# Modeling a RLC Circuits with Differential Equations

Teja Aluru and Aaron Osier

May 16, 2014

### Abstract

This paper will explain basic concepts in the field of signal processing. We are going to create and mathematically model an AM Radio tuner, using an RLC Circuit and our knowledge of differential equations. In order to create the AM Radio Tuner, we must make an RLC circuit, which can be known as a "tuning" circuit or band-pass filter. We will then model our specific circuit using an ordinary differential equation that models the capacitance, inductance, resistance, and driving voltages as functions of time. Using our differential equation and our model we will be able to tune our circuit to certain AM frequencies. We will then put our circuit into the frequency domain, and then derive a transfer function.

# Signal Processing

## Signal Processing in Society

Signal processing is an area of electrical engineering that focuses on analyzing digital or analog signals that depend on time. This is very important throughout modern society, especially in telecommunications, medicine, finance, seismology, and image processing. Signal Processing is most prominent in telecommunications, where it is required in order for any form of telecommunications to work. A phone speaker processes the audio wave from a person's mouth which is then sent to an antennae which processes the signal again before being sent to another phone. The same can be said for any forms of data sent through a phone, including text messages, and even mobile internet usage. Signal processing is also extremely important in medicine for electrocardiograms. This is very important for monitoring the electrical condition of the heart. They are vital for diagnosing heart diseases, and researching the heart. In finance, signal processing is used to predict the movement of stock prices. Isaac Newton introduced signal processing techniques in finance after he lost money on the South Sea Company investment bubble. In seismology signal processing is used in order to create seismograms which can the be used to further analyze earthquakes. Image processing is a different type of signal processing where the "signal" is always an image. Image processing is very important for everyday computing, and is what actually allows one to see the images on their monitor. It is vital for anything that has gone through a computer, and is also integral in modern day entertainment like movies and videogames.

## Signal Processing in an AM Radio Tuner

While its great to know about signal processing in modern society, an AM radio tuner provides a more concrete example of signal processing. It starts with the someone speaking into a microphone or music being played. Every radio station has a different

sine wave that varies by frequency whenever they broadcast. This is known as a carrier wave. When a song starts or a DJ starts talking, that audio wave is put onto the same frequency as the station's carrier wave by modifying the amplitude. An antenna at the radio station then sends the signal into space. The antenna in your radio receives the signal which then sends it to an AM radio tuner. The tuner uses a principal called resonance to tune out all other signals that aren't of the frequency that was selected by the radio dial. The tuner resonates and amplifies one particular frequency, while ignoring all others. The RLC Circuit that is being built and modeled in this project is a resonator. This is the signal processing that occurs inside an AM radio tuner. After one signal is amplified the radio demodulates whatever sound is on the resonated frequency, and then amplifies that sound and sends it to the speaker.

## Introduction to RLC Circuits

An RLC circuit always consists of a resistor, inductor, and capacitor. They can be modeled based on the configuration of the circuit, but all models require the use of a second-order ordinary differential equation, in order to analyze each component through time. In order to tune our circuit we need to be able to filter out other radio waves, based on their frequencies. Thus, we use a parallel RLC circuit to create a band-pass filter, which reduces the amplitude of all waves that are not at the set frequency. There are some initial concepts that need to be understood first before we start our discussion. It will be necessary to model resistance, capacitance, and inductance. It is also necessary to understand the concepts of voltage and current. Voltage is simply a measure of the potential energy between different points. Current is the rate at which a coulomb of electrons travels through a circuit with respect to time. A resistance is the measure of opposition of passage of current. A capacitor stores electrical charge, and the amount of charge that can be stored is measured by capacitance. Inductance is the measure of how much voltage is "induced" into a capacitor when the flow of the

current changes. Since we are using a parallel RLC Circuit we must use an ordinary differential equation in relation to voltage. This is a result of Kirchoff's Voltage Law, and also means that analyzing the circuit with a constant voltage (DC) is trivial.

