II a)

x_i	-1	$-1/2$	$1/2$	1
$f(x_i)$	$-1/2$	$-5/4$	$1/4$	$5/2$

$n+1 = 4 \iff \underline{n=3}$

Vil finne polynom $P_3(x) = a_3 x^3 + \dots + a_1 x + a_0$

$$l_i(x) = \prod_{j=0,\ j\neq i}^{3} \frac{x - x_j}{x_i - x_j}$$

$$l_0(x) = \frac{(x+1/2)(x-1/2)(x-1)}{(-1+1/2)(-1-1/2)(-1-1)} = -\frac{2}{3}\left(x^3 - x^2 - \cancel{1/2x^2} + \cancel{1/2x} + \cancel{1/2x^2} - \cancel{1/2x} - 1/4x + 1/4\right)$$

$$= -\frac{2}{3}x^3 + \frac{2}{3}x^2 + \frac{1}{6}x - \frac{1}{6}$$

$$l_1(x) = \frac{(x+1)(x-1/2)(x-1)}{(-1/2+1)(-1/2-1/2)(-1/2-1)} = \frac{4}{3}\left(x^3 - \cancel{x^2} - 1/2x^2 + \cancel{1/2x} + \cancel{x^2} - x - \cancel{1/2x} + 1/2\right)$$

$$= \frac{4}{3}x^3 - \frac{2}{3}x^2 - \frac{4}{3}x + \frac{2}{3}$$

$$(1)\ = -\frac{4}{3}x^3 - \frac{2}{3}x^2 + \frac{4}{3}x + \frac{2}{3}$$

$$l_2(x) = \frac{(x+1)(x+1/2)(x-1)}{(1/2+1)(1/2+1/2)(1/2-1)} = -\frac{4}{3}\left(x^3 - \cancel{x^2} + 1/2x^2 - \cancel{1/2x} + \cancel{x^2} + \cancel{1/2x} - x - 1/2\right) = \cancel{-\frac{3}{4}x^3 - \frac{3}{8}x^2 + \frac{3}{4}x + \frac{3}{8}}$$

$$l_3(x) = \frac{(x+1)(x+1/2)(x-1/2)}{(1+1)(1+1/2)(1-1/2)} = \frac{2}{3}\left(x^3 + x^2 - 1/4x - 1/4\right) = \frac{2}{3}x^3 + \frac{2}{3}x^2 - \frac{1}{6}x - \frac{1}{6}$$

$$P_n(x) = \sum_{i=0}^{3} y_i l_i(x) = -1/2\left(-\frac{2}{3}x^3 + \frac{2}{3}x^2 + \frac{1}{6}x - \frac{1}{6}\right) - 5/4\left(\frac{4}{3}x^3 - \frac{2}{3}x^2 - \frac{4}{3}x + \frac{2}{3}\right)$$

$$+ 1/4\left(\cancel{-\frac{3}{8}x^3 - \frac{3}{8}x^2 + \frac{3}{4}x + \frac{3}{8}}\right) + 5/2\left(\cancel{\frac{3}{2}x^3 + \frac{3}{2}x^2 - \frac{3}{8}x - \frac{3}{8}}\right)$$

(1) $\left(\frac{2}{3}x^3 + \frac{2}{3}x^2 - \frac{1}{6}x - \frac{1}{6}\right)$

$$= 0x^3 + 2x^2 + \frac{3}{2}x - 1$$

$$= 2x^2 + \frac{3}{2}x - 1 \qquad f(0) \approx -1$$

$\boxed{3}$ a) $f(x) = x^2 \cos x \qquad x \in [-1, 2]$

Med ekvidistribuert noder

x_i	-1	0	1	2
$f(x_i)$	1	0	1	4

$$\ell_0(x) = \frac{(x-0)(x-1)(x-2)}{(-1-0)(-1-1)(-1-2)} = \underline{-\frac{1}{6}x^3 - \frac{1}{2}x^2 + \frac{1}{3}x}$$

$$\ell_1(x) = \frac{(x+1)(x-1)(x-2)}{(0+1)(0-1)(0-2)} = \underline{\frac{1}{2}x^3 - x^2 - \frac{1}{2}x + 1}$$

$$\ell_2(x) = \frac{(x+1)(x+0)(x-2)}{(1+1)(1+0)(1-2)} = \underline{-\frac{1}{2}x^3 + \frac{1}{2}x^2 + x}$$

$$\ell_3(x) = \frac{(x+1)(x+0)(x-1)}{(2+1)(2+0)(2-1)} = \underline{\frac{1}{6}x^3 - \frac{1}{6}x}$$

