1

$$y'' + \omega^2 y = \cos\beta t$$

$$y_p = A\sin\beta t + B\cos\beta t$$

$$y' = A\beta\cos\beta t - B\beta\sin\beta t$$

$$y'' = -A\beta^2\sin\beta t - B\beta^2\cos\beta t$$

$$-A\beta^2\sin\beta t - B\beta^2\cos\beta t + \omega^2(A\sin\beta t + B\cos\beta t) = \cos\beta t$$

$$-A\beta^2\sin\beta t + A\omega^2\sin\beta t = 0 \Leftrightarrow (\omega^2-\beta^2)A = 0$$

$$-B\beta^2\cos\beta t + B\omega^2\cos\beta t = \cos\beta t \Leftrightarrow (\omega^2-\beta^2)B = 1$$

$$\Rightarrow \boxed{B = \frac{1}{\omega^2-\beta^2}}$$

$$y_p(t) = 0\cdot\sin\beta t + \frac{1}{\omega^2-\beta^2}\cos\beta t = \frac{1}{\omega^2-\beta^2}\cos\beta t$$

1 $y_p(t) = \frac{1}{\omega^2 - \beta^2} \cos \beta t$

$y_H = C_1 \cos \omega t + C_2 \sin \omega t$

$y'' + \omega^2 y = 0 \Rightarrow r^2 + \omega^2 = 0$

$\mathcal{L}(y'' + \omega^2 y) = s^2 Y - s y(0) - y'(0) + \omega^2 Y = 0$

$(s^2 + \omega^2) Y = s y(0) - y'(0)$

$Y = \frac{s\, y(0)}{s^2 + \omega^2} - \frac{y'(0)}{s^2 + \omega^2}$

$\boxed{-\frac{y'(0)}{\omega} = C_2}$

$\boxed{y(0) = C_1}$

$Y = C_1 \frac{s}{s^2 + \omega^2} + C_2 \frac{\omega}{s^2 + \omega^2}$

$y = \mathcal{L}^{-1}(Y) = C_1 \cos \omega t + C_2 \sin \omega t = y_h$

$\boxed{1}$

$$y(t) = y_p(t) + y_h(t)$$

$$y(t) = \frac{1}{\omega^2 - \beta^2} \cos \beta t + C_1 \cos \omega t + C_2 \sin \omega t$$

$\boxed{2}$ a)

$$x^2 = \sum_{n\in\mathbb{Z}} c_n e^{inx}$$

$$c_n = \frac{1}{2}(a - bi) = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x) e^{-inx}\,dx$$

x^2 er en like funksjon, altså vil b_n bli 0.

$$\Rightarrow a_n = \frac{1}{\pi}\int f(x)\cos nx\,dx \iff \pi a_n = \int_{-\pi}^{\pi} x^2 \cos nx\,dx$$

$$\Rightarrow \left[\frac{x^2 \sin nx}{n}\right]_{-\pi}^{\pi} - \frac{2}{n}\int_{-\pi}^{\pi} x \sin nx\,dx$$

$$\Rightarrow \frac{x^2 \sin nx}{n} - \frac{2}{n}\left(-\frac{x\cos nx}{n} + \frac{\sin nx}{n^2}\right)\Bigg|_{-\pi}^{\pi}$$

$$\Rightarrow \frac{x^2 \sin nx}{n} + \frac{2x\cos nx}{n^n} - \frac{2\sin nx}{n^3}\Bigg|_{-\pi}^{\pi}$$

$$\left[\left(\cancel{\frac{\pi^2 \cdot 0}{n}} + \frac{2\pi(-1)^n}{n^2} - \cancel{\frac{2\cdot 0}{n^3}}\right) - \left(\cancel{\frac{(-\pi)^2\cdot 0}{n}} - \frac{2\pi(-1)^n}{n^2} - \cancel{\frac{2\cdot 0}{n^3}}\right)\right]$$

$$a_n = \frac{1}{\cancel{\pi}}\,\frac{4\cancel{\pi}(-1)^n}{n^2} = \frac{4(-1)^n}{n^2}$$

$$c_n = \frac{1}{2}(a_n - ib_n) = \frac{1}{2}\,\frac{4(-1)^n}{n^2} = \underline{\underline{\frac{2(-1)^n}{n^2}}}$$

