variable x_m in the last equation using row operations.

Not that $m = n - 1$ and n is the number of the equations/variables.

Example

- System of linear equations:

$$\begin{cases} 9x_1 + 8x_2 + 9x_3 + 2x_4 = 42 \\ 5x_1 + 2x_2 + 7x_3 + 3x_4 = 45 \\ 6x_1 + 4x_2 + 3x_3 + 6x_4 = 53 \\ 8x_1 + 2x_2 + 5x_3 + 6x_4 = 63 \end{cases}$$

- Matrix form $(AX = B)$:

$$\begin{bmatrix} 9 & 8 & 9 & 2 \\ 5 & 2 & 7 & 3 \\ 6 & 4 & 3 & 6 \\ 8 & 2 & 5 & 6 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 42 \\ 45 \\ 63 \\ 53 \end{bmatrix}$$

- Augmented Matrix:

$$\left[\begin{array}{cccc|c} 9 & 8 & 9 & 2 & 42 \\ 5 & 2 & 7 & 3 & 45 \\ 6 & 4 & 3 & 6 & 63 \\ 8 & 2 & 5 & 6 & 53 \end{array}\right]$$

Gauss Elimination

- Step 1: Keep the 1st equation (1st row) unchanged and eliminate the terms that include the first variable x_1 in all the other equations using row operations.

Row2 = Row2 - 5/9*Row1
Row3 = Row3 - 6/9*Row1
Row4 = Row4 - 8/9*Row1

$$Ab := \mathrm{stack}\left(Ab^{\langle 1\rangle}, Ab^{\langle 2\rangle} - \frac{Ab_{2,1}}{Ab_{1,1}}\cdot Ab^{\langle 1\rangle}, Ab^{\langle 3\rangle} - \frac{Ab_{3,1}}{Ab_{1,1}}\cdot Ab^{\langle 1\rangle}, Ab^{\langle 4\rangle} - \frac{Ab_{4,1}}{Ab_{1,1}}\cdot Ab^{\langle 1\rangle}\right) = \begin{bmatrix} 9 & 8 & 9 & 2 & 42 \\ 0 & -2.444 & 2 & 1.889 & 21.667 \\ 0 & -1.333 & -3 & 4.667 & 25 \\ 0 & -5.111 & -3 & 4.222 & 25.667 \end{bmatrix}$$

- Step 2: Keep the 1st and 2nd equations (1st and 2nd rows) unchanged and eliminate the terms that include the first two variable x_1, x_2 in all the other equations (3rd and 4th rows) using row operations

Row3 = Row3 - (-1.333)/(-2.444)*Row2
Row4 = Row4 - (-5.111)/(-2.444)*Row2

$$Ab := \mathrm{stack}\left(Ab^{\langle 1\rangle}, Ab^{\langle 2\rangle}, Ab^{\langle 3\rangle} - \frac{Ab_{3,2}}{Ab_{2,2}}\cdot Ab^{\langle 2\rangle}, Ab^{\langle 4\rangle} - \frac{Ab_{4,2}}{Ab_{2,2}}\cdot Ab^{\langle 2\rangle}\right) = \begin{bmatrix} 9 & 8 & 9 & 2 & 42 \\ 0 & -2.444 & 2 & 1.889 & 21.667 \\ 0 & 0 & -4.091 & 3.636 & 13.182 \\ 0 & 0 & -7.182 & 0.273 & -19.636 \end{bmatrix}$$

- Step 2: Keep the 1st,2nd and 3rd equations (1st,2nd and 3rd rows) unchanged and eliminate the terms that include the first three variable x_1, x_2, x_3 in all the other equations (4th row) using row operations

