1 a) $Ax = b$ $\quad A = \begin{bmatrix} 3 & 1 & 1 \\ 1 & 3 & -1 \\ 3 & 1 & -5 \end{bmatrix} \quad b = \begin{bmatrix} 5 \\ 3 \\ -1 \end{bmatrix}$

Metode 1:

$$x^{(0)} = 0 \qquad x_i^{(k+1)} = \frac{1}{a_{ii}}\left(b_i - \sum_{j\neq i} a_{ij}x_j^{(k)}\right) \quad , i = 1, 2, \dots, n$$

$$x_1^{(1)} = \frac{1}{3}\left(5-(3\cdot 0 + 1\cdot 0 + 1\cdot 0)\right) = \underline{\frac{5}{3}}$$

$$x_2^{(1)} = \frac{1}{3}(3-0) = \underline{1} \qquad x_3^{(1)} = \frac{1}{-5}(-1-0) = \underline{\frac{1}{5}}$$

$$x^{(1)} = \begin{bmatrix} 5/3 \\ 1 \\ 1/5 \end{bmatrix} \rightarrow x_1^{(2)} = \frac{1}{3}\left(5 - (\cancel{\ }+1\cdot 1 + 1\cdot 1/5)\right) = \underline{\frac{19}{15}}$$

$$x_2^{(2)} = \frac{1}{3}\left(3-(1\cdot 5/3 + \cancel{\ } - 1\cdot 1/5)\right) = \underline{\frac{23}{45}}$$

$$x_3^{(2)} = \frac{1}{-5}\left(-1-(3\cdot 5/3 + 1\cdot 1 - \cancel{\ })\right) = \frac{7}{5}$$

Etter 2 iterasjoner er $x^{(2)} = \begin{bmatrix} -2/5 \\ -22/45 \\ 6/5 \end{bmatrix}$

Metode 2:

$$x_i^{(k+1)} = \frac{1}{a_{ii}}\left(b_i - \sum_{j=1}^{i-1} a_{ij}x_j^{(k+1)} - \sum_{j=i+1}^{n} a_{ij}x_j^{(k)}\right) \quad , i = 1, 2, \dots, n$$

$$x^{(0)} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \qquad x_1^{(1)} = \frac{1}{3}(5-0) = \underline{5/3}$$

$$x_2^{(1)} = \frac{1}{3}\left(3-(1\cdot 5/3 + 0 + 0)\right) = 4/9$$

$$x^{(1)} = \begin{bmatrix} 5/3 \\ 4/9 \\ 58/45 \end{bmatrix} \qquad x_3^{(1)} = \frac{1}{-5}\left(-1-(3\cdot 5/3 + 1\cdot 4/9 + 0\cdot(-5))\right) = 58/45$$

$\boxed{1}$ a) $x^{(1)} = \begin{bmatrix} 5/3 \\ 4/9 \\ 58/45 \end{bmatrix} \quad A = \begin{bmatrix} 3 & 1 & 1 \\ 1 & 3 & -1 \\ 3 & 1 & -5 \end{bmatrix} \quad b = \begin{bmatrix} 5 \\ 3 \\ -1 \end{bmatrix}$

$$x_1^{(2)} = \frac{1}{3}\left(5 - \left(1 \cdot 4/9 + 1 \cdot 58/45\right)\right) = \frac{49}{45} \qquad x^{(2)} = \begin{bmatrix} -26/45 \\ 53/45 \\ -6/5 \end{bmatrix}$$

$$x_2^{(2)} = \frac{1}{3}\left(3 - \left(1 \cdot (49/45) - 1(58/45)\right)\right) = \frac{16}{15}$$

$$x_3^{(2)} = \frac{1}{-5}\left(-1 - \left(3(49/45) + 1(53/45)\right)\right) = -\frac{6}{5}$$

Etter 2 iterasjoner med gode gamle Gauss-Seidel ble

$$x^{(2)} = \begin{bmatrix} -26/45 & 53/45 & -6/5 \end{bmatrix}^T$$

$\boxed{1}$ b) $|a_{ii}| > \sum_{j \neq i} |a_{ij}|$ for alle $i = 1, 2, \ldots, n$

strengt diagonalt dominant

$i = 1 \quad |a_{11}| = 3 \qquad \sum_{j \neq i} |a_{1j}| = 1 + 1 = 2 \qquad \underline{3 > 2}$

