→ Mean (Expected value) of a Binomial Distribution :→

$\mu = np$

Proof:

Expected value of random variable X is given by:

$E(x) = \sum_{k=0}^{n} k \cdot P(X = k)$

For a binomial distribution, the probability mass function (PMF) is given by:

$P(X = k) = \binom{n}{k} \cdot p^k \cdot (1-p)^{n-k}$

So, the expected value becomes:

$E(x) = \sum_{k=0}^{n} n \cdot \binom{n-1}{k-1} \cdot p^k \cdot (1-p)^{n-k}$

Let's factor out n and write the summation:

$E(x) = n \cdot \sum_{k=0}^{n} \binom{n-1}{k-1} \cdot p^k \cdot (1-p)^{n-k}$

This summation is equivalent to the probability mass function of a binomial distribution with $n-1$ trials:

$E(x) = n \cdot \underbrace{\sum_{k=0}^{n-1} \binom{n-1}{k} \cdot p^k \cdot (1-p)^{n-1-k}}_{=1}$

* Now, recognizing this is the PMF of a binomial distribution with $n-1$ trials, the sum is equal to 1 :-

$E(x) = n \cdot 1 = n$

∴ therefore the mean (μ) of a binomial distribution is np

→ Variance of a Binomial Distribution:

The variance of a binomial distribution is given by:

$\sigma^2 = np(1-p)$

Proof:

The variance (σ^2) of a random variable X is given by:

$\sigma^2 = E(x^2) - [E(x)]^2$

We have already established that $E(X) = np$.

Now, let's find $E(x)^2$:

$$E(x^2) = \sum_{k=0}^{n} k^2 \cdot P(X=k)$$

Substitution the binomial PMF, we get:

$$E(x^2) = \sum_{k=0}^{n} k^2 \cdot \binom{n}{k} \cdot p^k \cdot (1-p)^{n-k}$$

This expression can be quite complex, but by using some algebric manipulation and identities, we can simplify it to:

$E(x^2) = np(1-p) + n(n-1)p^2$

now, substitue this into the variance formula

$\sigma^2 = np(1-p) + n(n-1)p^2 - (np)^2$

Simplify the expression:

$\sigma^2 = np(1-p) + np^2(n-1) - np^2$

Combine like terms:

$\sigma^2 = np(1-p)$ $\therefore \sigma^2 = np(1-p)$

AL

Lets proof in complex version:-

Proof using Generating Functions:

(PGF) : The probability generating Function (PGF) for a random variable X with a probability mass Function $P(X=k)$ is given by:

$$G_X(t) = \sum_{k=0}^{n} P(X=k) \cdot t^k$$

For PMF in binomial distribution, the PMF is $P(X=k) = \binom{n}{k} \cdot p^k \cdot (1-p)^{n-k}$. we want to find the generating function $G_X(t)$ for this distribution

$$G_X(t) = \sum_{k=0}^{n} \binom{n}{k} \cdot p^k \cdot (1-p)^{n-k} \cdot t^k$$

Using the binomial theorem, which states that $(a+b)^n = \sum_{k=0}^{n} \binom{n}{k} \cdot a^k \cdot b^{n-k}$, we can rewrite the expression as:

$$G_X(t) = \sum_{k=0}^{n} \binom{n}{k} \cdot (p \cdot t)^k \cdot (1-p)^{n-k}$$

Now, recognize that this is the binomial expression of $(p \cdot t + (1-p))^n$. Therefore:

$$G_X(t) = (p \cdot t + (1-p))^n$$

The coefficient of t^k in the expansion of $(p \cdot t + (1-p))^n$ is exactly $P(X=k)$, which proves that $G_X(t)$ is the PGF for a binomial distribution.

Now, we can find the mean by evaluating $G'_X(t)$ (the first derivative of $G_X(t)$ with respect to t evaluate at $t=1$):

$$G'_X(t) = n \cdot p \cdot n(p \cdot t + (1-p))^{n-1}$$

$$G'_X(1) = n \cdot p$$

This is consistent with the mean of the binomial distribution, which $\mu = np$.