$\boxed{2}$ b) $c_n = \frac{2(-1)^n}{n^2}$ $\quad |c_n| = \frac{2}{n^2}$ $\quad |c_n|^2 = \frac{4}{n^4}$

$$2\pi \sum_{n\in\mathbb{Z}} \frac{4}{n^4} = \underline{8\pi \sum_{n\in\mathbb{Z}} \frac{1}{n^4}} = \int_{-\pi}^{\pi} |x^2|^2 = \frac{1}{5}\left[x^5\right]_{-\pi}^{\pi}$$

$$\Rightarrow \frac{1}{5}\left(\pi^5 - (-\pi)^5\right) = \frac{2}{5}\pi^5 = 8\pi \sum_{n\in\mathbb{Z}} \frac{1}{n^4}$$

$$\Rightarrow \underline{\underline{\sum_{n\in\mathbb{Z}} \frac{1}{n^4} = \frac{1}{20}\pi^4}}$$

$\boxed{3}$ a) $x^2 = a_0 + \sum_{k=1}^{\infty} \left(a_k \cos \frac{k x \pi}{L} + b_k \sin \frac{k x \pi}{L} \right)$

$\cancel{a_0 = \frac{4(-1)^n}{n^2}}$ $\qquad b_n = 0 \qquad a_0 = \frac{1}{2L} \int_{-L}^{L} x^2 \, dx$

$a_n = \int_{-L}^{L} f(x) \cos \frac{n x \pi}{L} \, dx = \qquad\qquad = \frac{1}{2L} \frac{1}{3} \left[x^3 \right]_{-L}^{L} = \frac{1}{6L} \left\{ L^3 - (-L)^3 \right\}$

$L a_n = \int_{-L}^{L} \underbrace{x^2}_{u} \underbrace{\cos \frac{n\pi}{L} x}_{v'} \, dx \qquad\qquad = \frac{L^2}{3} = a_0$

$= \frac{L}{n\pi} x^2 \sin \frac{n\pi}{L} x \Big|_{-L}^{L} - \frac{2L}{n\pi} \int_{-L}^{L} x \sin \frac{n\pi}{L} x \, dx$

$= \left[\cancel{\frac{L}{n\pi} x^2 \sin \frac{n\pi}{L} x} - \frac{2L}{n\pi} \left\{ - \frac{L x \cos \frac{n\pi}{L} x}{n\pi} + \cancel{\frac{L^2 \sin \frac{n\pi}{L} x}{n^2 \pi^2}} \right\} \right]_{-L}^{L}$

$\cancel{L a_n = \frac{2L^2 (L) \cos n\pi}{n^2 \pi^2}} \; \cancel{2L^3 \sin}$

$L a_n = \left[\frac{2L^2 x \cos \frac{n\pi}{L} x}{n^2 \pi^2} \right]_{-L}^{L} = \frac{2L^3 \cos\left(\frac{n\pi}{L} \cdot L\right)}{n^2 \pi^2} - \left(\frac{-2L^3 \cos\left(\frac{n\pi}{L}(-L)\right)}{n^2 \pi^2} \right)$

$L a_n = \frac{4L^3 (-1)^n}{n^2 \pi^2} \iff a_n = \frac{4L^2 (-1)^n}{n^2 \pi^2}$

$\boxed{3}$ b)

$$x^2 = \frac{L^2}{3} + \sum_{k=1}^{\infty} \left((-1)^k \frac{4L^2}{n^2\pi^2} \cos \frac{kx\pi}{L} \right)$$

$$L = \pi \qquad \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{(2n-1)^3}$$

$$f(x) = a_0 + \sum_{k>0} a_k \cos kx + b_k \sin kx \qquad \text{på } (-\pi, \pi)$$

