$\boxed{1}$ a) $F''(x) = kF(x)$ $\quad F(-\pi) - F(\pi)$, $F'(-\pi) = F'(\pi)$ $\quad [-\pi, \pi]$

for $k = 0$:

$F''(x) = 0 \cdot F(x) = 0$

$F = \iint F'' = \iint 0 = \int C_1 = \underline{C_1 x + C_2 = F(x)}$

Setter inn randbetingelser:

$F(-\pi) - F(\pi) = 0 \iff -C_1\pi + \cancel{C_2} - C_1\pi - \cancel{C_2} = 0$

$\Rightarrow -2C_1\pi = 0 \Rightarrow \underline{C_1 = 0}$

$\underline{F(x) = C_2}$ $\quad F'(-\pi) - F'(\pi) = \underline{C_2 - C_2 = 0}$

$C = C_2 \Rightarrow \underline{\underline{F(x) = C}}$

III b) $F''(x) = kF(x)$, $F(-\pi) = F(\pi)$, $F'(-\pi) = F'(\pi)$

$k > 0 \Rightarrow r^2 - k = 0$

$\sqrt{r^2} = \sqrt{k} = r = \pm\sqrt{k}$

Generell løsning : $F(x) = Ae^{\sqrt{k}x} + Be^{-\sqrt{k}x}$

$F(-\pi) = Ae^{-\sqrt{k}\pi} + Be^{\sqrt{k}\pi}$ $F(\pi) = Ae^{\sqrt{k}\pi} + Be^{-\sqrt{k}\pi}$

$F(-\pi) = F(\pi) \Rightarrow A(e^{\sqrt{k}\pi} - e^{-\sqrt{k}\pi}) - B(e^{\sqrt{k}\pi} - e^{-\sqrt{k}\pi}) = 0$

$$\cancel{2}(A-B)\underbrace{\sinh\sqrt{k}\pi}_{\neq 0} = 0$$

$A - B = 0$

$A = B$

$F(x) = A\underbrace{(e^{\sqrt{k}x} + e^{-\sqrt{k}x})}_{\neq 0} = 2A\cosh\sqrt{k}x = \underline{A = 0 = B}$

Altså finnes kun triviell løsning

1 c) $F''(x) = kF(x) \quad F(-\pi) = F(\pi), \; F'(-\pi) = F'(\pi)$

$k < 0$: $\quad k = -p^2 \quad (p > 0)$. Hvis p ikke er et heltall så har (1) bare den trivielle løsning. (Vil vise at)

$$F''(x) - kF(x) = 0$$

$$F''(x) + p^2F(x) = 0$$

$$r^2 + p^2 = 0$$

$$r = \pm\sqrt{-p^2} = \pm pi$$

$\Rightarrow F(x) = A\cos px + B\sin px$ ← generell løsning.

$$F'(x) = -Ap\sin px + Bp\cos px$$

Setter inn randbetingelser

$$F(-\pi) = A\cos p\pi - B\sin p\pi$$

$$F(\pi) = A\cos p\pi + B\sin p\pi$$

$\Rightarrow F(-\pi) - F(\pi) = 0 = 2B\sin p\pi \quad , \sin p\pi = 0 \; , p = 0, \pm 1, \pm 2 \ldots$

$$F'(-\pi) - F'(\pi) = 0 = Ap\sin p\pi + \cancel{Bp\cos p\pi} + Ap\sin p\pi - \cancel{Bp\cos p\pi}$$

$$0 = 2Ap\sin p\pi \Rightarrow p = 0, \pm 1, \pm 2 \ldots$$

Sinus(πn) er kun 0 når n er et heltall.