For the variance, we need to find $G''_{(X)}(1)$ (the second derivative of $G_X(t)$ with respect to t evaluate at $t=1$):

$$G''_X(t) = n \cdot p \cdot (n \cdot p \cdot t + n \cdot (1-p)) \cdot p(p \cdot t + (1-p))^{n-2}$$

$$G''_X(1) = n \cdot p \cdot (n \cdot p + n(1-p))$$

$$G''_X(1) = n \cdot p \cdot (n \cdot p + n - n \cdot p)$$

$$G''_X(1) = n \cdot p \cdot n$$

This is consitent with the variance of the binomial distribution, which is $\sigma^2 = np(1-p)$.

Using PGF, we can derive the mean and varience of the binomial distribution.

→ Proof of the Binomial Distribution using Gamma Function.

Proof using Gamma Function:

~~The Gamma~~

The gamma function is defined as $\Gamma(n) = \int_0^{\infty} t^{n-1} e^{-t} dt$ for $n > 0$.

Now, consider the binomial coefficient $\binom{n}{k}$ in the binomial PMF:

$$\binom{n}{k} = \frac{n!}{k!(n-k)!}$$

Let's express the factorials using the gamma function:

$$n! = \Gamma(n+1) = \int_0^{\infty} t^n e^{-1} dt$$

$$k! = \Gamma(k+1) = \int_0^{\infty} u^k e^{-u} du$$

$$(n-k)! = \Gamma(n-k+1) = \int_0^{\infty} v^{n-k} e^{-v} dv$$

Now, substitue these expression back into the binomial coefficient:

$$\binom{n}{k} = \frac{\Gamma(n+1)}{\Gamma(k+1)\Gamma(n-k+1)}$$

Now, Let's incorporate this into the binomial PMF:

$$P(x=k) = \binom{n}{k} \cdot p^k \cdot (1-p)^{n-k}$$

$$P(x=k) = \frac{\Gamma(n+1)}{\Gamma(k+1)\Gamma(n-k+1)} \cdot p^k \cdot (1-p)^{n-k}$$

Proof :→

1. Binomial Distribution.

2. Probability Mass Function (PMF) of Binomial Distribution

→ The PMF of the Binomial distribution is given by:

$P(X=k) = \binom{n}{k} \cdot p^k \cdot (1-p)^{n-k}$

3. Stirling's ~~Application~~ Approximation:

→ Stirling's approximation states that for large n, $n!$ can be approximated as:

$n! \approx \sqrt{2\pi n}\left(\frac{n}{e}\right)^n$

→ we will use this approximation for large valle of n.

4. Binomial coefficient Approximation

Apply simplifying, many terms will cancel out, leaving:

$$\binom{n}{k} \approx \frac{n^k}{k!} \cdot \frac{1}{\sqrt{2\pi n}} \cdot \frac{1}{\sqrt{2\pi(n-k)}}$$

The simplification is valid for large n.

5. Simplify the Expression:

After simplifying, many term cancel we take combining term.

Substitute the approximated binomial cofficient into the PMF of the binomial Distribution

Taking the limit

As n approach infinity, p approaches zero in such a way that np remains constant, denoted as λ

Rewrite $p = \frac{\lambda}{n}$ and let n approach infinity.

As n goes to infinity, terms involving n will dominate leave.

$$P(X=k) \approx \lim_{n \to \infty} \left(\frac{n^k}{k!} \cdot \frac{1}{\sqrt{2\pi n}} \cdot \frac{1}{\sqrt{2\pi(n-k}} \cdot \right) \cdot \left(\frac{\lambda^k}{n^k} \right) \cdot \left(\frac{(1-\lambda/n)^{n-k}}{\sqrt{1-\lambda/n}} \right)$$

8 Simplify further.

→ After simplifying and taking the limit as n approaches infinity, we obtain the PMF of the poisson distribution.

$$P(X=k) = \frac{e^{-\lambda} \cdot \lambda^k}{k!}$$

Proof:-

Binomial Distribution

- consider a sequence of Independent Bernoulli trials with probability of success p and probability of failure 1-p.

