$\boxed{1}$ $y - y * t = t$

$\overset{\mathcal{L}}{\Rightarrow} \mathcal{L}(y - y * t) = \mathcal{L}(t)$

$\Rightarrow Y - Y \cdot \frac{1}{s^2} = \frac{1}{s^2}$

$\Rightarrow Y\left(1 - \frac{1}{s^2}\right) = Y\left(\frac{s^2 - 1}{s^2}\right) = \frac{1}{s^2}$

$$Y = \frac{\frac{1}{s^2}}{\frac{s^2-1}{s^2}} = \frac{1}{\cancel{s^2}} \cdot \frac{\cancel{s^2}}{s^2 - 1} = \frac{1}{s^2 - 1}$$

$y = \mathcal{L}^{-1}(Y) = \mathcal{L}^{-1}\left(\frac{1}{s^2 - 1}\right) = \underline{\underline{-\sin(t)}}$

$\boxed{2}$ $x' = 2x - y \qquad x(0) = 0 \quad y(0) = 1$

$y' = 3x - 2y$

$\overset{\mathcal{L}}{\Rightarrow} sX - \cancel{x(0)} = 2X - Y \Rightarrow \underline{Y = X(2 - s)}$

$sY - y(0) = 3X - 2Y \Rightarrow \underline{3X = Y(s + 2) - 1}$

$X = \frac{1}{3}\left(X(2-s)\right)(s+2) - \frac{1}{3} \Leftrightarrow \frac{3X + 1}{X} = 4 - s^2$

$\Leftrightarrow \frac{3X}{X} + \frac{1}{X} = 4 - s^2 \Leftrightarrow \frac{1}{X} = 1 - s^2 \Leftrightarrow \underline{\underline{X = \frac{-1}{s^2 - 1}}}$

$\boxed{2}$ forts.

$$Y = X(2-s)$$
$$3X = Y(s+2) - 1 \quad \Rightarrow \quad X = \frac{-1}{s^2-1}$$

$$Y = \frac{-1}{s^2-1}(2-s) = \frac{s-2}{s^2-1} = \frac{s}{s^2-1} - \frac{2}{s^2-1}$$
$$= \frac{s}{s^2-1} + 2\,\frac{-1}{s^2-1}$$

$$x = \mathcal{L}^{-1}\left(\frac{-1}{s^2-1}\right) = \frac{1}{i}\sin(it)$$

$$y = \mathcal{L}^{-1}\left(\frac{s}{s^2+i^2}\right) + \frac{2}{i}\mathcal{L}^{-1}\left(\frac{i}{s^2+i^2}\right)$$

$$y = \cos(it) + \frac{2}{i}\sin(it)$$

$$x = \frac{1}{i}\sin(it), \quad y = \cos(it) + \frac{2}{i}\sin(it)$$

$\boxed{3}$ a) For $n = 0, 1, 2 \ldots$ er

$$\mathcal{L}(t^n) = \frac{n!}{s^{n+1}} = \frac{\Gamma(n+1)}{s^{n+1}}$$

$$\frac{n!}{\cancel{s^{n+1}}} = \frac{\Gamma(n+1)}{\cancel{s^{n+1}}} \Leftrightarrow n! = \int_0^\infty t^{(n+1)-1} e^{-t}\, dt = \int_0^\infty t^n e^{-t}\, dt$$

Vi altså vise at $\int_0^\infty t^n e^{-t}\, dt = n!$ for de ikke-negative heltall.

Base case: $n = 0$

$$n! \Rightarrow \underline{0! = 1}$$

$$\Gamma(0+1) = \int_0^\infty t^0 e^{-t}\, dt = -e^{-t}\Big|_0^\infty = \underline{1}$$

✓

Fakultet er definert som produktet $n \cdot (n-1) \cdot (n-2) \cdot \ldots 1$.
Vil vise at Γ og så oppfører seg sånn for de ikke-negative heltall:

n: $\int_0^\infty t^{n-1} e^{-t}\, dt$ (1)

$$n+1: \int_0^\infty t^n e^{-t}\, dt = \underbrace{-te^{-t}\Big|_0^\infty}_{0} + n \underbrace{\int_0^\infty t^{n-1} e^{-t}\, dt}_{=(1)}$$

✓

$\square$

$\boxed{3}$ b) 4/7

$$\Gamma(x+1) = \int_0^\infty t^{(x+1)-1} e^{-t} dt = -t^{x} e^{-t} \Big|_0^\infty + x \int_0^\infty t^{x-1} e^{-t} dt \quad \square$$

$$= \underline{x\Gamma(x)}$$

$$\Gamma\left(\frac{2k+1}{2}\right) = \Gamma(k+1/2) = \int_0^\infty t^{k+1/2-1} e^{-t} dt$$

$$= \int_0^\infty t^{k-1/2} e^{-t} dt = -t^{k-1/2} e^{-t} \Big|_0^\infty + (k-1/2) \int_0^\infty t^{k-3/2} e^{-t} dt$$

$$= k \int_0^\infty t^{k-3/2} e^{-t} dt - 1/2 \int_0^\infty t^{k-3/2} e^{-t} dt$$

$$\Rightarrow \underline{\Gamma(k+1/2) = k\Gamma(k-1/2) - 1/2\,\Gamma(k-1/2)} \quad \underline{k > 1/2}$$

c) $\Gamma(1/2) = \int_0^\infty t^{-1/2} e^{-t} dt$

$\overset{(*)}{\Rightarrow} \Gamma(1/2) = \int_0^\infty \cancel{t^{-1/2}} e^{-p^2} 2\cancel{t^{1/2}} dp$