$$P_3(x) = \sum_{j=0}^{3} y_i \ell_i(x) = -\frac{1}{6}x^3 - \frac{1}{2}x^2 + \frac{1}{3}x - \frac{1}{2}x^3 + \frac{1}{2}x^2 + x$$

$$+ \frac{2}{3}x^3 - \frac{2}{3}x = \underline{x^2}$$

$\underline{P_3(x) = x^2 \text{ med jevnt fordelte noder.}}$

3 a) $f(x) = x^2 \cos x$ med Chebyshev-punkter

$x \in [-1, 2] \rightarrow a = -1 \; , \; b = 2$

$$X_i = \frac{b-a}{2} \cos\left(\frac{(2i+1)\pi}{2(n+1)}\right) + \frac{b+a}{2}$$

$$X_i = \frac{3}{2} \cos\left(\frac{(2i+1)\pi}{2(n+1)}\right) + \frac{1}{2}$$

$\underline{X_0 = -0,89}$ $\underline{X_1 = -0,07}$

$\underline{X_2 = 1,07}$ $\underline{X_3 = 1,89}$

X_i	$-0,89$	$-0,07$	$1,07$	$1,89$
$f(x_i)$	$0,50$	$0,005$	$0,55$	$-1,12$

$$l_0(x) = \frac{(x+0,07)(x-1,07)(x-1,89)}{(-0,89+0,07)(-0,89-1,07)(-0,89-1,89)} = -6,64x^3 + 19,2x^2 - 12,06x - 0,94$$

$$l_1(x) = \frac{(x+0,89)(x-1,07)(x-1,89)}{(-0,07+0,89)(-0,07-1,07)(-0,89-1,89)} = 2,72x^3 - 5,64x^2 - 1,67x + 4,9$$

$$l_2(x) = \frac{(x+0,89)(x+0,07)(x-1,89)}{(1,07+0,89)(1,07+0,07)(1,07-1,89)} = -0,48x^3 + 0,44x^2 + 0,84x + 0,06$$

$$l_3(x) = \frac{(x+0,89)(x+0,07)(x-1,07)}{(1,89+0,89)(1,89+0,07)(1,89-1,07)} = 0,58x^3 - 0,06x^2 - 0,56x - 0,04$$

$$P_3(x) = \sum_{i=0}^{3} l_i(x)\, y_i = \underline{\underline{-4,22x^3 + 9,89x^2 - 4,95x - 0,37}}$$

3 b) Equidistributed nodes

Feilen er gitt ved $e(x) = f(x) - p(x)$

Den maksimale feilen er gitt ved:

$$|e(x)| \leq \frac{h^{n+1}}{4(n+1)} \cdot M \quad , \quad M = \max_{x\in[a,b]} \left| f^{(n+1)}(x) \right|$$

$$h = \frac{b-a}{n} = 1 \quad n = 3 \quad b = 2 \quad a = -1$$

$$\frac{d^m}{dx^m} x^2 \cos(x) = (-1)^{m/2}\left(x^2 \cos x + 2m \sin x - m(m-1)\cos x\right) \text{ for } m \text{ like}$$

$m = n+1 = 4$:

$$\underline{\frac{d^4}{dx^4} x^2 \cos(x) = x^2 \cos x + 8 \sin x - 12 \cos x}$$

$f^{(5)}(x) = 0 \Rightarrow x = 0$ $|f^{(4)}(x)|$ har altså maksimal verdi i punktet $x = 0$, $f^{(4)}(0) = 0^2 \cdot \cos 0 + 8 \cdot \sin 0 - 12 \cos 0 = \underline{-12}$

$$\cancel{|f^{(4)}(0)|} \max_{x\in[1,2]} |f^{(4)}(x)| = \underline{12} = M$$

$$|e(x)| \leq \frac{1^4}{4(3+1)} \cdot 12 = \underline{\underline{\frac{3}{4}}}$$

3 b) Chebyshev points

Feilen $|e(x)|$ er gitt ved:

$$|f(x) - P(x)| \leq \frac{(b-a)^{n+1}}{2^{2n+1}(n+1)!} M_{n+1}$$

$$M_{n+1} = \max_{x \in [a,b]} |f^{(n+1)}(x)| = \underline{12}$$

Denne ble regnet ut på forrige side

$$|e(x)| \leq \frac{(2-(-1))^{3+1} \cdot 12}{2^{2\cdot 3+1}(3+1)!} = \frac{3^4 \cdot 12}{2^7(\cancel{3+1}\,4)!} = \underline{\underline{0,3164}}$$

For $n=3$ på intervallet $[-1,2]$ kan altså ikke feilen bli verre enn 0,3164, noe som er betydelig bedre enn $\frac{3}{4} = 0,75$ med jevnt fordelte noder, men fortsatt ikke særlig nøyaktig.