Row4 = Row4 - (-7.182)/(-4.091)*Row3

$$Ab := \mathrm{stack}\left(Ab^{\langle 1\rangle}, Ab^{\langle 2\rangle}, Ab^{\langle 3\rangle}, Ab^{\langle 4\rangle} - \frac{Ab_{4,3}}{Ab_{3,3}}\cdot Ab^{\langle 3\rangle}\right) = \begin{bmatrix} 9 & 8 & 9 & 2 & 42 \\ 0 & -2.444 & 2 & 1.889 & 21.667 \\ 0 & 0 & -4.091 & 3.636 & 13.182 \\ 0 & 0 & 0 & -6.111 & -42.778 \end{bmatrix}$$

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Back Substitution

Back substitution

$$m_{4,1} := \frac{Ab_{4,5}}{Ab_{4,4}} = 7$$

$$m_{3,1} := \frac{\left(Ab_{3,5} - Ab_{3,4} \cdot m_{4,1}\right)}{Ab_{3,3}} = 3$$

$$m_{2,1} := \frac{\left(Ab_{2,5} - Ab_{2,3} \cdot m_{3,1} - Ab_{2,4} \cdot m_{4,1}\right)}{Ab_{2,2}} = -1$$

$$m_{1,1} := \frac{\left(Ab_{1,5} - Ab_{1,2} \cdot m_{2,1} - Ab_{1,3} \cdot m_{3,1} - Ab_{1,4} \cdot m_{4,1}\right)}{Ab_{1,1}} = 1$$

$$\begin{bmatrix} w \\ x \\ y \\ z \end{bmatrix} := m = \begin{bmatrix} 0 \\ -1 \\ 3 \\ 7 \end{bmatrix}$$

Row Echelon Form

1. If a row doesn't consist entirely of zeros, then the first nonzero number in the row is a 1. (some textbooks exclude this condition)
2. All zero rows are at the bottom of the matrix.
3. In any two successive rows that don't consist entirely of zeros, the leading 1 in the lower row occurs farther to the right than the leading 1 in the higher row.

- Note that Row Echelon form is **not unique**.

Reduced Row Echelon Form

1. If a row doesn't consist entirely of zeros, then the first nonzero number in the row is a 1. (some textbooks exclude this condition)
2. All zero rows are at the bottom of the matrix.
3. In any two successive rows that don not consist entirely of zeros, the leading 1 in the lower row occurs farther to the right than the leading 1 in the higher row.
4. Each column containing a leading 1 has zeros in all its other entries.

- Note that Reduced Row Echelon form is **unique**.
- In MATLAB, you can use "rref" function to get the reduced row Echelon form.

Examples

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 3 & 1 \\ 0 & -3 & 2 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 2 & 0 & 3 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & -7 & 5 & 5 \\ 0 & 1 & 3 & 2 \end{bmatrix}$$

Examples

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$ RREF

$$\begin{bmatrix} 1 & 3 & 1 \\ 0 & -3 & 2 \\ 0 & 0 & 1 \end{bmatrix}$$?

$$\begin{bmatrix} 1 & 2 & 0 & 3 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$ RREF

$$\begin{bmatrix} 1 & -7 & 5 & 5 \\ 0 & 1 & 3 & 2 \end{bmatrix}$$ REF

Unique Solution

- Solve for x
- $A = \begin{bmatrix} 9 & 5 & 8 \\ 7 & 8 & 5 \\ 9 & 4 & 0 \end{bmatrix}, b = \begin{bmatrix} 3 \\ 8 \\ 4 \end{bmatrix}$

No Solution

- Solve for x
- $A = \begin{bmatrix} 9 & 5 & 24 \\ 7 & 8 & 31 \\ 9 & 4 & 21 \end{bmatrix}, b = \begin{bmatrix} 3 \\ 8 \\ 4 \end{bmatrix}$

Infinitely Many Solutions

- Solve for x
- $A = \begin{bmatrix} 9 & 5 & 24 \\ 7 & 8 & 31 \\ 9 & 4 & 21 \end{bmatrix}, b = \begin{bmatrix} 210 \\ 262 \\ 186 \end{bmatrix}$