Project Report

QF605 Fixed Income Securities

Singapore Management University
MSc. In Quantitative Finance

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Part I – Bootstrapping Swap Curves

1. OIS discount factor

The following methodology is adopted to bootstrap the OIS discount factor:

$$D_0(0,0.5y)=\frac{1}{1+0.5\times OIS_{0.5y}} \tag{1.1}$$

$$D_0(0,1y)=\frac{1}{1+1\times OIS_{1y}} \tag{1.2}$$

For other tenor years, $PV_{fix}=PV_{flt}$:

$$[D_0(0,1y)+D_0(0,2y)]\times OIS_{2y}=D_0(0,1y)[(1+\frac{f_0}{360})^{360}-1]+D_0(0,2y)[(1+\frac{f_1}{360})^{360}-1] \tag{1.3}$$

Where $(1+\frac{f_0}{360})^{360}=\frac{1}{D_0(0,1y)},(1+\frac{f_1}{360})^{360}=\frac{1}{D_0(1y,2y)}$, equation 1.3 can be simplified as:

$$\begin{aligned}[D_0(0,1y)+D_0(0,2y)]\times OIS_{2y}&=D_0(0,1y)(\frac{1}{D_0(0,1y)}-1)+D_0(0,2y)(\frac{1}{D_0(1y,2y)}-1)\\&=1-D_0(0,1y)+D_0(0,2y)(\frac{D_0(0,1y)}{D_0(0,2y)}-1)\\&=1-D_0(0,1y)+D_0(0,1y)-D_0(0,2y)\\&=1-D_0(0,2y)\end{aligned} \tag{1.4}$$

$$\vdots$$

$$[D_0(0,1y)+D_0(0,2y)+\cdots D_0(0,Ny)]\times OIS_{Ny}=1-D_0(0,Ny) \tag{1.5}$$

Where $N\in[0,30]$

We were provided with scattered information on the OIS and hence, interpolation was performed using the derived discount factors from the given data. The reason behind using discount rates for interpolation is that they are generally linearly and monotonically decaying, and therefore only a negligible amount of noise will be introduced in contrast with using OIS. To solve for discount factors in year [7,10], we used the following equation:

$$\begin{aligned}&[D_0(0,1y)+D_0(0,2y)+\cdots D_0(0,7y)+D_0(0,8y)+D_0(0,9y)+D_0(0,10y)]\times OIS_{10y}\\&=1-D_0(0,10y)\end{aligned} \tag{1.6}$$

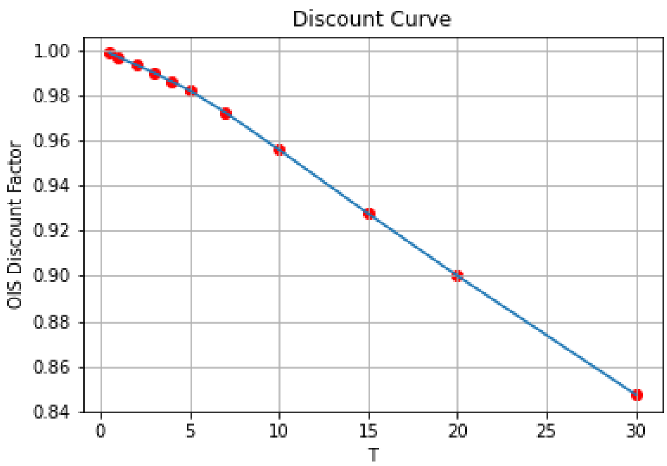
In equation 1.6, all discount factors prior to year 7 have been calculated, while $D_0(0,8y)$. and $D_0(0,9y)$ are dependent variables that can be expressed in terms of $D_0(0,7y)$ and $D_0(0,10y)$. We can solve for the discount factors by setting up these constraints and pivoting on the root search method to solve for $D_0(0,10y)$ since its OIS rate is known. The same method was used to solve for the remaining discount factors.

