2 a) $y'(x) = f(y(x))$, $y(x_0) = y_0$

$k_1 = f(y_n)$ $\qquad k_2 = f(y_n + h a_{21} k_1)$

$y_{n+1} = y_n + h(b_1 k_1 + b_2 k_2)$

~~Taylorutvidelsen av $y(x_0+h)$~~

Derivarte av y:

$y'(x) = f(y(x)) = \underline{f}$

$y''(x) = (y'(x))' = (f(y(x)))' = f' \cdot y'(x) = \underline{f' \cdot f}$

$y'''(x) = (f'(y(x)) \cdot f(y(x)))' = f''(y(x)) \cdot y'(x) \cdot f(y(x)) + f'(y(x)) f'(y(x)) \cdot y'(x)$

$= \underline{f'' \cdot f^2 + (f')^2 f}$

Taylorutvidelse av $y(x_0+h)$:

$y(x_0+h) = y_0 + hf + \frac{h^2}{2}(f' \cdot f) + \frac{h^3}{6}(f'' f^2 + (f')^2 f)$

2 a) forts.

Taylorrekken til den numeriske tilnærmingen:

$k_1 = f$

$k_2 = f(y_0 + h a_{21} k_1)$

$= f + f' h a_{21} k_1 + \frac{1}{2}(h^2 a_{21}^2 k_1^2)(f'' \cdot f^2 + (f')^2 f)$

$= f + f'f h a_{21} + \frac{h^2}{2} a_{21}^2 k^2 (f'' \cdot f^2 + (f')^2 f)$

$y_1 = y_0 + h(b_1 k_1 + b_2 k_2)$

$= y_0 + b_1 h f + h b_2 (f + f'f h a_{21} + \frac{h^2}{2} a_{21}^2 (f'' f^4 + (f')^2 f^3))$

$= y_0 + h f (b_1 + b_2) + h^2 f' f b_2 a_{21} + \frac{h^3}{2}(f'' f^4 + (f')^2 f^3) a_{21}^2 b_2^2$

Lokal avbruddsfeil:

$y(x_0 + h) - y_1 = \cancel{y_0} + hf + \frac{h^2}{2} f' f + \frac{h^3}{6}(f'' f^2 + (f')^2 f)$
$- (\cancel{y_0} + hf(b_1 + b_2) + h^2 f' f b_2 a_{21} + \frac{h^3}{2}(f'' f^4 + (f')^2 f^3) a_{21}^2 b_2^2)$

$\Rightarrow hf(1 - (b_1 + b_2)) + h^2 f' f (\frac{1}{2} - b_2 a_{21}) + \frac{h^3}{2}(f'' f^2 + (f')^2 f)(\frac{1}{3} + f^2 a_{21}^2 b_2^2)$

For at metoden skal være av 1. orden må: $\underline{b_1 + b_2 = 1}$

Kravet for 2. orden: $h^2 f' f (\frac{1}{2} - b_2 a_{21}) = 0 \Rightarrow \underline{b_2 a_{21} = \frac{1}{2}}$

b_2 og a_{21} er allerede låst i kravet ovenfor så nei, den kan ikke være av 3. orden – man trenger flere grader av frihet.

b) Et valg av noder som gir en metode av 2. orden (optimalt) er $\underline{b_1 = b_2 = \frac{1}{2}}$ $\underline{a_{21} = 1}$

3 a) Testligning: $y' = \lambda y \quad y(0) = y_0 \qquad \lambda \in \mathbb{R} \quad \lambda < 0$ Side 3

$y(x) = e^{\lambda x} y_0$

Ett steg med Runge-Kutta brukt på testligningen kan skrives $y_{n+1} = R(z) y_n$

$z = \lambda h$

$$k_1 = \lambda y_n$$

$$k_2 = \lambda\left(y_n + \frac{h}{2}\lambda y_n\right) = \lambda\left(1 + \frac{z}{2}\right) y_n$$

$$k_3 = \lambda\left(y_n + \frac{3}{4}h \cdot \lambda\left(1 + \frac{z}{2}\right) y_n\right) = \lambda\left(1 + \frac{3z}{4}\left(1 + \frac{z}{2}\right)\right) y_n = \lambda\left(1 + \frac{3 \cdot z}{4} + \frac{3z^2}{8}\right) y_n$$

$$y_{n+1} = y_n + \frac{h}{9}\left(2k_1 + 3k_2 + 4k_3\right)$$

$$= y_n + \frac{h}{9}\left(2 \cdot \lambda y_n + 3 \cdot \lambda\left(1 + \frac{z}{2}\right) y_n + 4 \cdot \lambda\left(1 + \frac{3z}{4} + \frac{3z^2}{8}\right) y_n\right)$$

$$= y_n + \frac{1}{9}\left(2z + 3z + \frac{3z^2}{2} + 4z + \frac{12z^2}{4} + \frac{12z^3}{8}\right) y_n$$

$$= y_n\left(1 + \frac{z}{9} + \frac{3z}{9} + \frac{3z^2}{18} + \frac{4z}{9} + \frac{3z^2}{9} + \frac{z^3}{6}\right)$$

$$= y_n\left(1 + z + \frac{1}{2}z^2 + \frac{1}{3}z^3\right) \longrightarrow R(z) = 1 + z + \frac{1}{2}z^2 + \frac{1}{6}z^3$$

$$S \approx [-2{,}51 \, , \, 0]$$

$\boxed{3}$ b) $y' = Ay$, $A = \begin{bmatrix} -41 & 38 \\ 19 & -22 \end{bmatrix}$

