A Visual Proof of the Parallelogram Law

Vincent Conitzer

Abstract. In this note, I give a novel proof of the parallelogram law, relying on two superimposed tilings of the plane.

Visual proofs have always had a controversial status in mathematics [1]. On the one hand, we celebrate them for the intuition that they provide; on the other hand, we worry about their rigor. Consequently, we celebrate the large variety of visual proofs of the Pythagorean theorem, as well as the activity of finding new such proofs that continues to this day (e.g., [2]); but then, once we move on to deriving additional results in elementary geometry, we tend to rely on applications of the Pythagorean theorem as much as possible, minimizing the further use of visual intuition. A good example is the parallelogram law.

Theorem 1 (Parallelogram law). *Given an arbitrary parallelogram with side lengths a and b and diagonal lengths c and d, we have $2a^2 + 2b^2 = c^2 + d^2$.*

It is straightforward to prove the parallelogram law by application of the Pythagorean theorem, which can be done in any of a number of ways.[1] But we should worry that something is lost when confining the use of visual proofs in this way. Visual arguments often provide intuition that algebraic derivations do not. Moreover, proving more results visually will likely result in new types of visual proofs, thereby providing more examples to consider in debates of what should count as a visual proof.

To illustrate these points, in this note, I provide a—to my knowledge, novel—visual proof of the parallelogram law. This theorem is a natural candidate for such an exercise, as the Pythagorean theorem is the special case of the parallelogram law where the parallelogram is a rectangle. To the extent that such a proof is not significantly more complex than corresponding proofs of the Pythagorean theorem, it suggests that the parallelogram law is in fact the theorem that corresponds more naturally to the proof technique, and thereby it perhaps better inspires generalizations.

The proof. Figure 1 shows an arbitrary parallelogram in orange, and its dual parallelogram in blue. Based on these parallelograms, Figure 2 provides the visual proof. Ignoring for now the teal (blue-green) squares in it, the figure exhibits two tilings of the plane, superimposed: one (the "upper" tiling) uses the orange parallelograms, and the transparent squares that complement them, to fill the plane; the other (the "lower" tiling) uses the blue parallelograms, and the transparent squares that complement them, to fill the plane.

The superimposition of the two tilings demonstrates the following: (1) The first tiling requires twice as many parallelograms as the second; since the latter (blue) parallelograms are twice as large in area, this establishes that both tilings leave the same amount of transparent space. (2) There are also twice as many transparent squares in the first tiling. Together, these observations suffice to establish the parallelogram law.

[1] In fact, this includes *visual* proofs of the parallelogram law that rely on applying the Pythagorean theorem along the way [3, 4]. In contrast, the proof in this note does not presuppose the Pythagorean theorem, and uses only the parallelogram, its dual, and the four relevant squares as shapes in the proof. That said, this proof is not the only known direct visual proof of the parallelogram law; see a proof by Wise [5, 6].

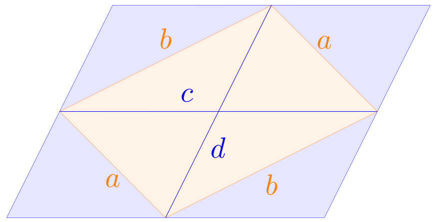

Figure 1. Orange/inside: an arbitrary parallelogram with one of its diagonals placed horizontally. Blue/outside: its dual parallelogram with side lengths c and d, and diagonal lengths $2a$ and $2b$.

The proof uses two tilings of the (infinite) plane; such tilings have been shown to provide proofs of a number of theorems in geometry, including the Pythagorean theorem [7, 8]. One may worry whether comparisons of numbers of parallelograms or squares, or the areas they occupy, are meaningful, given that these are all infinite. This concern can be addressed in several ways, for example by considering finite parts of the plane—such as those contained in the finite figures in this note—and letting these grow larger, so that what happens at the edges of the figure shrinks in relative importance. Alternatively, one may consider the teal (blue-green) squares, which all have the same contents. These correspond to quotienting the plane, resulting in a flat torus—i.e., a single teal square where by crossing one of its boundaries, one ends up on the corresponding point on the other side of the square. Focusing on a single such teal square (say, the one colored more darkly), one can see that it contains, in the orange tiling, exactly four parallelograms and two of each type of transparent square in that tiling; and, in the blue tiling, exactly two parallelograms and one of each type of transparent square in that tiling. Since the parallelograms in each tiling take up the same area, so do the transparent squares, proving the parallelogram law.