Table 1.1 summarizes the OIS discount factors and its curve is plotted in Figure 1.1.

Table 1.1 OIS Discount Factors

Tenor	OIS	OIS DF
0	0.00250	0.998752
1	0.00300	0.997009
2	0.00325	0.993531
3	0.00335	0.990015
4	0.00350	0.986117
5	0.00360	0.982184
7	0.00400	0.972406
10	0.00450	0.955977
15	0.00500	0.927611
20	0.00525	0.900076
30	0.00550	0.847407

Figure 1.1 OIS Discount Curve



2. LIBOR discount factor

Since the swap market is collateralized in cash and overnight interest is paid on collateral posted, the following methodology is adopted:

$$\begin{aligned} D_0(0,0.5y)\times IRS_{0.5y}\times 0.5 &= D_0(0,0.5y)\times L(0,0.5y)\times 0.5 \\ &= D_0(0,0.5y)\times\frac{1-D(0,0.5y)}{D(0,0.5y)} \end{aligned} \tag{1.7}$$

$$\begin{aligned} (D_0(0,0.5y)+D_0(0,1y))\times IRS_{1y}\times 0.5 &= D_0(0,0.5y)\times L(0,0.5y)\times 0.5+D_0(0,1y)\times L(0.5y,1y)\times 0.5 \\ &= D_0(0,0.5y)\times\frac{1-D(0,0.5y)}{D(0,0.5y)}+D_0(0,1y)\times\frac{D(0,0.5y)-D(0,1y)}{D(0,1y)} \end{aligned} \tag{1.8}$$

$$\begin{aligned} &(D_0(0,0.5y)+D_0(0,1y)+D_0(0,1.5y)+\cdots+D_0(0,Ny))\times IRS_{Ny}\times 0.5 \\ &= D_0(0,0.5y)\times\frac{1-D(0,0.5y)}{D(0,0.5y)}+D_0(0,0.5y)\times\frac{D(0,0.5y)-D(0,1y)}{D(0,1y)}+\cdots+D_0(0,Ny)\times\frac{D(0,(N-0.5)y)-D(0,Ny)}{D(0,Ny)} \end{aligned} \tag{1.9}$$

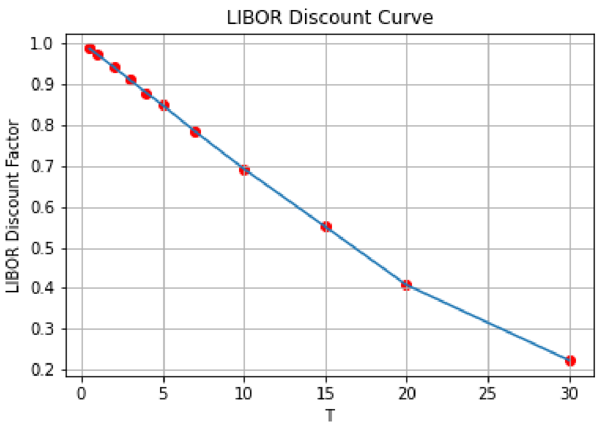
Where $N\in[0,30]$

Table 1.2 summarizes the LIBOR discount factors and its curve is plotted in Figure 1.2.

Table 1.2 LIBOR Discount Factors

Tenor	OIS DF	LIBOR
0	0.998752	0.987654
1	0.997009	0.972577
2	0.993531	0.942179
3	0.990015	0.910482
4	0.986117	0.878981
5	0.982184	0.848989
7	0.972406	0.784216
10	0.955977	0.692710
15	0.927611	0.551081
20	0.900076	0.408218
30	0.847407	0.223662

Figure 1.2 LIBOR Discount Curve



3. Forward swap rates

$$\begin{aligned} &(D_0(0,1.5y)+D_0(0,2y))\cdot FRS_{1\times 1} \\ &= D_0(0,1.5y)\times L(1,1.5y)\times 0.5+D_0(0,2y)\times L(1.5y,2y)\times 0.5 \\ &= D_0(0,1.5y)\times\frac{D(0,1y)-D(0,1.5y)}{D(0,1.5y)}\times 0.5+D_0(0,2y)\times\frac{D(0,1.5y)-D(0,2y)}{D(0,2y)}\times 0.5 \end{aligned} \tag{1.10}$$

Forward swap rates for each expiry\tenor pair can be derived using the formula above and are summarized in Table 1.3.