1 d) $k = -n^2 \qquad n = 1, 2, \dots$

Vil vise at $F_n(x) = C_n \sin nx + D_n \cos nx$

$$F''(x) = kF(x)$$

$$\Rightarrow F''(x) + n^2 F(x) \Rightarrow r^2 = -n^2 \Leftrightarrow r = \pm ni$$

$$F(x) = Ce^{inx} + De^{-inx} = C\cos nx + iC\sin nx$$
$$+ D\cos nx - iD\sin nx$$

$$\frac{1}{2}\left(e^{inx} + e^{-inx}\right) = \cos nx$$

$$\frac{1}{2i}\left(e^{inx} - e^{-inx}\right) = \sin nx$$

$$\Rightarrow \underline{F_n(x) = C\cos nx + D\sin nx} \quad \square$$

Viste i forrige oppgave at det stemmer om n er et heltall.

$\boxed{2}$ $u_{tt}(t,x) = u_{xx}(t,x)$, $u(t,-\pi) = u(t,\pi)$ $t \geq 0$

$u_x(t,-\pi) = u_x(t,\pi)$ $-\pi \leq x \leq \pi$

Antar at $u(t,x) := G(t)\,F(x)$ er en ikke-triviell løsning

a) $u(t,x) = G(t)\,F(x)$

$$u_{tt}(t,x) = \frac{\partial^2 u}{\partial t^2} = \ddot{G}(t)\,F(x)$$

$$u_{xx}(t,x) = \frac{\partial^2 u}{\partial x^2} = G(t)F''(x)$$

$$u_{tt}(t,x) = u_{xx}(t,x) \Leftrightarrow \ddot{G}(t)\,F(x) = G(t)\,F''(x)$$

$$\Leftrightarrow \frac{\ddot{G}(t)}{G(t)} = \frac{F''(x)}{F(x)} = k$$

Vet at begge sider må være konstante fordi VS avhenger av t og HS av x, altså vil kun den ene siden påvirkes om enten t eller x varieres.

$$\Rightarrow \frac{\ddot{G}(t)}{G(t)} = k \Leftrightarrow \underline{\underline{G''(t) = k\,G(t)}}$$

$$\frac{F''(x)}{F(x)} = k \Leftrightarrow \underline{\underline{F''(x) = k\,F(x)}}$$

$\square$

$\boxed{2}$ b) Viste i oppgave $\boxed{1}$ at $F''(x) = kF(x)$ kun har ikke-triviell løsning for $k < 0$.

Setter $k = -p^2$ $(p > 0)$

(1) $F''(x) = k\,F(x) \Rightarrow F''(x) + p^2F(x) = 0$ som har generell løsning:

(2) $\underline{F(x) = A\cos px + B\sin px}$

$$u(t,-\pi) = u(t,\pi) \Leftrightarrow G(t)F(-\pi) = G(t)F(\pi) \Leftrightarrow \boxed{F(-\pi) = F(\pi)}$$

$$u_x(t,-\pi) = u_x(t,\pi) \Leftrightarrow G(t)F'(-\pi) = G(t)F'(\pi) \Leftrightarrow \boxed{F'(-\pi) = F'(\pi)}$$

Viste i oppgave $\boxed{1}$ at $F''(x) = -p^2F(x)$ kun har ikke-trivielle løsninger for $p = 0, 1, 2, \ldots$

(Vil vise at dette holder for $G''(t) = -p^2G(t)$ også :)

c) $u(t,x) = G(t)F(x)$ Har funnet ut at generell løsninger er

$G_n(t) = (K_1)_n \cos nt + (K_2)_n \sin nt$

$F_n(x) = (K_3)_n \cos nx + (K_4)_n \sin nx$, $\underline{k_{1,2,3,4} \text{ konstant}}$

$$u(t,x) = G(t)F(x) = \left((k_1)_n \cos nt + (k_2)_n \sin nt\right) \cdot \left((k_3)_n \cos nx + (k_4)_n \sin nx\right)$$

Definerer: $A_n := (k_1 \cdot k_3)_n$ $B_n := (k_2k_4)_n$ $C_n := (k_1k_4)_n$ $D_n := (k_2k_3)_n$

$$u(t,x) = A_n \cos nt \cos nx + B_n \sin nt \sin nx + C_n \cos nt \sin nx + D_n \sin nt \cos nx$$