$i = 2 \quad |a_{22}| = 3 \qquad \sum |a_{2j}| = 1 + 1 = 2 \qquad \underline{3 > 2}$

$i = 3 \quad |a_{33}| = 5 \qquad \sum |a_{3j}| = 3 + 1 = 4 \qquad \underline{5 > 4}$

Matrisen A er altså strengt diagonalt dominant.

1 c) $|e_i^{(k+1)}| = |x_i - x_i^{(k)}| = \left|\frac{1}{a_{ii}}\left(b_i - \sum_{j=1\, j\neq i}^{n} a_{ij}x_j\right) - \frac{1}{a_{ii}}\left(b_i - \sum_{j=1\, j\neq i}^{n} a_{ij}\, x_j^{(k)}\right)\right|$

$$= \left|\frac{1}{a_{ii}}\left(\sum_{j=1\, j\neq i}^{n}(x_i^{(k)} - x_i)a_{ij}\right)\right| = \left|\sum_{j=1\, j\neq i}^{n} \frac{a_{ij}}{a_{ii}} e_i^{(k)}\right| \quad \left(d_{ij} := \frac{a_{ij}}{a_{ii}}\right)$$

$$= \left|\sum_{j=1\, j\neq i}^{n} d_{ij} e_i^{(k)}\right| \overset{\text{trekant ulikheten}}{\leq} \sum_{j=1\, j\neq i}^{n} |d_{ij}||e_i^{(k)}| \leq \|e^{(k)}\| \sum_{j=1\, j\neq i}^{n} |d_{ij}|$$

Definerer nå $L := \max\{L_i \mid i = 1,2..n\}$, hvor $L_i = \sum_{j=1\, j\neq i}^{n} |d_{ij}|$

Per definisjon for en matrise A som er strengt diagonalt dominant vil summen av alle elementer per rad, bortsett fra elementet på a_{ii}, være mindre enn a_{ii}, altså $a_{ii} > \sum_{j=1\, j\neq i}^{n} a_{ij} \quad \Big| \cdot \frac{1}{a_{ii}}$

$$\Rightarrow \underline{1 > \sum_{j=1\, j\neq i}^{n} \frac{a_{ij}}{a_{ii}}}$$, altså er ~~summen~~ av alle $\underline{L_i < 1}$.

$$\|e^{(k)}\| \sum_{j=1\, j\neq i}^{n} |d_{ij}| = \|e^{(k)}\| L_i \leq \underline{L\|e^{(k)}\|}$$

Har vist at $|e_i^{(k+1)}| \leq L\|e^{(k)}\|$. Siden alle elementer i $e^{(k+1)}$ er mindre en $L\|e^{(k)}\|_\infty$, må også maksnormen til $e^{(k+1)}$ være det:

$$\underline{\|e^{(k+1)}\| \leq L\|e^{(k)}\|}$$

Har nå vist at maksnormen for feilen etter k+1 steg er mindre eller lik maksnormen til maksnormen til feilen etter k steg, ganget med en konstant L, . Som jeg har vist er strengt mindre hvis matrisen A er strengt diagonalt dominant, altså vil Jacobi-iterasjonene konvergere om så er tilfelle. $\square$

2 a)

Find the exact solution:

$$y' - xy^2 = 0 \qquad y(0) = 1$$

$$\Rightarrow \frac{dy}{dx} = xy^2 \Rightarrow \int y^{-2}\,dy = \int x\,dx$$

$$\Rightarrow -y^{-1} + C_1 = \frac{1}{2}x^2 + C_2 \Rightarrow -1\cdot\left(\frac{x^2 + C_3}{2}\right) = y^{-1}$$

$$\Rightarrow y = \frac{-2}{x^2 + C}, \quad y(0) = 1 = \frac{-2}{0^2 + C} = 1 \Rightarrow C = -2$$

$$y(x) = \frac{-2}{x^2 - 2} \qquad y(0{,}4) = \frac{25}{23}$$

b) Eulers: $y_{n+1} = y_n + hf(x_n, y_n)$ $\quad x_{n+1} = x_n + h$

$$y' - xy^2 = 0 \Rightarrow y' = xy^2, \quad f(x,y) = xy^2, \quad h = 0{,}1$$

$y_0 = 1 \qquad x_0 = 0$

$y_1 = y_0 + hf(x_0, y_0) = 1 + 0{,}1(0\cdot 1^2) = 1 \qquad x_1 = 0 + 0{,}1 = 0{,}1$