## Basic Electric Equations and Relationships

Before deriving the equation for our specific RLC Circuit, it is necessary to know some basic relationships. For each different equation it will be necessary to solve for the voltage as this will be what determines the values for each different circuit element. The first equation is $V = IR$, otherwise known as Ohm's Law where $V$ is the voltage, $i$ is the current, and $R$ is the resistance. Next we look at the relationship for capacitance, which is $C = Q/V$, where $Q$ is the electric charge, $C$ is the capacitance and $V$ is the voltage. Solving for $V$ we get $V = Q/C$. Finally we have the relationship for inductance, which is $V(t) = Ldi/dt$. Now that we have all of our relationships we can properly derive our equation.

## Modeling the RLC Circuit

Now that we have all of the basic relationships for the elements in the circuit, we can find a model for the circuit. The next step involves using Kirchoff's Voltage Law in order to find the circuit. By substituting the circuit elements into Kirchoff's law, we get:

$$v_r + v_l + v_c = v(t) \tag{1}$$

Where $v_r$ is the voltage drop across the resistor, $v_l$ is the voltage drop across the inductor, $v_c$ is the voltage drop across the capacitor, and $v_t$ is a time varying voltage. It is now necessary to find the voltage drops across each circuit element. To find the voltage drop across the resistor, we take the resistance equation and modify it for a

time varying voltage giving us:

$$Ri(t) = V(t)$$

Where $R$ is the resistance, $i(t)$ is a time varying current, and $v(t)$ is the time varying voltage.

$$L\frac{di}{dt} = V(t)$$

To find the voltage drop across the capacitor, it is necessary to examine the actual component in physical sense. The current through any circuit element is defined by the rate of charge passing through it with respect to time. However, electrons do not pass through a capacitor, but instead electrons stick to the negative plate of the capacitor for each electron that leave the positive plate. The charge on plate is equal and opposite to the charge on the other plate of the capacitor, which means that the charge on each plate is the integral of the time varying current function. The equation is then as follows:

$$\frac{1}{C} \int_0^t i(\tau)d\tau + V(0) = V(t)$$

Where $C$ is the capacitance, $t$ is the time interval. $i(t)$ is the time varying current, and $V(0)$ is an initial condition, of the voltage when the system is turned on. The last thing that needs to be determined is the time varying voltage function. This function will have a sinusoidal curve given the nature of radio waves, and will also be time dependent. Thus, the equation is

$$V_0 \sin \omega t$$

Substituting into equation (1) we get the following equation:

$$L\frac{di}{dt} + Ri(t) + \frac{1}{C} \int_0^t i(\tau)d\tau + V(0) = V_0 \sin \omega t \tag{2}$$

Differentiating this equation with respect to time gives:

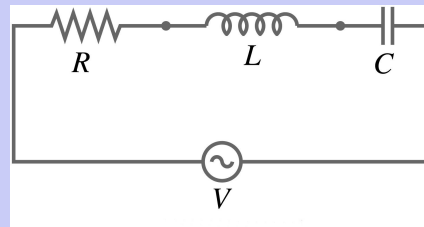$$L\frac{di^2}{d^2t} + R\frac{di}{dt} + \frac{1}{C}i(t) = V_o\omega\cos\omega t \qquad (3)$$



Figure 1: A circuit diagram of a basic closed loop RLC Circuit in series.

# Finding the General Solution to the Equation

In order to find the general solution to this equation to this equation it is necessary to find both a homogenous and particular solution to the equation. In this case, we will start by finding the homogenous solution, then using the complex exponential waveform, find the particular solution to the equation, which will allow us to tune the circuit.

## Finding the Homogenous Solution

In order to find the homogenous solution for equation (3) simply set it equal to zero then solve the resulting equation. In order to make the analysis less cluttered the paper

will use prime notation rather than Leibniz notation from here on out.

$$Li'' + Ri' + \frac{1}{C}i = 0$$

First divide the whole equation by L in order to isolate the highest order derivative.

$$i'' + \frac{R}{L}i' + \frac{1}{LC}i = 0$$

Next find the characteristic polynomial using the coefficients of the differential equation, and then find the eigenvalues.

$$\lambda^2 + \frac{R}{L}\lambda + \frac{1}{LC} = 0$$

Solve for lambda using the quadratic formula.

$$\lambda = \frac{-\frac{R}{L} \pm \sqrt{\left(\frac{R}{L}\right)^2 - \frac{4}{LC}}}{2}$$

Simplifying the equation we get

$$\lambda = -\frac{R}{2L} \pm \sqrt{\left(\frac{R}{2L}\right)^2 - \frac{1}{LC}} \tag{4}$$

Now, we can look at the discriminant in order to determine the behavior of the circuit, based on whether or not the discriminant is positive, negative, or zero.