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VINCENT CONITZER received his Ph.D. in Computer Science from Carnegie Mellon University in 2006. After 16 wonderful years on the faculty at Duke University, he returned to CMU's Computer Science Department, where he directs the Foundations of Cooperative AI Lab (FOCAL). He also serves part-time as Head of Technical AI Engagement for the Institute for Ethics in AI at the University of Oxford.
Computer Science Department, Carnegie Mellon University, Pittsburgh PA 15213, USA.

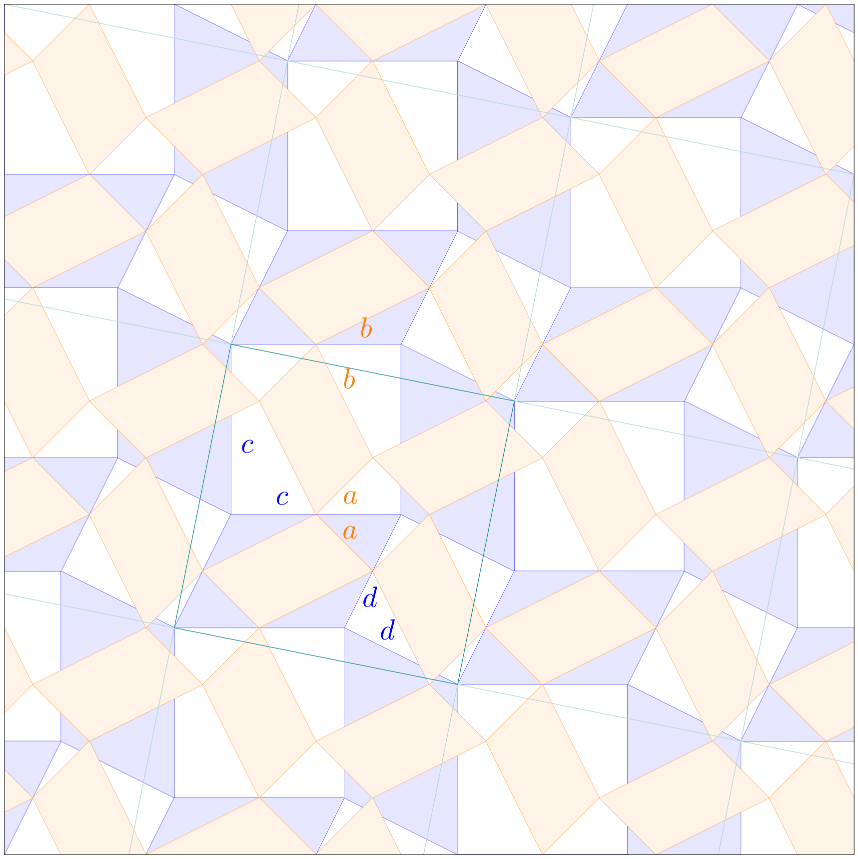

Figure 2. Two tilings of the plane, superimposed. One (the "upper") tiling consists of orange parallelograms and the transparent squares that complement them. The other (the "lower") tiling consists of blue parallelograms and the transparent squares that complement them. a, b, c, and d are the lengths of the sides of the (transparent) squares in which they appear. For a and b, these are the transparent squares in the upper tiling that uses the orange parallelograms. (Through these squares, we can in places see parts of the blue parallelograms in the lower tiling). For c and d, these are the transparent squares in the lower tiling that uses the blue parallelograms. (These squares are in places obscured by the orange parallelograms from the upper tiling.) The teal (blue-green) squares all have identical contents; a single such square, naturally seen as a flat torus, also suffices for the proof.

conitzer@cs.cmu.edu