$\underline{\text{for } n \neq 0}$

$\boxed{2}$ c) Forts.:

$k = 0$: $\quad F(x) = C_3$

$G''(t) = 0 \cdot G'(t) = 0$

$G = \iint G'' = \iint 0 = \int C_1 = \underline{C_1 t + C_2 = G(t)}$

$u(t,x) = F(x)\,G(t) = (C_1 t + C_2)\,C_3$

$$\boxed{\begin{aligned} A_0 &= C_1 \cdot C_3 \\ B_0 &= C_2 \cdot C_3 \end{aligned}}$$

$\Rightarrow \underline{\underline{u(t,x) = A_0 t + B_0}}$

$\boxed{3}$ (2) $u_{tt}(t,x) = u_{xx}(t,x)$, $u(t,-\pi) = u(t,\pi)$, $u_x(t,-\pi) = u_x(t,\pi)$

a) $u := \sum_{n=0}^{\infty} u_n$, $u_n(t,x) = A_n \cos nt \cos nx + B_n \sin nt \, \cancel{\cos}\sin nx$
$+ C_n \cos nt \sin nx + D_n \sin nt \cos nx$

$(u_n)_{tt} = -A_n n^2 \cos nt \cos nx - B_n n^2 \sin nt \sin nx - C_n n^2 \cos nt \sin nx - D_n n^2 \sin nt \cos nx$

$(u_n)_{xx} = -A_n n^2 \cos nt \cos nx - B_n n^2 \sin nt \sin nx - C_n n^2 \cos nt \sin nx - D_n n^2 \sin nt \cos nx$

$(u_n)_{tt} = (u_n)_{xx}$ ✓ Ser at likheten holder for alle $n \in \mathbb{N}$

~~$u_n(t,\pi) = A_n \cos nt \cos n\pi + B_n \sin nt \sin n\pi + C_n \cos nt \sin n\pi + D_n \sin nt \cos n\pi$~~

$\cos(n(-\pi)) = \cos(n\pi) = (-1)^n$
$\sin((-\pi)n) = \sin(n\pi) = 0$
$\rightarrow u_n(t,-\pi) = u_n(t,\pi)$
og $u_{n_x}(t,-\pi) = u_{n_x}(t,\pi)$

u_n løser (2) for alle n, dermed er $u = \sum_{n=0}^{\infty} u_n$ en løsning av (2)

$\boxed{3}$ b $\quad u := \sum_{n=0}^{\infty} u_n$

$$u_n(t,x) = A_n \cos nt \cos nx + B_n \sin nt \sin nx + C_n \cos nt \sin nx + D_n \sin nt \cos nx$$

$$f(x) = u(0,x) = \sum_{n=0}^{\infty} A_n \cos nx + C_n \sin nx = \underbrace{A_0 + \sum_{n=1}^{\infty} A_n \cos nx + C_n \sin nx}_{\text{Fourierrekken til } u(0,x)}$$

Altså må A_0, A_n og C_n være Fourierkoeffisienter gitt ved:

$$A_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\,dx \qquad A_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,dx \qquad C_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\,dx$$

$$u_t(0,x) = g(x)$$

$$u_{t_n}(t,x) = -A_n n \sin nt \cos nx + B_n n \cos nt \sin nx$$
$$- C_n n \sin nt \sin nx + D_n n \cos nt \cos nx$$

$$g(x) = \sum_{n=0}^{\infty} u_{t_n}(0,x) = \underbrace{\sum_{n=0}^{\infty} B_n n \sin nx + D_n n \cos nx}_{\text{Fourierrekken til } u_{t_n}}$$

Altså må B_n og D_n være Fourierkoeffisienter.