Egenverdier:

$$|A - \lambda I| = 0 \Rightarrow \begin{vmatrix} -41-\lambda & 38 \\ 19 & -22-\lambda \end{vmatrix} = 0 \Rightarrow (-41-\lambda)(-22-\lambda) - 38 \cdot 19 = 0$$

$$\Rightarrow \lambda^2 + 63 + 180 = 0 \Rightarrow (\lambda + 3)(\lambda + 60) = 0 \Rightarrow \underline{\lambda_1 = -3} \quad \underline{\underline{\lambda_2 = -60}}$$

For å få en stabil løsning må man finne en h s.a. $z = \lambda h \in S$:

~~$S = [-1{,}87, 0] \rightarrow -1{,}87 \leq z \leq 0$~~

~~$\Rightarrow -1{,}87 \leq \lambda h \leq 0 \Rightarrow \frac{-1{,}87}{\lambda} \geq h \geq 0$~~

~~Velger største $|\lambda|$ $\Rightarrow h \leq \frac{-1{,}87}{\lambda_2} = \frac{-1{,}87}{-60} = 0{,}0312$~~

~~Maksimal h~~

$$S \approx [-2{,}51, 0] \rightarrow -2{,}51 \leq z \leq 0$$

$$\Rightarrow -2{,}51 \leq \lambda h \leq 0 \Rightarrow \frac{-2{,}51}{\lambda} \geq h \geq 0$$

Velger største $|\lambda|$ $\Rightarrow h \leq \frac{-2{,}51}{-60} \approx \underline{\underline{0{,}042}}$

$\boxed{4}$ a)

$$y_{n+1} = y_n + h f\left(x_n + \frac{h}{2}, \frac{1}{2}(y_n + y_{n+1})\right)$$

Hvis den skal være A(0) - stabil må $|R(z)| \leq 1$ for alle $z \leq 0$.

$$y' = \lambda y$$

$$y_{n+1} = y_n + h\lambda \frac{1}{2}(y_n + y_{n+1}) = y_n + \frac{z}{2} y_n + \frac{z}{2} y_{n+1}$$

$$\Rightarrow y_{n+1}\left(1 - \frac{z}{2}\right) = y_n\left(1 + \frac{z}{2}\right) \Rightarrow y_{n+1} = y_n \frac{1 + \frac{z}{2}}{1 - \frac{z}{2}}$$

$$\underline{R(z) = \frac{2 + z}{2 - z}}$$

$$\lim_{z \to -\infty} R(z) = \lim_{z \to -\infty} \frac{\frac{2}{z} + 1}{\frac{2}{z} - 1} = \underline{-1} \qquad R(0) = \underline{1}$$

$$R'(z) = \frac{1 \cdot (2 - z) - (2 + z) \cdot (-1)}{(2 - z)^2} = \frac{4}{(2 - z)^2} > 0 \text{ for alle } z$$

R er strengt økende på hele $\mathbb{R}$, samt at $\lim_{z \to -\infty} R(z) = -1$ og $R(0) = 1$, altså er $|R(z)| \leq 1$ for alle $z \leq 0$ og den implisitte midtpunktregelen er A(0) - stabil. $\square$

$\boxed{5}$ a) $P(\theta) = a\theta^3 + b\theta^2 + c\theta + d$

$P(0) = y_n = d$

$p'(0) = hy'_n = c$

$P(1) = a + b + c + d = a + b + y'_n + y_n = y_{n+1}$

$\Rightarrow a = y_{n+1} - b - hy'_n - y_n$

$P'(1) = 3a + 2b + c = 3a + 2b + y'_n = hy'_{n+1}$

$\Rightarrow 3(y_{n+1} - b - hy'_n - y_n) + 2b + y'_n$

$b = -3y_n - 2hy'_n - 3y_{n+1} - hy'_{n+1}$

$a = y_n - (-3y_n - 2hy'_n - 3y_{n+1} - hy'_{n+1}) - hy'_n - y_n$

$a = 4y_{n+1} + 2y_n + hy'_n + hy'_{n+1}$

$$P(\theta) = (2y_n + hy'_n + 4y_{n+1} + hy'_{n+1})\theta^3$$
$$+ (-3y_n - 2hy'_n - 3y_{n+1} - hy'_{n+1})\theta^2$$
$$+ hy'_n\theta + y_n$$

b) $y_n = 6.03 \quad y_{n+1} = 5.91, \quad y'_n = 5.41 \quad y'_{n+1} = -8.29 \quad h = 0.5$

$$P(\theta) = (2 \cdot 6.03 + 0.5 \cdot 5.41 + 4 \cdot 5.91 + 0.5 \cdot (-8.29))\theta^3$$
$$+ (-3 \cdot 6.03 - 2 \cdot 0.5 \cdot 5.41 - 3 \cdot 5.91 - 0.5 \cdot (-8.29))\theta^2$$
$$+ 0.5 \cdot 5.41 \cdot \theta + 6.03$$

$$P(\theta) = 34.26\theta^3 - 37.085\theta^2 + 2.705\theta + 6.03$$