Side 1

$\boxed{1}$ a)

x_i	-2	1	6
y_i	1	2	3

$n+1 = 3 \Rightarrow \underline{n=2}$

$$\begin{array}{r|llll} -2 & 1 & & & \\ & & 1/3 & & \\ 1 & 2 & & -1/60 & \\ & & 1/5 & & \\ 6 & 3 & & & \end{array}$$

$P_2(x) = c_2(x-\cancel{x_0}) + c_1(x-x_0)(x-x_1) + c_0$

$c_0 = f[x_0] = \underline{1}$ $\quad c_1 - f[x_0 x_1] = \dfrac{2-1}{1-(-2)} = \underline{\dfrac{1}{3}}$

$\cancel{c_2 = f[}$

$f[x_1, x_2] = \dfrac{3-2}{6-1} = 1/5$

$c_2 = f[x_0, x_1, x_2] = \dfrac{f[x_1,x_2] - f[x_0,x_1]}{6-(-2)} = \dfrac{1/5 - 1/3}{8} = \underline{-\dfrac{1}{60}}$

$\cancel{P_2(x) = -\frac{1}{60}(x+2) + \frac{1}{3}(x+2)(x-1) + 1}$

$\cancel{= 3/10 + 19/60\,x + 1/3\,x^2}$

$P_2(x) = -\frac{1}{60}(x+2)(x-1) + \frac{1}{3}(x+2) + 1$

$\underline{\underline{= 17/10 + 19/60\,x - 1/60\,x^2}}$

b)

$$\begin{array}{r|lllll} -2 & 1 & & & & \\ & & 1/3 & & & \\ 1 & 2 & & -1/60 & & \\ & & 1/5 & & 1/315 & \\ 6 & 3 & & -4/315 & & \\ & & 2/9 & & & \\ -3/4 & 3/2 & & & & \end{array}$$

x_i	-2	1	6	$-3/4$
y_i	1	2	3	$3/2$

$f[x_2, x_3] = \dfrac{3/2 - 3}{-3/4 - 6} = \underline{2/9}$ $\quad f[x_1, x_3] - \dfrac{f[x_2,x_3] - f[x_1,x_2]}{x_3 - x_1} = -\dfrac{4}{315}$

$f[x_0, x_3] = \dfrac{f[x_1,x_3] - f[x_0,x_2]}{x_3 - x_0} = \underline{1/315}$

I b) $c_3 = 1/315 \quad c_2 = -1/60 \quad c_1 = 1/3 \quad c_0 = 1$

$$P_3(x) = \frac{1}{315}(x+2)(x-1)(x-6) - \frac{1}{60}(x+2)(x-7) + \frac{1}{3}(x+2) + 1$$

$$\underline{\underline{P_3(x) = \frac{1}{315}x^3 - \frac{41}{1260}x^2 + \frac{367}{1260}x + \frac{73}{42}}}$$

II c) $y = f(x) = x^2 - 3 \qquad [a,b] = [1,3] \qquad x_0, x_1, x_2 = 1, 2, 3$

$$y = x^2 - 3 \iff x^2 = y+3 \iff \underline{x = f^{-1}(y) = \sqrt{y+3}}$$

$\underline{f(1) = -2} \quad \underline{f(2) = 1} \quad \underline{f(3) = 6}$

y_i	-2	1	6
$f^{-1}(y_i)$	1	2	3

Fant ut i oppgave a) at interpolasjonspolynomet til tabellen over er gitt ved:

$$P_2(x) = 17/10 + 19/60\,x - 1/60\,x^2 \Rightarrow P_2(0) = \frac{17}{10} = \underline{1,7}$$

$$f(x) = 0 = x^2 - 3 \Rightarrow x = \pm\sqrt{3} \Rightarrow x = \sqrt{3}$$ Ser kun på intervallet $[1,3]$

$\underline{\underline{\sqrt{3} - 1,7 = 0,0320508}}$ Feilen ved $n=2$

$\underline{n=3}$ (fra a)) $P_3(x) = \frac{73}{42} + \frac{367x}{1260} - \frac{41x^2}{1260} + \frac{x^3}{315} \Rightarrow P_3(0) = \underline{\frac{73}{42}}$

Feilen ved $n=3$: $\underline{\underline{\sqrt{3} - \frac{73}{42} = -6.04443 \cdot 10^{-3}}}$

Ca. 5 ganger bedre enn ved $n=2$

3 a) Simpsons regel på et intervall $[a,b]$:

$$S(a,b) = \frac{b-a}{6}\left[f(a) + 4f(c) + f(b)\right], \quad c = \frac{b+a}{2}$$