Table 1.3 Forward Swap Rates

	1Y	2Y	3Y	5Y	10Y
1Y	0.032007	0.033259	0.034011	0.035255	0.038428
5Y	0.039274	0.040075	0.040072	0.041093	0.043634
10Y	0.042189	0.043116	0.044097	0.046249	0.053458

Part II – Swaption Calibration

Using the derived forward swap rate in Part I, together with the given lognormal market volatility, we aim to calibrate the Displaced-Diffusion model & SABR model to match the swaption market data prices.
Since there is very limited liquidity on in-the-money options, the market tends to trade at-the-money or the out-of-the-money options. This means that we should focus on high strike call swaptions (when $K \geq F$) and low strike put swaptions ($K \leq F$).

1. Displaced Diffusion Model Calibration

Given the lognormal market volatility across all the strikes in the "Swaption" tab of the IR Data, our objective of the calibration is achieved by using "least square" function to minimize the squared error between the given lognormal market volatilities and the implied volatilities by the Black76 model. The volatilities used in the displaced diffusion model are at-the-money ATM volatilities which were obtained directly from the ATM column of the given IR data. As mentioned above, root search was utilized to get implied volatilities (σ_{K_i}) that best matched the swaptions pricing by DD model and Black76 model. Optimized parameters β were also observed for each expiry\tenor category of the swaptions. The calibrated DD model parameters are summarized in Table 2.1 below.

Table 2.1 Calibrated Displaced Diffusion Model Parameters

Sigma					
Expiry\Tenor	1Y	2Y	3Y	5Y	10Y
1Y	0.2250	0.2872	0.2978	0.2607	0.2447
5Y	0.2726	0.2983	0.2998	0.2660	0.2451
10Y	0.2854	0.2928	0.2940	0.2674	0.2437
Beta					
Expiry\Tenor	1Y	2Y	3Y	5Y	10Y
1Y	5.801818e-08	3.133166e-13	1.087411e-12	1.475881e-10	0.000007
5Y	1.310269e-11	5.503210e-08	2.277114e-06	1.432631e-04	0.055462
10Y	1.396061e-07	7.489292e-06	8.155062e-05	1.387962e-06	0.001745

