1 a) $f(t) = \sinh(At) = \frac{1}{2}(e^{At} - e^{-At})$

$$\mathcal{L}(f) = \frac{1}{2}\mathcal{L}(e^{At}) - \frac{1}{2}\mathcal{L}(e^{-At}) = \frac{1}{2}\left(\frac{1}{S+A} - \frac{1}{S-A}\right)$$

$$= \frac{1}{2}\left(\frac{S - A + S + A}{S^2 - A^2}\right) = \underline{\underline{-\frac{A}{S^2-A^2}}}$$

b) $f(t) = \cosh(At) = \frac{1}{2}(e^{At} + e^{-At})$

$$\mathcal{L}(f) = \frac{1}{2}\left(\mathcal{L}(e^{At}) + \mathcal{L}(e^{-At})\right) = \frac{1}{2}\left(\frac{1}{S+A} + \frac{1}{S-A}\right)$$

$$= \frac{1}{2}\left(\frac{S-A+S+A}{S^2-A^2}\right) = \underline{\underline{\frac{S}{S^2-A^2}}}$$

c) $f(t) = \begin{cases} 0, & 0 < t < \pi \\ 1, & \text{ellers} \end{cases} \Leftarrow f(t) = u(t-\pi)$

$$\mathcal{L}(f) = \int_0^\infty e^{-st} u(t-\pi)\,dt = \int_\pi^\infty e^{-st} \cdot 1\,dt = \left[-\frac{1}{s}e^{-st}\right]_\pi^\infty$$

$$= 0 - \left(-\frac{1}{s}e^{-s\pi}\right) = \underline{\underline{\frac{e^{-s\pi}}{s}}}$$

1d) $f(t) = \begin{cases} 0, & 0 < t < \pi \\ \cancel{\ } \cos t, & \text{ellers} \end{cases} = \cos(t)\,u(t\cancel{\ }-\pi)$

$$\mathcal{L}(f) = \int_{\pi}^{\infty} e^{-st}\cos(t)\,dt = \left[\frac{e^{-st}(\sin t - s\cdot\cos t)}{s^2+1}\right]_{\pi}^{\infty}$$

$$\Rightarrow 0 - \left(\frac{e^{-s\pi}\cdot(\cancel{\ }s)}{s^2+1}\right) = \underline{\underline{\frac{se^{-s\pi}}{s^2+1}}}$$

e) $f(t) = t^2 e^t$

$$\mathcal{L}(f) = \mathcal{L}((t-1)^2) = \underline{\underline{\frac{2}{(t-1)^3}}}$$

f) $f(t) = e^t \cos(t) \cancel{\Rightarrow}$

$$\mathcal{L}(f) = \mathcal{L}(\cos(t-1)) = \underline{\underline{\frac{s-1}{(s-1)^2+1}}}$$

g) $f(t) = e^t \sin(t)$

$$\mathcal{L}(f) = \mathcal{L}(\sin(t-1)) = \underline{\underline{\frac{1}{(s-1)^2+1}}}$$

$\boxed{2}$ a) $y'' - 2y' + 2y = 6e^{-t}$; $y(0) = 0$ $y'(0) = 1$

$$\overset{\mathcal{L}}{\Longrightarrow} s^2Y - 2s\underbrace{y(0)}_{=0} - \underbrace{y'(0)}_{=1} - 2(sY - \underbrace{y(0)}_{=0}) + 2Y = 6\,\frac{1}{s+1}$$

$$Y(s^2 - 2s + 2) - 1 = \frac{6}{s+1}$$

$$Y = \frac{s+7}{(s^2-2s+2)(s+1)} = \frac{As+B}{s^2-2s+2} + \frac{C}{s+1}$$

$$\Rightarrow As^2 + As + Bs + B + Cs^2 - 2Cs + 2C = s + 7$$

$$\Rightarrow A + C = 0 \qquad A + B - 2C = 1 \qquad B + 2C = 7$$

$$A = -C \qquad -C + 7 - 2C - 2C = 1 \qquad B = 7 - 2C$$

$$-5C = -6$$

$$\underline{A = -\frac{6}{5}} \qquad \underline{C = \frac{6}{5}} \qquad \underline{B = \frac{23}{5}}$$

$$\Rightarrow \frac{-\frac{6}{5}s + \frac{23}{5}}{(s-1)^2+1} + \frac{\frac{6}{5}}{s+1}$$

$$\overset{\mathcal{L}^{-1}}{\Longrightarrow} y(t) = \frac{6}{5}e^{-t} - \frac{6}{5}e^{t}\cos t + \frac{6}{5}e^{t}\sin t$$