$f(x) = e^{-x}$

$$S_1[f](1,3) = \frac{3-1}{6}\left[e^{-1} + 4e^{-2} + e^{-3}\right] \approx 0{,}31967$$

$$S_2[f](1,3) = \frac{2-1}{6}\left[e^{-1} + 4e^{-1{,}5} + e^{-2}\right] + \frac{3-2}{6}\left[e^{-2} + 4e^{-2{,}5} + e^{-3}\right]$$

$$= \frac{1}{6}\left[e^{-1} + 4e^{-1{,}5} + 2e^{-2} + 4e^{-2{,}5} + e^{-3}\right] \approx 0{,}3181997$$

Den faktiske feilen:

$$S_1: \int_1^3 e^{-x}\,dx - S_1(1,3) = -1{,}578 \cdot 10^{-3}$$

$$S_2: \int_1^3 e^{-x}\,dx - S_2(1,3) = -1{,}073 \cdot 10^{-4}$$

Feilestimat:

$$E_1(a,b) = I(a,b) - S_1(a,b) \approx \frac{16}{15}\left(S_2(a,b) - S_1(a,b)\right) = \mathcal{E}_1(a,b)$$

$$\mathcal{E}_1 = \frac{16}{15}\left(S_2(1,3) - S_1(1,3)\right) = -1{,}56832 \cdot 10^{-3}$$

$$\mathcal{E}_2 = \frac{1}{15}\left(S_2(1,3) - S_1(1,3)\right) = -9{,}802 \cdot 10^{-5}$$

3 $\boxed{b}$ $\quad |I(a,b) - S_m(a,b)| \leq 10^{-8}$

$$E_m(a,b) = -\frac{(b-a)h^4}{180} f^{(4)}(\xi) \qquad f(x) = e^{-x}$$

$$f^{(4)} = e^{-x} - \max_{x \in [1,3]} e^{-x} = \underline{e^{-1}}$$

$$h = \frac{b-a}{2m}$$

$$\max_{x \in [1,3]} E_m(a,b) = \frac{(b-a)^5}{180 \cdot 2^4 \cdot m^4} \cdot e^{-1} < 10^{-8}$$

$$\Rightarrow \quad m = \sqrt[4]{\frac{(b-a)^5 e^{-1}}{180 \cdot 2^4 \cdot 10^{-8}}} = \sqrt[4]{\frac{\cancel{2}\, e^{-1}}{90 \cdot 10^{-8}}} = \underline{25,28}$$

m må altså være $\underline{\underline{26}}$ eller større for å garantere feil mindre enn 10^{-8}.

$\boxed{4}$ a) Gauss - Legendre på $[-1, 1]$, $m=3$

Finner nodene:

$$L_3(t) = \frac{d^3}{dt^3}(t^2-1)^3 = \frac{d^3}{dt^3}(t^6 - 3t^4 + 3t^2 - 1)$$

$$\Rightarrow \frac{d^2}{dt^2}(6t^5 - 12t^3 + 6t) = \frac{d}{dt}(30t^4 - 36t^2 + 6) = \underline{120t^3 - 72t}$$

$$L_3(t) = 0 \Rightarrow 120t^3 - 72t = 0 \Rightarrow \underline{t = 0} \vee \underline{t = \pm\frac{\sqrt{15}}{5}}$$

Noder: $t_0 = -\frac{\sqrt{15}}{5}$ $t_1 = 0$, $t_2 = \frac{\sqrt{15}}{5}$

Lager kardinalfunksjoner:

$$l_0(t) = \frac{(t-0)\left(t - \frac{\sqrt{15}}{5}\right)}{\left(-\frac{\sqrt{15}}{5} - 0\right)\left(-\frac{\sqrt{15}}{5} - \frac{\sqrt{15}}{5}\right)} = \frac{5}{6}t^2 - \frac{1}{6}\sqrt{15}\,t$$

$$l_1(t) = \frac{\left(t + \frac{\sqrt{15}}{5}\right)\left(t - \frac{\sqrt{15}}{5}\right)}{\left(0 + \frac{\sqrt{15}}{5}\right)\left(0 - \frac{\sqrt{15}}{5}\right)} = -\frac{5}{3}t^2 + 1$$

$$l_2(t) = \frac{\left(t + \frac{\sqrt{15}}{2}\right)(t-0)}{\left(\frac{\sqrt{15}}{5} + \frac{\sqrt{15}}{5}\right)\left(\frac{\sqrt{15}}{5} - 0\right)} = \frac{5}{6}t^2 + \frac{1}{6}\sqrt{15}\,t$$