2. SABR Calibration

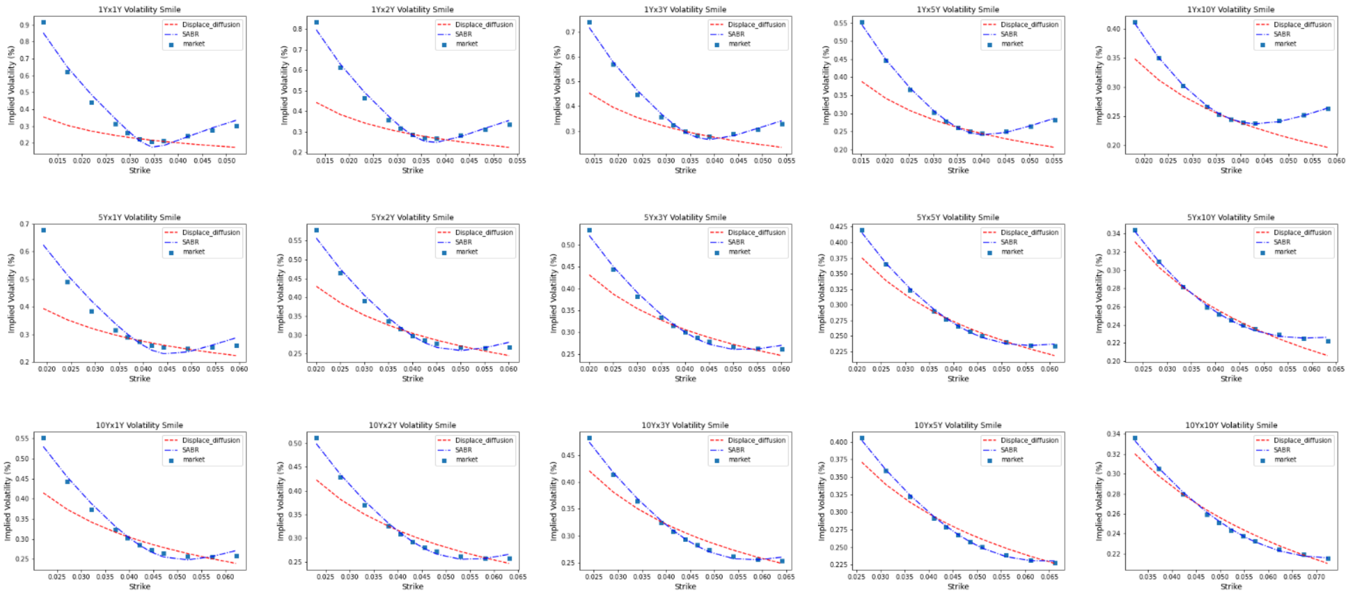
SABR is a stochastic volatility model which allows volatility to be obtained directly from the Black76 Model. From Part I, we had derived the forward swap rates for each combination of expiry and tenor, and the range of strikes used is the corresponding forward swap rate + basis points as in the swaption data. With this information, we were able to directly calibrate the SABR model using the given lognormal swaption implied volatilities to get our SABR parameters α, ρ and ν (while fixing β at 0.9). The calibrated SABR model parameters are summarized in Table 2.2 below.

Table 2.2 Calibrated SABR Model Parameters

Alpha					
Expiry\Tenor	1Y	2Y	3Y	5Y	10Y
1Y	0.139070	0.184649	0.196850	0.178052	0.171151
5Y	0.166532	0.199498	0.210256	0.191125	0.177184
10Y	0.177533	0.195216	0.207121	0.201610	0.181153
Nu					
Expiry\Tenor	1Y	2Y	3Y	5Y	10Y
1Y	2.049441	1.677413	1.438147	1.064877	0.777785
5Y	1.339647	1.061912	0.936615	0.671289	0.495764
10Y	1.007239	0.925448	0.868935	0.719956	0.577963
Rho					
Expiry\Tenor	1Y	2Y	3Y	5Y	10Y
1Y	-0.633211	-0.525117	-0.482845	-0.414426	-0.264779
5Y	-0.585168	-0.546877	-0.549411	-0.511652	-0.438724
10Y	-0.545807	-0.544358	-0.550638	-0.562790	-0.511985

The implied volatility smiles for Displaced-Diffusion and SABR are plotted against the market data and are shown below in Figure 2.2.

Figure 2.2 Volatility Smile for calibrated Displaced-Diffusion and SABR Model



3. Pricing Swaptions using calibrated DD and SABR model

This section documents the pricing of swaptions using our calibrated displaced-diffusion and SABR model. Firstly, we derived the forward swap rate of the forward-starting swap using the generalized formula below, where LIBOR forward rate (computed from LIBOR discount factors) and OIS discounting factors from Part I were used (assuming the swap market is collateralized, and swaptions are semi-annually compounded).

$$S_{forward} = \frac{\sum_{i=1}^{n}[D(0,T_i)\times\Delta_{i-1}\times L(T_{i-1},T_i)]}{\Delta[D(0,T_i)+D(0,T_{i+1})+\cdots+D(0,T_N)]}\ \text{, where}\ \ L(T_{i-1},T_i) = \frac{1}{\Delta_{i-1}}\frac{D(0,T_{i-1})-D(0,T_i)}{D(0,T_i)}$$