Så :

$$\int_{-\pi/2}^{\pi/2} f(x)\,dx = \pi a_0 + \sum_{n=1}^{\infty} 2(-1)^{n+1} \frac{a_{2n-1}}{2n-1}$$

$$f(x) = x^2 = \underset{\substack{\uparrow \\ a_0}}{\frac{\pi^2}{3}} + \sum_{k=1}^{\infty} \underbrace{(-1)^k \frac{4}{k^2}}_{a_n} \cos kx$$

$$\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} x^2\,dx = \frac{1}{3}\left[x^3\right]_{-\frac{\pi}{2}}^{\frac{\pi}{2}} = \frac{1}{3}\left\{ \frac{\pi^3}{8} - \left(-\frac{\pi^3}{8}\right) \right\} = \frac{\pi^3}{12}$$

$$= \pi \frac{\pi^2}{3} + \sum_{n=1}^{\infty} 2(-1)^{n+1} \frac{(-1)^{2n-1} \frac{4}{(2n-1)^2}}{(2n-1)}$$

$\boxed{3}$ b) Ants.

$$\frac{\pi^3}{12} = \frac{\pi^3}{3} + 8 \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{(2n-1)^3}$$

$$\frac{-3\pi^3}{12} \cdot \frac{1}{8} = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{(2n-1)^3} = \underline{\underline{-\frac{\pi^3}{32}}}$$

4 a) $f(x) = \begin{cases} 0 & x \leq 0 \\ e^{-x} & x > 0 \end{cases}$

$$e^{-x} \cdot e^{-i\omega x} = e^{-x-i\omega x} = e^{-x(1+i\omega)}$$

9/12

$$\hat{f}(\omega) = \mathcal{F}(f) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) e^{-i\omega x} dx$$

$$= \frac{1}{\sqrt{2\pi}} \int_{0}^{\infty} e^{-x(1+i\omega)} dx = \frac{1}{\sqrt{2\pi}} \cdot \left(-\frac{1}{1+i\omega}\right) \left[e^{-x(1+i\omega)} \right]_0^{\infty} \qquad e^{-x} \xrightarrow[x \to \infty]{} 0$$

$$= \frac{1}{\sqrt{2\pi}(1+i\omega)}$$

$\boxed{2}$ b) $f(x) = \begin{cases} 1-x^2 & -1 \le x \le 1 \\ 0 & |x| > 1 \end{cases}$ $\quad \hat{f}(w) = \frac{1}{\sqrt{2\pi}} \int_{-1}^{1} (1-x)e^{-iwx}\,dx$

$$= \cancel{\frac{1}{\sqrt{2\pi}}}\left[\int_{-1}^{1} e^{-iwx}\,dx - \int_{-1}^{1} x^2 e^{-iwx}\,dx\right]$$

$$= \frac{e^{-iwx}}{-iw}\Bigg|_{-1}^{1} - \left[\frac{x^2 e^{-iwx}}{-iw} - \left(\frac{2x e^{-iwx}}{w^2} - \frac{2e^{-iwx}}{-iw^3}\right)\right]_{-1}^{1}$$

$$= \cancel{\frac{e^{-iw}}{-iw}}$$

$$= \cancel{\frac{e^{-iw} - e^{iw}}{-iw}} - \cancel{\frac{e^{-iw} - e^{iw}}{-iw}} - \frac{2e^{-iw} + 2e^{iw}}{w^2} + \frac{2e^{-iw} - 2e^{iw}}{-iw^3}$$

$$= \frac{2(\cos iw - \cancel{i \sin iw} + \cos iw + \cancel{i \sin iw})}{w^2}$$

$$+ \frac{2(\cancel{\cos iw} - i\sin iw - (\cancel{\cos iw} - i \sin iw))}{-iw^3}$$

$$= \frac{4\cos iw}{w^2} + \frac{4 \sin iw}{w^3} = \frac{4}{w^3}(w \cos iw + \sin iw)$$

$$\sqrt{2\pi}\,\hat{f}(w) \Leftrightarrow \cancel{\sqrt{2\pi}}\,\hat{f}(w) = \frac{1}{\sqrt{2\pi}} \cdot \frac{4}{w^3}\left(w\cos(iw) + \sin(iw)\right)$$