$y_2 = y_1 + h(0{,}1\cdot 1^2) = 1{,}01 \qquad x_2 = 0{,}2$

$y_3 = y_2 + h(x_2 \cdot y_2^2) = 1{,}030402 \qquad x_3 = 0{,}3$

$y_4 = y_3 + h(x_3 \cdot y_3^2) = 1{,}062254 \qquad x_4 = 0{,}4$

Euler gir ca $1{,}062254 \approx y(0{,}4)$

Feilen er $y(0{,}4) - y_4 \approx 0{,}0247$

$\boxed{2}$ c) $y' - xy^2 = 0 \Rightarrow f(x,y) = xy^2$ $h = 0{,}2$ Side 5

$(x_0, y_0) = (0, 1)$

Heuns metode:

$u_{n+1} = y_n + hf(x_n, y_n)$ $y_{n+1} = y_n + \frac{h}{2}\left(f(x_n, y_n) + f(x_{n+1}, u_{n+1})\right)$

$u_1 = y_0 + hf(x_0, y_0) = 1 + 0{,}2(0 \cdot 1^2) = \underline{1}$

$y_1 = y_0 + \frac{h}{2}\left(f(x_0, y_0) + f(x_1, u_1)\right) = 1 + \frac{0{,}2}{2}\left(0 + 0{,}2 \cdot 1\right) = \underline{\frac{51}{50}}$

$x_1 = 0{,}2$ $y_1 = \frac{51}{50}$ $x_2 = 0{,}4$

$u_2 = y_1 + hf(x_1, y_1) = \frac{51}{50} + 0.2\left(0{,}2 \cdot \left(\frac{51}{50}\right)^2\right) = \underline{1{,}061616}$

$y_2 = y_1 + \frac{h}{2}\left(f(x_1, y_1) + f(x_2, u_2)\right) = \frac{51}{50} + \frac{0{,}2}{2}\left(0{,}2\left(\frac{51}{50}\right)^2 + 0{,}4 \cdot (1{,}061616)^2\right)$

$= \underline{1{,}085889}$

Feilen er $y(0{,}4) - y_4 \approx \underline{1{,}0674 \cdot 10^{-3}}$ betydelig bedre enn forrige

$\boxed{2}$ d) $y' - xy^2 = 0 \qquad y(0) = 1 \qquad h = 0{,}4$

$f(x,y) = xy^2$

$k_i = f(x_n + c_i h,\ y_n + h\sum_{j=1}^{s} a_{ij}k_j) \quad i = 1,2,\dots s$

$y_{n+1} = y_n + h\sum_{i=1}^{s} b_i k_i$

	c	1	2	3	4
1	0	0	0	0	0
2	1/2	1/2	0	0	0
3	1/2	0	1/2	0	0
4	1	0	0	1	0
	b	1/6	1/3	1/3	1/6

$k_1 = f(x_0 + hc_1,\ y_0 + h\sum_{j=1}^{1} a_{1j}k_j)$

$= (0 + 0{,}4\cdot 0)(1 + 0{,}4(0\cdot k_j))^2 = \underline{0}$

$k_2 = (0 + 0{,}4\cdot 1/2)(1 + 0{,}4(0+0))^2 = \underline{0{,}2}$

$k_3 = (0 + 0{,}4\cdot 1/2)(1 + 0{,}4(0 + 0{,}2\cdot 1/2 + k_3\cdot 0))^2 = \underline{0{,}21632}$