4 a)

$$\left(\hat{f}(\omega)\right)' = \left(\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-i\omega x} e^{-x^2/2}\, dx\right)' = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} \frac{d}{d\omega} e^{-i\omega x} e^{-x^2/2}\, dx$$

10/13

$$= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} -ix e^{-i\omega x} e^{-x^2/2}\, dx = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} \underbrace{-ix e^{-x^2/2}}_{(1)} e^{-i\omega x}\, dx$$

$$(1) \Rightarrow \left(i e^{-x^2/2}\right)' = i e^{-x^2/2} \frac{d}{dx}$$

$$= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} i\underbrace{\left(e^{-x^2/2}\right)'}_{u'} \underbrace{e^{-i\omega x}}_{v}\, dx = \frac{i}{\sqrt{2\pi}}\left(\underbrace{e^{-x^2/2} e^{-i\omega x}\Big|_{-\infty}^{\infty}}_{=0} - (-i\omega)\int_{-\infty}^{\infty} e^{-x^2/2} e^{-i\omega x}\, dx\right)$$

$$= \frac{i}{\sqrt{2\pi}}\left(i\omega\int_{-\infty}^{\infty} e^{-x^2/2} e^{-i\omega x}\, dx\right) = -\omega \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-i\omega x} e^{-x^2/2}\, dx = -\omega \hat{f}(\omega)$$

$\square$

4b)

$$\hat{f}'(\omega) = -\hat{f}(\omega)\omega$$

$$\hat{f}(\omega) = Ce^{-\frac{\omega^2}{2}} = c f(\omega) \quad *$$

$$\cancel{\hat{f}'(\omega) + \hat{f}} \quad \hat{f}'(\omega) + \omega \hat{f}(\omega) = 0 \quad \Big| \cdot e^{\omega^2/2}$$

$$\hat{f}'(\omega)e^{\omega^2/2} + \omega e^{\omega^2/2}\hat{f}(\omega) = 0$$

$$\int \frac{d}{d\omega}\left(\hat{f}(\omega)e^{\omega^2/2}\right) = 0 = \hat{f}(\omega)e^{\omega^2/2} = C$$

$$\hat{f}(\omega) = Ce^{-\omega^2/2} \Rightarrow f(x) = e^{-x^2/2} \Rightarrow \hat{f}(\omega) = \underline{e^{-\omega^2/2}}$$

$\boxed{4}$ c) $f(x) = \mathcal{F}^{-1}(\hat{f}(\omega)) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \hat{f}(\omega) e^{+i\omega x} \, d\omega$

$\hat{f}(\omega) = C f(\omega) \quad \rightarrow \quad \frac{C}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\omega^2/2} e^{+i\omega x} \, d\omega$

$f(x) = e^{-x^2/2} = \frac{C}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\omega^2/2} e^{i\omega x} \, d\omega$

Setter $x = 0$:

$1 = \frac{C}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\omega^2/2} \, d\omega$

$y = -\omega^2/2$

$\frac{dy}{d\omega} = -\omega$

$\boxed{4}$ e)

$$u(t,x) = \frac{1}{\sqrt{2\pi t}} e^{\frac{-x^2}{2t}}$$

$$u_t(t,x) = \frac{d}{d_t}\left(\frac{1}{\sqrt{2\pi t}}\right)e^{-x^2/2t} + \frac{1}{\sqrt{2\pi t}}\frac{d}{dt}\left(e^{-x^2/2t}\right)$$

$$= -\frac{e^{-x^2/2t}}{2\sqrt{2}\,\pi^{1/2}\,t^{3/2}} + \frac{x^2 e^{-x^2/t^2}}{2t^2} \cdot \frac{1}{\sqrt{2\pi t}}$$

$$= \underline{\frac{e^{-x^2/2t}(x^2-t)}{2\sqrt{2}\,\sqrt{\pi}\,t^{5/2}}}$$

$$u_{xx}(t,x) = \underline{\frac{e^{-x^2/2t}(x^2-t)}{\sqrt{2}\,\sqrt{\pi}\,t^{5/2}}}$$

Hopper over kalkulasjonene

$$\underline{\underline{u_t(t,x) = u_{xx}(t,x)}}$$