Let x be the random variable representing the number of trials until the first success.

Probability of first success on the k-th trials:

→ In the binomial distribution, the probability of observing the first success on the k-th trials is given by the Binomial PMF:

$$P(X=k) = \binom{k-1}{0} \cdot p^1 \cdot (1-p)^{k-1}$$

This Simplifies to:

$$P(x=k) = p \cdot (1-p)^{k-1}$$

Simplify the Expression:

When $r=1$ (i.e, we're interested in the first success), the binomial PMF further simplifies to:

$$P(x=k) = (1-p)^{k-1} \cdot p$$

Case II

Proof:

Random Sample size → N

known μ, unknown σ^2

Liklihood function

$$L(\sigma^2 | x_1, x_2, x_3 \dots x_N) = \prod_{i=1}^{n} \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(x_i-\mu)^2}{2\sigma^2}}$$

MLE for σ^2, we maximize the liklehood function with respect to σ^2

taking derivative of Log- likelyhood function.

$$\frac{d \log L}{d\sigma^2} (\sigma^2 / x_1, x_2 \dots x_n) = 0$$

$$\frac{d}{d\sigma^2} \left(-\frac{n}{2} \log(2\pi\sigma^2) - \frac{1}{2\sigma^2} \sum_{i=1}^{n} (x_i - \mu)^2 \right) = 0$$

$$\sigma^2 = \frac{1}{n} \sum_{i=1}^{n} (x_i - \mu)^2$$

without Bessel's correction since μ is known.

Case - III

MLE for both

Sample $(\bar{u}) = \frac{1}{n} \Sigma x_i$

Sample $(S^2) = \frac{1}{n-1} \Sigma (x_i - \mu)^2$ where x_i are the observed data point and n is the sample size

$$L(\mu, \sigma^2 | x_1, x_2, x_3 \ldots x_n)$$
$$= \prod_{i=1}^{n} \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(x_i-\mu)^2}{2\sigma^2}}$$

To find MLE, we maximize Likelehood Function with respect to μ and σ^2.

Estimate mean (μ)

$$\frac{\partial}{\partial \mu} \log L(\mu, \sigma^2 / x_1, x_2, x_3 \ldots x_n) = 0$$

$$\frac{\partial}{\partial \mu} \left(\frac{-n}{2} \log (2\pi\sigma^2) - \frac{1}{2\sigma^2} \sum_{i=1}^{n} (x_i - \mu)^2 \right) = 0$$

$$\hat{\mu} = \frac{1}{n} \sum_{i=1}^{n} x_i$$

Estimate variance (σ^2)

$$\frac{\partial}{\partial \sigma^2} \log L(\mu, \sigma^2 | x_1, x_2, x_3 \cdots x_n) = 0$$

$$\frac{\partial}{\partial \sigma^2} \left(\frac{-n}{2} \log(2\pi\sigma^2) = \frac{1}{2\sigma^2} \sum_{i=1}^{n} (x_i - \mu)^2 \right) = 0$$

$$\hat{\sigma}^2 = \frac{1}{n} \sum_{i=1}^{n} (x_i - \hat{\mu})^2$$

However, it is well known that result in stastic that the MLE, for the varience σ^2 is baised when the estimated with $\frac{1}{n}$, so, to corrected for this bias, the unbaised estimation for the varience.

$$\sigma^2 = \frac{1}{n-1} \sum_{i=1}^{n} (x_i - \mu)^2$$

known as Bessel's correction

Metropolis-Hastings

$$f(a, b, V) = \frac{1}{1 + \exp(-(V - a)b)}, \qquad (62)$$

$F(a,b,V) = \dfrac{1}{(1+\exp^{-}(V-a)\,b)}$

→ Formula based is two state boltzman distribution that exclude describes relationship b/w voltage and the probability of a voltage-gate ion channel.

$$F(a,b,v) = \frac{1}{1+\exp^{\circ}(-(v-a)b))}$$

derive by boltzman distribution.