Lager vektene:

$$\omega_0 = \int_{-1}^{1} l_0(t)\,dt = 2\cdot\left[\frac{1}{3}\cdot\frac{5}{6}\cdot t^3\right]_0^1 = \underline{\frac{5}{9}}$$

$$\omega_1 = \int_{-1}^{1} l_1(t)\,dt = \left[\frac{1}{3}\left(-\frac{5}{3}\right)t^3 + t\right]_{-1}^{1} = \underline{\frac{8}{9}}$$

$$\omega_2 = \int_{-1}^{1} l_2(t)\,dt = 2\left[\frac{1}{3}\frac{5}{6}\cdot t^3\right]_0^1 = \underline{\frac{5}{9}}$$

$$\Rightarrow \int_{-1}^{1} f(t)\,dt \approx \int_{-1}^{1} P_2(t)\,dt = \sum_{i=0}^{2} \omega_i f(t_i) = \underline{\underline{\frac{1}{9}\left[5f(t_0) + 8f(t_1) + 5f(t_2)\right]}}$$

4 b) Presisjonsgrad:

Side 6

$n=0$: $\int_{-1}^{1} t^0 dt = \underline{2}$

$n=1$: $t^1 = t \to \int_{-1}^{1} t\,dt = \underline{0}$

$n=2$: $t^2 \to \int_{-1}^{1} t^2 dt = \underline{\frac{2}{3}}$

$n=3$: $t^3 \to \int_{-1}^{1} t^3 dt = \underline{0}$

$n=4$: $t^4 \to \int_{-1}^{1} t^4 dt = \underline{\frac{2}{5}}$

$n=5$: $t^5 \to \int_{-1}^{1} t^5 dt = \underline{0}$

$n=6$: $t^6 \to \int_{-1}^{1} t^6 dt = \underline{\frac{2}{7}}$

$G[1](-1,1) = \frac{1}{9}\left[5\cdot 1 + 8\cdot 1 + 5\cdot 1\right] = \frac{18}{9} = \underline{2}$

$G[t](-1,1) = \frac{1}{9}\left[5\left(-\frac{\sqrt{15}}{5}\right) + 8\cdot 0 + 5\left(\frac{\sqrt{15}}{5}\right)\right] = \underline{0}$

$G[t^2](-1,1) = \frac{1}{9}\left[5\left(-\frac{\sqrt{15}}{5}\right)^2 + 8\cdot 0 + 5\left(\frac{\sqrt{15}}{5}\right)^2\right] = \frac{6}{9} = \underline{\frac{2}{3}}$

$G[t^3](-1,1) = \frac{1}{9}\left[5\left(-\frac{\sqrt{15}}{5}\right) + 0 + 5\left(\frac{\sqrt{15}}{5}\right)\right] = \underline{0}$

$G[t^4](-1,1) = \frac{1}{9}\left[5\cdot\left(-\frac{\sqrt{15}}{5}\right)^4 + 0 + 5\left(\frac{\sqrt{15}}{5}\right)^4\right] = \underline{\frac{2}{5}}$

$G[t^5](-1,1) = \frac{1}{9}\left[5\left(\frac{\sqrt{15}}{5}\right)^5(1-1) + 0\right] = \underline{0}$

$G[t^6](-1,1) = \frac{1}{9}\left[10\left(\frac{\sqrt{15}}{5}\right)^6 + 0\right] = \frac{6}{25}$

$\int_{-1}^{1} t^n dt$ er lik $G[t^n](-1,1)$ for n opp til og med 5, altså har Gauss-Legendre kvadratur presisjonsgrad 5.

4 c) $X = \frac{b-a}{2}t + \frac{b+a}{2} \qquad dx = \frac{b-a}{2}\,dt$

$$\int_a^b f(x)\,dx = \frac{b-a}{2}\int_{-1}^{1} f\left(\frac{b-a}{2}t + \frac{b+a}{2}\right)dt$$

$$\approx \frac{b-a}{2}\cdot\frac{1}{9}\left[5\cdot f\left(\frac{b-a}{2}\cdot\left(-\frac{\sqrt{15}}{5}\right)+\frac{b+a}{2}\right)+9\cdot f\left(\frac{b+a}{2}\right)+5\cdot f\left(\frac{b-a}{2}\cdot\frac{\sqrt{15}}{5}+\frac{b+a}{2}\right)\right]$$

Lar $c = \frac{b+2}{2}$ og $h = \frac{b-a}{2}$:

$$= \underline{\underline{\frac{h}{9}\left[5\cdot f\left(-\frac{\sqrt{15}}{5}h + c\right)+9\cdot f(c) + 5\cdot f\left(\frac{\sqrt{15}}{5}h+c\right)\right]}}$$

$\underline{\int_1^3 e^{-x}\,dx = -e^{-3}+e^{-1}} \qquad [a,b] = [1,3] \to h = 1,\ c = 2$