The corresponding σ, β, α, ρ and ν parameters were derived by interpolating our calibrated results for 10Y tenor in the previous section to get parameters for 2Y & 8Y expiries, respectively. We observed

$$L(2,10) = 0.039634,\ PVBP1 = 9.699536,$$
$$L(8,10) = 0.048711,\ PVBP2 = 9.375970.$$

Prices of the swaptions were then computed by simply plugging in the necessary parameters into our defined DD and Black76 pricing function (where the latter uses SABR model for volatility). Table 2.3.1 and 2.3.2 depicts the results of our pricing using the calibrated models.

i) **Table 2.3.1** Payer 2Y×10Y Swap

Strikes	DD	SABR
0.01	0.288142	0.289605
0.02	0.194936	0.198311
0.03	0.112326	0.115202
0.04	0.051345	0.052181
0.05	0.017366	0.021375
0.06	0.004106	0.010725
0.07	0.000651	0.006602
0.08	0.000067	0.004622

ii) **Table 2.3.2** Receiver 8Y×10Y Swap

Strikes	DD	SABR
0.01	0.018985	0.019243
0.02	0.033904	0.038409
0.03	0.056649	0.061080
0.04	0.088980	0.090172
0.05	0.132050	0.130219
0.06	0.186136	0.186031
0.07	0.250582	0.257305
0.08	0.323971	0.338549

Part III – Convexity Correction

1. Present value of CMS product

A CMS leg is a collection of CMS rates paid over a period. The PV is the sum of the discounted values of the CMS rates, multiplied by the day count fraction.

Each CMS rate is calculated using equation 3.1 below:

$$E^T\left[S_{n,N}(T)\right] = g(F) + \frac{1}{D(0,T)}\left[\int_0^F h''(K)V^{rec}(K)dK + \int_F^\infty h''(K)V^{pay}(K)dK\right] \tag{3.1}$$

Here, the IRR-settled option price (V^{pay} or V^{rec}) is given by:

$$V(K) = D(0,T)\ \ IRR(S_{n,N}(0)Black76\left(S_{n,N}(0), K, \sigma_{SABR}, T\right))$$

where

i) $F = S_{n,N}(0)$ is the forward swap rate calculated based on the curves we bootstrapped in Part I.

ii) σ_{SABR} is the volatility obtained from the SABR model. Parameters required for corresponding expiry\tenor were interpolated from the calibrated α, ρ and ν values from Part II.

iii) IRR (i.e. sum of all the discount factors for a given range N×M) = $\sum_{i=1}^{N \times M} \frac{\frac{1}{m}}{(1+\frac{S}{m})^i}$

$$h''(K) = \frac{-IRR''(K).K - 2\ .\ IRR'(K)}{IRR(K)^2} + \frac{2\ .IRR'(K)^2\ .K}{IRR(K)^3} \tag{3.2}$$

Hence,

a) The PV of a leg receiving CMS10y semi-annually over the next 5 years:

$$PV = D(0,6m) \times 0.5 \times E^T\left[S_{6m,10y6m}(6m)\right] + D(0,1y) \times 0.5 \times E^T\left[S_{1y,11y}(1y)\right]$$

$$\ldots\ldots$$

$$+D(0,5y) \times 0.5 \times E^T\left[S_{5y,15y}(5y)\right]$$

= **0.204486**

b) The PV of a leg receiving CMS2y quarterly over the next 10 years:

$$PV = D(0,3m) \times 0.25 \times E^T\left[S_{3m,2y2m}(3m)\right] + D(0,6m) \times 0.25 \times E^T\left[S_{6m,2y6m}(6m)\right]$$

$$\ldots\ldots$$

$$+D(0,10y) \times 0.25 \times E^T\left[S_{10y,10y2m}(10y)\right]$$

= **0.582685**

2. Comparison of Forward Swap Rates and CMS Rates

CMS Rates obtained using equation 3.1 are documented in Table 3.2,

Table 3.2 CMS Rates

	1Y	2Y	3Y	5Y	10Y
1Y	0.032029	0.033345	0.034158	0.035458	0.038848
5Y	0.040797	0.042536	0.043012	0.044087	0.047919
10Y	0.052270	0.055583	0.058432	0.060288	0.071961