$\boxed{2}$ $\underline{b)}$ $y'' + y = f(t)$, $f(t) = u(t-\pi)$

$y(0) = 0 \quad y'(0) = 0$

$\overset{\mathcal{L}}{\Longrightarrow} \cancel{*}\, s^2 Y - \underbrace{s f(0) - f'(0)}_{=0} + Y = \int_{\pi}^{\infty} e^{-st}\, dt$

$(s^2+1)Y = \frac{1}{s} e^{-s\pi} \iff Y = \frac{e^{-s\pi}}{(s^2+1)\cdot s}$

$\underline{\frac{1}{(s^2+1)s} = \frac{As+B}{s^2+1} + \frac{C}{s}} \Rightarrow As^2 + Bs + Cs^2 + C$

$\Rightarrow (A+C)s^2 + Bs + C \Rightarrow C = 1$

$A = -1$

$B = 0$

$\Rightarrow Y = e^{-s\pi}\left(-\frac{s}{s^2+1} + \frac{1}{s}\right)$

$\overset{\mathcal{L}^{-1}}{\Longrightarrow} y = -\cos(t)u(t-\pi) + u(t-\pi)$

$\underline{\underline{= (1-\cos(t))u(t-\pi)}}$

$\boxed{3}$

$$f(t+T) = f(t) \qquad T > 0$$

$$\mathcal{L}(f)(s) = \frac{1}{1-e^{-sT}} \int_0^T e^{-st} f(t)\,dt$$

$$\Rightarrow \sum_{n=0}^{\infty} \int_{nT}^{(n+1)T} e^{-st} f(t)\,dt \qquad \underline{t = z + nT}$$

$$\Rightarrow \sum_{n=0}^{\infty} \int_0^T e^{-s(z+nT)} f(z)\,dz = \underbrace{\sum_{n=0}^{\infty} e^{-snT}}_{\text{Geometrisk Rekke}} \int_0^T e^{-sz} f(z)\,dz$$

$$\left(\sum_{n=0}^{\infty} r^n = \frac{1-r^n}{1-r} \right)$$

$$\Rightarrow \lim_{n \to \infty} \frac{1-e^{-snT}}{1-e^{-sT}} \int_0^T e^{-sz} f(z)\,dz = \frac{1}{1-e^{-sT}} \int_0^T e^{-st} f(t)\,dt \quad \square$$

4

$$\mathcal{L}(h_m f)(s) = (-1)^m \frac{d^m \mathcal{L}(f)}{ds^m}(s)$$

$$\mathcal{L}(h_m)(s) = \frac{n!}{s^{n+1}}$$

$$h_n = t^n \quad , n = 1, 2, 3\ldots \qquad f(t) = 1$$

$$\mathcal{L}(h_1) = (-1)^1 \cdot \frac{d\,\mathcal{L}(1)}{ds}(s) \rightarrow \left(-\frac{1}{s}\right)\frac{d}{ds} = \underline{\frac{1}{s^2}}$$

$$\mathcal{L}(h_3) = (-1)^3 \cdot \frac{d^3 \mathcal{L}(1)}{(ds)^3}(s) = \left(-\frac{1}{s}\right)\left(\frac{d}{ds}\right)^3 = \frac{1}{s^2}\left(\frac{d}{ds}\right)^2$$

$$\left(-\frac{2}{s^3}\right)\left(\frac{d}{ds}\right) = \underline{\underline{\frac{6}{s^4}}}$$

Ser at å ta Laplace-transformasjonen av $h_m \cdot f$ tilsvarer å derivere $\mathcal{L}(f)$ m ganger.
Nærliggende å tro at det blir tilsvarende med $\mathcal{L}\left(\frac{f}{h_1}\right)$, bare med integrasjon.

$$\mathcal{L}\left(\frac{f}{h_1}\right), \quad f(t) = 1 \Rightarrow \mathcal{L}\left(\frac{1}{t}\right) = \mathcal{L}(t^{-1})$$

$$= \underline{\int_0^\infty e^{-st} \cdot t^{-1}\, dt} \quad (*)$$

I likhet med integralet $\int_0^\infty \frac{1}{t}\, dt$, divergerer $(*)$, og derfor

<u>finnes ikke Laplace-transformasjonen.</u>