## Positive Discriminant

When the discriminant is positive, there are two real negative eigenvalues. In this case, the circuit is over damped.

Home Page

Title Page

◀◀    ▶▶

◀    ▶

Go Back

Full Screen

Close

Quit

### Zero Discriminant

The system will have one real eigenvalue. The equilibrium point for this system is a degenerative node, and the solution will be critically damped. This means that there will be no oscillations whatsoever.

### Negative Discriminant

In this case, there are two complex eigenvalues. This yields an equilibrium point of a spiral sink. This means the circuit will oscillate till it finally reaches the equilibrium amplitude. This is known as the underdamped response.

## Finding the Particular Solution

In order to find the particular solution for the system use the method of undetermined coefficients. It is first necessary to discuss the complex exponential waveform, which is used extensively to simplify analysis.

### The Complex Exponential Waveform

Rather than use a periodic function to express the voltage and current waveforms, it is possible to use a complex exponential function. The complex exponential function is in the form $e^{j\omega t}$, where $j$ is the imaginary number $\sqrt{-1}$. The function is still a sinusoid due to Euler's Identity, which states that $e^{jt}$ is equivalent to $\sin t + j \cos t$. It is important to note that it is not physically possible to observe a function with the form $e^{j\omega t}$ due to the presence of the imaginary number. The complex exponential waveform is instead a mathematical abstraction used in place of the standard periodic function because it makes analysis much simpler. Integrating or differentiating the function only changes the coefficients.

Again consider equation (3). We first need to replace the sinusoidal function with the complex exponential waveform in order to simplify the analysis. This means that $V_0\omega sin(\omega t)$ will be replaced with $V_0 j\omega e^{j\omega t}$.

$$Li'' + Ri' + \frac{1}{C}i = V_0 j\omega e^{j\omega t}$$

This means the particular solution is then

$$i_p = ae^{j\omega t}$$

Where a is a constant that needs to be solved for. The first and second derivatives for the proposed particular solution are

$$i_p' = a\omega j e^{j\omega t} \tag{5}$$
$$i_p'' = -a\omega^2 e^{j\omega t} \tag{6}$$

Substituting into the original equation gives

$$-La\omega^2 e^{j\omega t} i'' + aR\omega j e^{j\omega t} + \frac{1}{C}ae^{j\omega t} = V_0 j\omega e^{j\omega t}$$

Since $e^{j\omega t}$ is a common term, it can be divided out from the whole equation.

$$-a\omega^2 L + Ra\omega j + \frac{1}{C}a = V_0 j\omega$$

Now isolate $V_0$. This reasoning for this step will become clearer later on, but it makes the equation more readable in a physical sense. After some simplification this gives

$$a(R + j(\omega L - 1/\omega C)) = V_0$$

Home Page

Title Page

◀◀   ▶▶

◀   ▶

Go Back

Full Screen

Close

Quit

Solving for $a$ gives

$$a = \frac{V_0}{R + j(\omega L - 1/\omega C)}$$

So the particular solution is then

$$i_p = \frac{V_0}{R + j(\omega L - 1/\omega C)} e^{j\omega t} \tag{7}$$

The denominator in equation (7) is the sum of the complex impedances. The complex impedance is the resistance to the current given a time-varying complex driving function. With the particular solution for the current, it is now possible to find the steady-state response for the circuit after the voltage is applied. It is important to note that when the voltage is first applied to the system, there will be a transient before the system settles into the steady-state response. A transient is when a graph exhibits behavior that is unrelated to the function of the graph, and the steady-state response is what a graph is supposed to be like given the function it is modeling. Equation (7) is the steady state output current of an RLC Circuit, which models the long-term behavior of the system.