$\boxed{4}$ c)

$$f(x) = \begin{cases} T + x & -T \leq x < 0 \\ T - x & 0 \leq x \leq T \\ 0 & |x| > T \end{cases}$$

$$\hat{f}(w) = \frac{1}{\sqrt{2\pi}} \int_{-T}^{T} f(x) e^{-iwx} dx$$

$$\sqrt{2\pi}\,\hat{f}(x) = \int_{-T}^{0} (T+x) e^{-iwx} dx + \int_{0}^{T} (T-x) e^{-iwx} dx$$

$$= T\int_{-T}^{0} e^{-iwx} dx + \int_{-T}^{0} \underbrace{x}_{u}\underbrace{e^{-iwx}}_{v'} dx + T\int_{0}^{T} e^{-iwx} dx - \int_{0}^{T} x e^{-iwx} dx$$

$$= \left[-\frac{T}{iw} e^{-iwx} - \frac{x e^{-iwx}}{iw} + \frac{e^{-iwx}}{w^2}\right]_{-T}^{0} + \left[-\frac{T}{iw} e^{-iwx} + \frac{x}{iw} e^{-iwx} - \frac{e^{-iwx}}{w^2}\right]_{0}^{T}$$

$$= \left[\left(-\frac{T}{iw} + \frac{1}{w^2}\right) - \left(\cancel{-\frac{T}{iw} e^{iwT}} + \cancel{\frac{Te^{iwT}}{iw}} + \frac{e^{iwT}}{w^2}\right)\right]$$

$$+ \left[\left(\cancel{-\frac{Te^{-iwT}}{iw}} + \cancel{\frac{Te^{-iwT}}{iw}} - \frac{e^{-iwT}}{w^2}\right) - \left(-\frac{T}{iw} - \frac{1}{w^2}\right)\right]$$

$$= \cancel{-\frac{T}{iw}} + \frac{1}{w^2} - \frac{e^{iwT} + e^{-iwT}}{w^2} + \cancel{\frac{T}{iw}} + \frac{1}{w^2}$$

$$= \frac{2}{w^2} - \frac{e^{iwT} + e^{-iwT}}{w^2} = \frac{2(1 - \overset{\cos}{\cancel{\sin}}\, wT)}{w^2} = \sqrt{2\pi}\,\hat{f}(w)$$

$$\Rightarrow \hat{f}(w) = \frac{2(1 - \overset{\cos}{\cancel{\sin}}\, wT)}{\sqrt{2\pi}\, w^2}$$

$\boxed{5}$ a) $a \neq b$

$$I(a,b) = e^{-at} * e^{-bt} = \int_0^t e^{-a\tau} e^{-b(t-\tau)} \, d\tau = \int_0^t e^{-a\tau - bt + b\tau} \, d\tau$$

$$= \int_0^t e^{-bt + (b-a)\tau} \, d\tau = e^{-bt} \int_0^t e^{(b-a)\tau} \, d\tau = \frac{e^{-bt}}{b-a} \left[e^{(b-a)\tau} \right]_0^t$$

$$= \frac{e^{-bt}}{b-a} \left[e^{(b-a)t} - 1 \right] = \underline{\underline{\frac{e^{-bt}\left(e^{bt} e^{-at} - 1\right)}{b-a}}} = \underline{\underline{\frac{e^{-at} - e^{-bt}}{b-a}}}$$

$\boxed{5}$ b)

$$I(a,a) = e^{-at} * e^{-at} = \int_0^t e^{-a\tau} e^{-a(t-\tau)} \, d\tau = \int_0^t e^{-a\tau - at + a\tau} \, d\tau$$

$$= e^{-at} \int_0^t e^{(a-a)\tau} \, d\tau = e^{-at} \int_0^t 1 \, d\tau = e^{-at} \left[\tau \right]_0^t = \underline{\underline{t e^{-at}}}$$