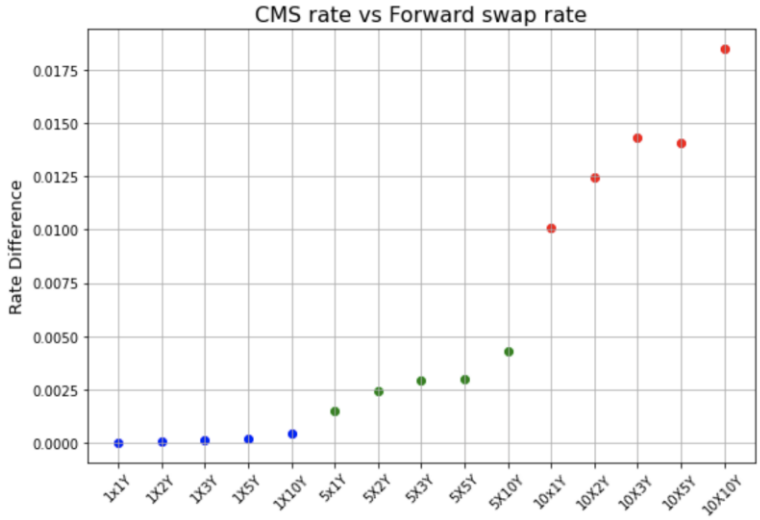
and Forward Swap Rate from Part I is included here for a direct comparison.

Table 1.3 Forward Swap Rates

	1Y	2Y	3Y	5Y	10Y
1Y	0.032007	0.033259	0.034011	0.035255	0.038428
5Y	0.039274	0.040075	0.040072	0.041093	0.043634
10Y	0.042189	0.043116	0.044097	0.046249	0.053458

The absolute differences between the values of the forward swap rates and CMS were computed and plotted in a graph for visual representation, as shown in Figure 3.2 below.

Figure 3.2 Difference between Forward swap rates and CMS



At first glance, we can see that the differences observed between FSR rates and CMS rates are higher when maturity increases, and besides some negligible outliers, the general trend also shows that the differences increase as tenor increases, but with lower variability. These dissimilarities are a result of convexity correction when valuing the CMS rates using the formula below.

$$E^T[S_{n,N}(T)] = g(F) + \boxed{\frac{1}{D(0,T)}\left[\int_0^F h''(K)V^{rec}(K)dK + \int_F^\infty h''(K)V^{pay}(K)dK\right]}$$

The convexity correction term arises from the boxed term from the Breeden-Litzenberg formula above, where g(F) is the payoff function for the underlying swap rates which can be replicated exactly.

CMS rate can only be observed at a swap settlement at time T in the future. As of current timestamp, CMS rate can be likened to a random variable whose value ultimately depends on the evolution of interest rates (that is likely stochastic) between current timestamp and the day it is realized.

In simple terms, swap (CMS) rate is simply the par-swap rate for a given tenor, and a CMS swap is the periodic exchange of a CMS rate versus either a fixed rate, or more typically versus LIBOR.

In contrast, forward swap rates (FSRs) and "forward" CMS rates are rates that are known with certainty. FSRs are used in pricing forward-starting swaps and European-style swaptions; while forward CMS rates are used in pricing CMS swaps and CMS caplets and floorlets.

To further understand the concept, consider sequence of dates:
$T_0 < T_1 < T_2 \ldots < T_n$, where time 0 is today.