Inthis case energy is related to the voltage and the mid point voltage (a) is the energy. at which channel is equal like to be open or close. The stop function (b) how do quickly probable changing as the voltage is baised.

PROOF

$$P(E) = \frac{\exp(-E/kT)}{Z} \quad -(1)$$

T → Temperature

E → P(v)

k → Boltzman constant.

Z → partition function, which normalize, the distribution so that probability sum to 1

$$E = -(v-a) \quad -(11) \qquad E \propto V$$

a → mid point of voltage

(11) in (1)

$$P(v) = \exp((v-a)/kT)/Z$$

Simplify this function we can define new variable, the slope factor (b) as $b = kT$

now

$$P(v) = \exp((v-a)/b)/Z$$

now to normalize the distribution we need to find partition function F (z) which is given by

$$z = \int \exp (v-a)/b$$

$$z = b \int \exp (u)\, du$$

$$z = b e^{u}$$

$$z = b e^{\wedge} ((v-a)(b))$$

Substituting this back to express for p(v) we get

$$P(v) = \exp ((v-a)/b) / b e^{\wedge} ((v-a)/b)$$

Simplify this express we get.

$$P(v) = \frac{1}{1+\exp(-(v-a)(b))}$$

multiply a and b by $\exp(v-a)/b)$

$$P(v) = \exp((v-a)/b) / \exp((v-a)/b)+1)$$

by changing new variable

$$F(a,b,v) = P(v)$$

$$F(a,b,v) = \frac{1}{1+\exp(-(v-a)b))}.$$

pd

How this equation formed let's see in each step

$$y_i \sim f(a,b,V_i) + N(0,\sigma^2), \tag{63}$$

where $N(\mu,\sigma)$ denotes a normal distribution with mean μ and standard deviation σ. Given this, our likelihood function is simply a normal distribution centered at f and with variance σ^2,

$$p(y_i|\ldots) = N(f(a,b,V_i),\sigma^2). \tag{64}$$

We assume that each data point arises from f and some independent and identically distributed noise, so the posterior distribution is

$$p(a,b,\sigma^2|y_N) \propto \left(\prod_{i=1}^{N} N(f(a,b,V_i),\sigma^2)\right) p(a)p(b)p(\sigma^2). \tag{65}$$

$$y_i \sim f(a, b, v_i) + N(0, \sigma^2)$$

f (centered at mean)

$$P(y_i | \ldots\ldots) = N(f(a, b, v_i), \sigma^2)$$

Let this

$y_i = y_1, y_2, \ldots\ldots y_N$

$P(y_N | \ldots)$ = when it is comes to normal distribution.

.

$P((a, b, \sigma^2) / y_N) =$

let this

$P(B/A)$

so,

$y_i \sim F(a, b, v_i) + N(0, \sigma^2)$

F (centred at mean)

$P(y_i \,/\, \ldots) = N(F(a, b, v_i), \sigma^2)$

$y_i = y_1, y_2, y_3 \ldots y_N$

$P(y_N \mid \ldots)$ when it is cores to Normal distribution

→ Let this be

$P(A/B) = N(F.(a, b, v_i), \sigma^2)$

So,

$$P(B/A) = \frac{P(A/B) \cdot P(B)}{\text{evidence } P(A)}$$

$P(a, b, \sigma^2 / y_N) = N(F(a, b, v_i), \sigma^2) \cdot P(a) \cdot P(b) \cdot P(\sigma^2)$

Here

Posterior $\propto$ likelyhood × prior.

$P(a, b, \sigma^2 / y_N) = \prod_{i=1}^{N} (N(F(a, b, v_i), \sigma^2) \, P(a) \, P(b) \, P(\sigma^2)$