$= \underline{2\int_0^\infty e^{-p^2} dp} \quad \square$

$p = t^{1/2}$

$\frac{dp}{dt} = t^{1/2}\frac{d}{dt} = 1/2\, t^{-1/2}$

$\Rightarrow dp = 1/2\, t^{-1/2} dt$

$\Rightarrow \underline{dt = 2t^{1/2}} \quad (*)$

4 a) $x = \sum_{n \neq 0} \frac{i(-1)^n}{n} e^{inx}$ for $-\pi < x < \pi$

$f(x) = x$ Den komplekse Fourierserien av f er gitt ved:

$$f(x) = \sum_{n=-\infty}^{\infty} c_n e^{inx}, \quad c_n = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x) e^{-inx}\,dx$$

$n = 0, \pm 1, \pm 2, \dots$

Finner c_n:

$$c_n = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x) e^{-inx}\,dx = \frac{1}{2\pi}\int_{-\pi}^{\pi} x e^{-inx}$$

$$= \frac{1}{2\pi}\left\{ \frac{-1}{in} x e^{-inx}\Big|_{-\pi}^{\pi} + \frac{1}{in}\int_{-\pi}^{\pi} e^{-inx}\,dx \right.$$

$$= \frac{1}{2\pi}\left\{\left(\frac{-1}{in}x - \frac{1}{(in)^2}\right) e^{-inx}\Big|_{-\pi}^{\pi}\right\}$$

$$e^{-i\pi n} = \cos\pi n \underbrace{- i\sin\pi n}_{=0} = (-1)^n$$

$$e^{-i\pi n} = \cos\pi n + i\sin\pi n = (-1)^n$$

$$= \frac{1}{2\pi}\left\{\frac{1}{n^2} - \frac{x}{in}\right\}_{-\pi}^{\pi}(-1)^n$$

$$= \frac{1}{\cancel{2\pi}}\left\{\cancel{\frac{1}{n^2}} - \frac{\cancel{\pi}}{in} - \left(\cancel{\frac{1}{n^2}} + \frac{\cancel{\pi}}{in}\right)\right\}(-1)^n = \frac{-(-1)^n}{in} \cdot \frac{i}{i} = \boxed{\frac{i(-1)^n}{n} = c_n}$$

$$f(x) = x = \sum_{n=-\infty}^{\infty} c_n e^{inx} = \sum_{n \neq 0} \frac{i(-1)^n}{n} e^{inx} \quad \square$$

$f(x) = x(2\pi - x)$ $\boxed{4}$ b)

$$f(x) = \sum_{n=-\infty}^{\infty} c_n e^{inx}$$

$$c_n = \frac{1}{2\pi}\int_{-\pi}^{\pi} x(2\pi - x)e^{-inx}\,dx = \frac{1}{2\pi}\left\{2\pi\int_{-\pi}^{\pi} xe^{-inx}\,dx - \underbrace{\int_{-\pi}^{\pi} x^2 e^{-inx}\,dx}_{(*)}\right\}$$

$$(*) \Rightarrow \int_{-\pi}^{\pi} x^2 e^{-inx}\,dx = \frac{-x^2}{in}e^{-inx}\Big|_{-\pi}^{\pi} + \frac{2}{in}\int_{-\pi}^{\pi} xe^{-inx}\,dx$$

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} xe^{-inx}\,dx = \frac{i(-1)^n}{n} \iff \boxed{\int_{-\pi}^{\pi} xe^{-inx}\,dx = \frac{2\pi i(-1)^n}{n}}$$

$$c_n = \frac{1}{\cancel{2\pi}}\left\{\cancel{2\pi}\,\frac{2\pi i(-1)^n}{n} + \underbrace{\frac{x^2}{in}e^{-inx}\Big|_{-\pi}^{\pi}}_{=0} - \frac{2}{in}\cdot\frac{\cancel{2\pi}\, i(-1)^n}{n}\right\}$$

$$= \left\{\frac{2\pi i(-1)^n}{n} + \frac{2i(-1)^n}{in^2}\right\} = \boxed{\left\{\frac{2\pi i(-1)^n}{n} + \frac{2(-1)^{n+1}}{n^2}\right\}}$$

$$c_0 = a_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} x(2\pi - x)\,dx = \frac{1}{2\pi}\left\{\underbrace{\int_{-\pi}^{\pi} 2\pi x\,dx}_{\substack{\text{odde} \\ \to\, =0}} - \int_{-\pi}^{\pi} x^2\,dx\right\}$$

4 b) forts.

$$\frac{1}{2\pi}\left\{-\frac{1}{3}x^3\Big|_{-\pi}^{\pi}\right\} = \frac{1}{2\pi}\left(-\frac{1}{3}\pi^3 - \left(-\frac{1}{3}(-\pi)^3\right)\right)$$

$$= -\frac{1}{\cancel{2}\pi}\cdot\frac{\cancel{2}}{3}\pi^{\cancel{3}} = -\frac{\pi^2}{3} = a_0 = c_0$$

$$\Rightarrow \cancel{f(x) = x(2\pi - x) = -\frac{\pi^2}{3} + \sum_{n\neq 0}\left(\frac{2\pi i(-1)^n}{n} + \frac{2(-1)^{(n+1)}}{n^2}\right)}$$

$\Rightarrow$ for $-\pi < x < \pi$ er $f(x) \sim$

$$-\frac{\pi^2}{3} + \sum_{n\neq 0}^{\infty}\left(\frac{2\pi i(-1)^n}{n} + \frac{2(-1)^{n+1}}{n^2}\right)e^{inx}$$

$\square$