$k_4 = (0 + 0{,}4\cdot 1)(1 + 0{,}4(0 + 0{,}2\cdot 0 + 0{,}21632\cdot 1 + k_4\cdot 0))^2 = \underline{0{,}47221724}$

$y_1 = y_0 + h\sum_{i=1}^{4} b_i k_i$

$= 1 + 0{,}4(1/6\cdot 0 + 1/3\cdot 0{,}2 + 1/3\cdot 0{,}21632 + 1/6\cdot 0{,}47221724)$

$= \underline{\underline{1{,}0869904}}$

Feilen for 4-ordens Runge-Kutta er ca

$|y(0{,}4) - y_1| = \left|\frac{25}{23} - 1{,}0869904\right| = \underline{\underline{3{,}39\cdot 10^{-5}}}$

4-ordens Runge-Kutta var klart best, og var ca. 2 størrelsesordner ~~[illegible]~~ bedre enn Heun.

$\boxed{3}$ a) $u_1'' = -\dfrac{1}{(u_1 - u_2)^2}$ $\qquad u_2'' = \dfrac{1}{(u_1 - u_2)^2}$

$u_1(0) = 0 \quad u_1'(0) = 1 \quad u_2(0) = 1 \quad u_2'(0) = 0$

$y_1(x) = u_1(x) \quad y_2(x) = u_1'(x) \quad y_3(x) = u_2(x) \quad y_4(x) = u_2'(x)$

$y_1'(x) = y_2(x) \qquad y_2'(x) = -\dfrac{1}{(y_1(x) - y_3(x))^2}$

$y_3'(x) = y_4(x) \qquad y_4'(x) = \dfrac{1}{(y_1(x) - y_3(x))^2}$

b)

$$f = \mathbb{y}' = \begin{bmatrix} y_2 \\ -1/(y_1 - y_3)^2 \\ y_4 \\ 1/(y_1 - y_3)^2 \end{bmatrix} \qquad \mathbb{y}_0 = \begin{bmatrix} 0 \\ 1 \\ 1 \\ 0 \end{bmatrix}$$

$$\boxed{\begin{aligned} \mathbb{k}_1 &= f(x_n, \mathbb{y}_n) \\ \mathbb{k}_2 &= f(x_n + h, \mathbb{y}_n + h\mathbb{k}_1) \\ \mathbb{y}_{n+1} &= \mathbb{y}_n + \frac{h}{2}(\mathbb{k}_1 + \mathbb{k}_2) \end{aligned}}$$

$$\mathbb{k}_1 = \mathbb{y}'(\mathbb{y}_0) = \begin{bmatrix} 1 \\ -1/(0-1)^2 \\ 0 \\ 1/(0-1)^2 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \\ 0 \\ 1 \end{bmatrix}$$

$$\underbrace{\mathbb{y}_0 + h\mathbb{k}_1}_{*} = \begin{bmatrix} 0 \\ 1 \\ 1 \\ 0 \end{bmatrix} + 0{,}1 \begin{bmatrix} 1 \\ -1 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 1/10 \\ 9/10 \\ 1 \\ 1/10 \end{bmatrix} \to \mathbb{y}'(*) = \begin{bmatrix} 9/10 \\ -1/(1/10 - 1)^2 \\ 1/10 \\ 1/(9/10 - 1)^2 \end{bmatrix} = \begin{bmatrix} 9/10 \\ -100/81 \\ 1/10 \\ 100/81 \end{bmatrix}$$

$$\mathbb{y}_1 = \mathbb{y}_0 + \frac{h}{2}(\mathbb{k}_1 + \mathbb{k}_2) = \begin{bmatrix} 0 \\ 1 \\ 1 \\ 0 \end{bmatrix} + \frac{0{,}1}{2}\left\{\begin{bmatrix} 1 \\ -1 \\ 0 \\ 1 \end{bmatrix} + \begin{bmatrix} 9/10 \\ -100/81 \\ 1/10 \\ 100/81 \end{bmatrix}\right\} = \begin{bmatrix} 19/200 \\ 1439/1620 \\ 201/200 \\ 181/1620 \end{bmatrix}$$

Stemmer såns noges lunde med numerisk løsning i jupyter..