## Tuning the RLC Circuit

Remember that in order to tune the circuit to a certain frequency we needed to attenuate the amplitudes of all other frequencies, and increase the amplitude of the signal with the specified frequency. We can then assume that by maximizing the amplitude of the output current at a certain frequency, we have successfully tuned the circuit. To find

the amplitude of the output current simply take the magnitude of equation (7).

$$|I_p| = \frac{|V_0 e^{j\omega t}|}{|R + j(\omega L - 1/\omega C)|}$$

$$|I_p| = \frac{V_0}{(R^2 + (\omega L - 1/\omega C)^2)^{1/2}}$$

The circuit can now be tuned by changing the inductance and capacitance. The max amplitude is achieved when $C$ and $L$ satisfy the following equation.

$$\omega L - \frac{1}{\omega C} = 0$$

Solving for the frequency $\omega$ in terms of $L$ and $C$ gives

$$\omega L = \frac{1}{\omega C}$$

$$\omega^2 = \frac{1}{LC}$$

The following equation maximizes the gain, that is the ratio of the input amplitude over the steady-state amplitude.

$$\omega^2 = \frac{1}{LC} \tag{8}$$

By maximizing the gain for a certain frequency, the amplitude for that frequency is also amplified effectively "tuning" the circuit to that signal's frequency. However, this model is not accurate given a large number of different signals with different frequencies.

## Optimizing the Model for Precision

So far, we have addressed how to tune the circuit to one frequency by maximizing the gain. However, in a more realistic scenario there will be multiple radio waves that the

radio's antenna will receive. In this case the equation for the driving voltage will look like this

$$\xi(t) = \sum_{k=1}^{N} V_k e^{j\omega_k t}$$

The derivative is

$$\xi'(t) = \sum_{k=1}^{N} V_k j\omega_k e^{j\omega_k t)}$$

Where N is the number of radio stations received, and k is the number for the radio station. Because the equation for current is linear, the steady state response for multiple broadcasts is

$$i_p = \sum_{k=1}^{N} \frac{V_k e^{j\omega_k t}}{R + j(\omega_k L - 1/\omega_k C)}$$

In the theoretical situation where one wants to tune to a certain station $k$, then the circuit will "filter" out all other amplitudes by making the amplitude of station $k$ much larger than all other, effectively attenuating all other signals, and accomplishing the goal of the AM radio tuner.

## The Frequency Domain

So far, we have been evaluating this circuit strictly in the time domain, with the exception of equation (8), which is a function of frequency rather than time. In order to do more elegant analysis on the system, it is necessary to put our circuit into the frequency domain. This will be accomplished using equations for the complex impedances.

Home Page

Title Page

◀◀　　▶▶

◀　　▶

Go Back

Full Screen

Close

Quit

## Finding the Complex Impedances

Recall that the complex impedance of a circuit is the resistance to a complex time-varying current. The impedance function can then be expressed as the ratio of the complex voltage over the complex current. This will be denoted by $\tilde{v}$ and $\tilde{i}$, respectively. With this information we can find the impedance functions for each circuit element.

### Resistance

Using Ohm's Law we can very easily derive the impedance function for resistance.

$$v = iR$$

Substitute in the complex waveforms for each equation.

$$\tilde{v} = \tilde{i}R$$

$$\frac{\tilde{v}}{\tilde{i}} = R$$

Because the complex impedance is the ratio of the complex voltage to the complex current we have finished deriving this function. It should be noted that rather than expressing the fraction of voltage over current, it can simply be replaced with **Z**. So the impedance function for resistance is.

$$\mathbf{Z}(\omega) = R \tag{9}$$

Notice that the impedance is a function of frequency rather than time.