~~$\int_1^3 e^{-x}\,dx \approx \frac{h}{9}\left[-\frac{\sqrt{15}}{5}+2\right.$~~

$$\int_1^3 e^{-x}\,dx \approx \frac{1}{9}\left[5\cdot e^{-\left(-\frac{\sqrt{15}}{5}+2\right)} + 9e^{-2} + 5\cdot e^{-\left(\frac{\sqrt{15}}{5}+2\right)}\right] = \underline{0{,}3331208}$$

$|I(1,3) - G(1,3)| = (-e^{-3}+e^{-1}) - (0{,}3331208) = \underline{\underline{-0{,}01503}}$

Feilen

4 d) Feilen for sammensatt Gauss-Legendre

$$E(a,b) = \int_a^b f(x)\,dx - G(a,b) = \frac{(b-a)^7}{2016000} f^{(6)}(\eta)$$

← Feilen for ett intervall

$$h = \frac{x_{j+1} - x_j}{\cancel{m}} \iff \cancel{m}h = x_{j+1} - x_j$$

← Lengde av ett delintervall

Sammensatt Gauss-Legendre: $G_m(a,b)$

$$\int_a^b f(x)\,dx - G_m(a,b) = \sum_{j=0}^{m-1}\left(\int_{j}^{j+1} f(x)\,dx - G(x_j, x_{j+1})\right)$$

$$= \sum_{j=0}^{m-1} \frac{(b-a)^7}{2016000} f^{(6)}(\eta) = \sum_{j=0}^{m-1} \frac{(h)^7}{2016000} f^{(6)}(\eta)$$

Middelverditeoremet: $\sum_{j=0}^{m-1} f^{(6)}(\eta_j) = m f^{(6)}(\xi)$ for en $\xi \in (a,b)$

$$\Rightarrow \sum_{j=0}^{m-1} \frac{(h)^7}{2016000} f^{(6)}(\eta) = \frac{(h)^7}{2016000} m f^{(6)}(\xi)$$

$$\Rightarrow \frac{mh\, h^6}{2016000} f^{(6)}(\xi) = \frac{(b-a)h^6}{37500 \cdot 2^6} f^{(6)}(\xi)$$

$$\boxed{h = \frac{(b-a)}{m} \iff hm = b-a}$$

$$\max_{x\in[1,3]} \left(e^{-x}\right)^{(6)} = e^{-1}$$

Feil mindre enn 10^{-8}:

$$E_m(a,b) = \frac{(b-a)h^6}{2016000} f^{(6)}(\xi)$$

$$h = \frac{b-a}{m}$$

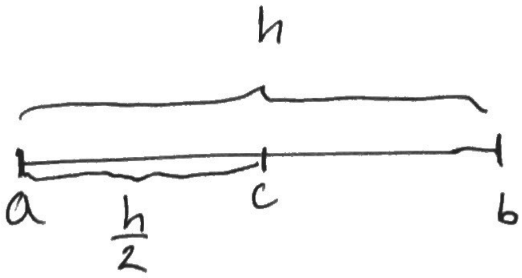
$m = \sqrt[6]{\frac{(b-a)^7 e^{-1}}{2016000 \cdot 10^{-8}}} = 3,64$, man trenger altså $m=4$ for å garantere feil mindre enn 10^{-8}, altså betydelig mindre enn med simpsons.

4 e) Feilestimat

$f^{(6)}(x)$ er tilnærmet konstant på $[a,b]$:

$$E(a,b) = \frac{(b-a)h^6}{31500\cdot 2^6} f^{(6)}(\xi)$$

$$C = \frac{f^{(6)}(x)}{31500\cdot 2^6} \qquad h = b-a$$



$$I(a,b) - S_1(a,b) \approx Ch^7$$

$$I(a,b) - S_2(a,b) = I(a,c) - S_1(a,c) + I(c,b) - S_1(c,b)$$

$$= C\left(\frac{h}{2}\right)^7 + C\left(\frac{h}{2}\right)^7 = 2Ch^7\frac{1}{2^7} = \frac{1}{64}Ch^7$$

$$\rightarrow S_2(a,b) - S_1(a,b) \approx \frac{63}{64}Ch^7 \Rightarrow Ch^7 \approx \frac{64}{63}\left(S_2(a,b) - S_1(a,b)\right)$$

$$E_1(a,b) \approx \mathcal{E}_1(a,b) = \frac{64}{63}\left(S_2(a,b) - S_1(a,b)\right)$$

$$E_2(a,b) \approx \mathcal{E}_2(a,b) = \frac{1}{63}\left(S_2(a,b) - S_1(a,b)\right)$$

Feilestimat for $Q_2(a,b)$