### Inductance

Consider a complex current equation $\tilde{i} = I_0 e^{j\omega t}$, and the inductance relationship $v = L\,di/dt$. Substitute the complex current to solve for the complex voltage, and then find

the ratio of the two to find the impedance function.

$$\tilde{v} = j\omega L I_0 e^{j\omega t}$$

Substituting $\tilde{i}$ for $I_0 e^{j\omega t}$ gives

$$\tilde{v} = j\omega L \tilde{i}$$

Solving for the impedance gives

$$\mathbf{Z}(\omega) = j\omega L \qquad (10)$$

### Capacitance

Finding the impedance function for capacitance requires the sames steps for the other two functions. This time the voltage relationship is $\frac{1}{C}\int_0^t i(\tau)d\tau = v$, and the complex current is still $\tilde{i} = I_0 e^{j\omega t}$. Integrating the complex current and solving for the voltage gives

$$\tilde{v} = \frac{I_0 e^{j\omega t}}{Cj\omega}$$

Simplifying and solving for the impedance yields

$$\mathbf{Z}(\omega) = \frac{1}{j\omega C} \qquad (11)$$

## Transforming our Circuit to the Frequency Domain

Since we are dealing with a series circuit we simply need to sum the impedance functions to put our circuit into the frequency domain. This gives the following

$$\tilde{v} = \frac{1}{j\omega C} + j\omega L + R \qquad (12)$$

So we now have an equation for voltage as a function of frequency rather than time.

Home Page

Title Page

◀◀    ▶▶

◀    ▶

Go Back

Full Screen

Close

Quit

# The Transfer Function

The transfer function for a circuit is simply the ratio of a certain complex exponential waveform from one point in the network to another point in the network. It can be expressed as a ratio of voltages, or currents, or a voltage to a current, but is commonly expressed as the ratio of the output voltage compared to the input voltage. The transfer function will be denoted by $H(\omega)$. The frequency transfer function allows one to characterize a given system more easily. The information conveyed by the transfer function is conveyed using Bode plots.

## Characterizing the System

The frequency transfer function is expressed by $A(\omega)e^{j\phi(\omega)}$. This equation describes not only the amplitude as a function of frequency but also the phase angle as a function of frequency. However, for the purposes of this paper we assumed the phase angle for our harmonic oscillator was a constant zero, which is an incorrect assumption. Rather than doing all those differential equations, we can simply look at the ratio of the output to input, and take the magnitude. Consider now our system, but with a complex driving voltage $\tilde{v_2} = V_0 j\omega e^{j\omega t}$. We can assume our input voltage to be the sum of complex impedances, so $\tilde{v_1}$ is just equation <span style="color:red">12</span>. The transfer function is then

$$H(\omega) = \frac{\tilde{v_2}}{\tilde{v_1}} = \frac{V_0 j\omega e^{j\omega t}}{1/j\omega Cd + j\omega L + R}$$

To find the amplitude of this equation simply take the magnitude, as was done for the particular solution. This gives

$$A(\omega) = \frac{V_0}{(R^2 + (\omega L - 1/\omega C)^2)^{(1/2)}}$$

Signal Processing

Introduction to RLC...

Finding the General...

Tuning the RLC Circuit

The Frequency Domain

The Transfer Function

Bode Plots

Conclusion

Home Page

Title Page

◀◀  ▶▶

◀  ▶

Go Back

Full Screen

Close

Quit

Which is equivalent to the amplitude of the steady-state response for current. This method was much simpler and less painful than finding all of the differential equations, and solving for it.

## Bode Plots

Using the MATLAB control systems toolbox, we can look at the amplitude as a function of frequency and the phase angle as a function of frequency . We can then graph the Bode plot which is the log log curve of the frequency with respect to both the gain, and the phase angle. The following code can be used to find the Bode plot of a simple bandpass filter.

```
R = 1; L = 1; C = 1;
G = tf([1/(R*C) 0],[1 1/(R*C) 1/(L*C)]);
```

The code first defines the resistance, capacitance and inductance. In this case we will make it simple and use one for each circuit element. The function tf takes a row vector of coefficients and specifies the transfer function for those values.

```
bode(G), grid
```

This plots the Bode plots for the specified transfer function. The graphs will be log log curves of the frequency vs the phase angle, and the frequency vs the gain. Note that for our circuit we assume the phase angle remains constant, but that is not the case otherwise.
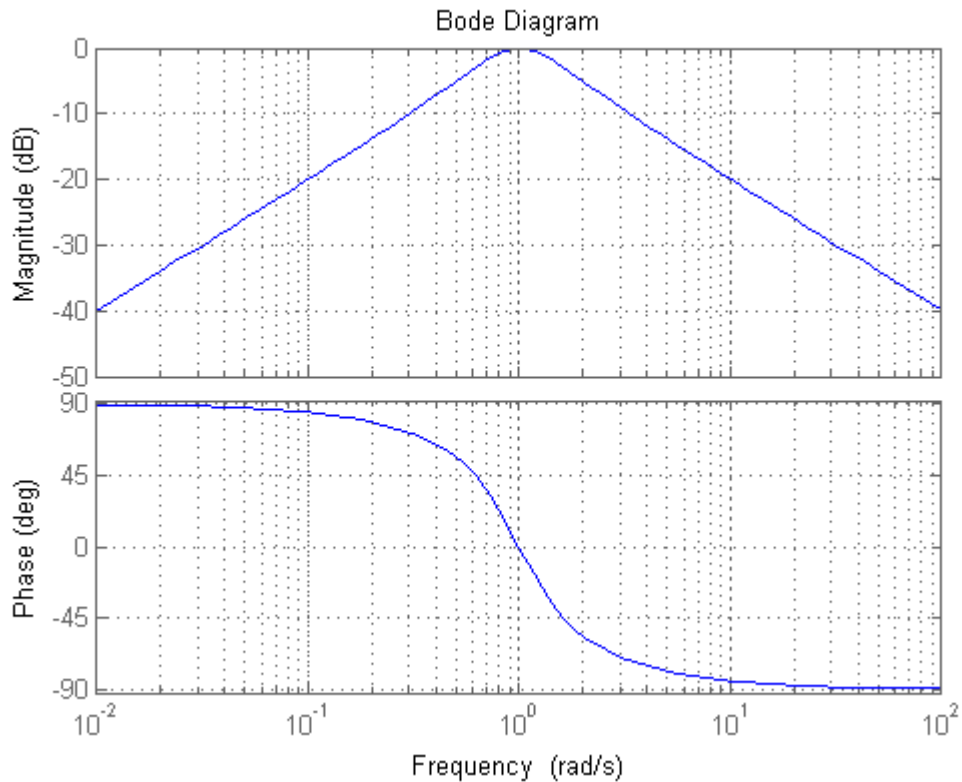
Full Screen

Close

Quit



Figure 2: The Bode Plots for a bandpass filter where R=1, L=1, and C=1.

# Conclusion

In order to tune the circuit it is first necessary to find an equation for the amplitude. This can be done by finding the particular solution of the differential equation using current, and then taking the magnitude of that function. You can also find the amplitude function using the transfer function. After finding the amplitude function as a function of frequency, simply maximize the amplitude by changing the inductance or capacitance of the circuit. The tuning of the AM Radio Circuit is just one of a very large amount of uses for signal processing. The techniques covered in this paper were very basic, however they serve as a springboard for more advanced and elegant analysis.

# References

[1] David Arnold. *Writing Scientific Papers in LaTex.* December 31, 2008 http://msemac.redwoods.edu/~darnold/math55/WritingScientificPapers/project_latex.pdf

[2] John Polking, Albert Boggess, David Arnold. *Differential Equations With Boundary Value Problems.* Pearson; 2 edition (August 7, 2005)

[3] Belle A. Shenoi (2006). Introduction to digital signal processing and filter design. John Wiley and Sons. p. 120. ISBN 978-0-471-46482-2.

[4] Wikibooks. *Circuit Theory/Second-Order Solution* Last modified Jan 3, 2014 http://en.wikibooks.org/wiki/Circuit_Theory/Second-Order_Solution

[5] Kenny Harwood *Modeling a RLC Circuit's Current with Differential Equations* May 17, 2011 http://home2.fvcc.edu/~dhicketh/DiffEqns/Spring11projects/Kenny_Harwood/ACT7/KennyHarwoodFinalProject.pdf

[6] Philip Denbigh *System Analysis and Signal Processing* Addison-Wesley; (1998)

[7] *Analyzing the Response of an RLC Circuit* http://www.mathworks.com/help/control/ug/analyzing-the-response-of-an-rlc-circuit.htmlzmw57dd0e14349