The present value of a payer swap is given by:

$$\sum_{i=1}^{n} V_i^{flt}(t) - \sum_{i=1,}^{n} V_i^{fix}(t) = \big(D(0,T_0) - D(0,T_n)\big) - K\sum_{i=1}^{n} \Delta_{i-1}\, D(0,T_i)$$

A fixed rate K would make the value of the swap zero:

$$\frac{D(0,T_0) - D(0,T_n)}{\sum_{i=1}^{n} \Delta_{i-1} D(0,T_i)}$$

This particular rate is known as the par swap rate. A forward-starting swap is a swap initiating in the future which pays the par or forward swap rate, whereas when settlement time $T_0 = 0$ (i.e. today), it will be a spot starting swap.
Let's consider now the value of swap at settlement time T_0:

$$\frac{1 \quad - D(0,T_n)}{\sum_{i=1}^{n} \Delta_{i-1} D(0,T_i)}$$

We can then define the corresponding CMS rate as the time T_0 swap rate. In other words, the CMS rate is a spot swap rate for a forward-starting swap.

To calculate expected future CMS rate, we need to first calculate forward swap rate and then adjusting them by a CMS convexity adjustment.

For a CMS-based swap, we are hedging a linear payoff (CMS rate) with a nonlinear/convex payoff (forward swap). The hedge for receiving the CMS-rate is to pay in a PV01- equivalent amount of a forward swap. This hedge, however, confers a systematic advantage to the CMS-receiver, the size of which depends on the curvature/convexity of the forward swap versus the swap rate, and the expected size of the deviation of the realized swap rate at reset date versus its forward value. The CMS-payer will charge the receiver for this benefit by adjusting the forward swap rate upwards by the convexity adjustment.

Part IV – Decompounded Options

The IRR-settled option price (V^{pay} and V^{rec}) is given by:

$$V^{pay}(K) = D(0,T)\int_K^\infty IRR(S)(S-K)f(S)dS \tag{4.1}$$

$$V^{rec}(K) = D(0,T)\int_0^K IRR(S)(K-S)f(S)dS \tag{4.2}$$

$$IRR(S) = \sum_{i=1}^{N\times m} \frac{\frac{1}{m}}{(1+\frac{S}{m})^i} \tag{4.3}$$

$$V_0 = D(0,T)g(F) + \int_0^F h''(K)V^{rec}(K)dK + \int_F^\infty h''(K)V^{pay}(K)dK \tag{4.4}$$

$$h(K) = \frac{g(K)}{IRR(K)}$$

$$h'(K) = \frac{IRR(K)g'(K) - g(K)IRR'(K)}{IRR(K)^2} \tag{4.5}$$

$$h''(K) = \frac{IRR(K)g''(K) - IRR''(K)g(K) - 2IRR'(K)g'(K)}{IRR(K)^2} + \frac{2IRR'(K)^2 g(K)}{IRR(K)^3}$$

$$F = S_{n,N}(0), \quad n = 5, \quad N = 15, \quad T = 5, \quad p = 4, \quad q = 2$$

$$g(K) = K^{\frac{1}{4}} - 0.2, \qquad g'(K) = \frac{1}{4}K^{\frac{-3}{4}}, \qquad g''(K) = -\frac{3}{16}K^{\frac{-7}{4}}$$

Static replication for valuation

1. $Payoff = CMS10y^{\frac{1}{p}} - 0.04^{\frac{1}{q}}$

With the defined variables, we can value the PV of this decompounded option payoff using the formulas above.
PV of payoff V_0 = **0.243024**

2. $Payoff = (CMS10y^{\frac{1}{p}} - 0.04^{\frac{1}{q}})^+$

We have $S_T^{\frac{1}{4}} > 0.04^{\frac{1}{2}}$, $S_T > 0.0016 = L$,

$$\begin{aligned} V_0^+ &= D(0,T)\int_L^\infty g(K)f(K)dK \\ &= \int_L^\infty h(K)\frac{\partial^2 V^{pay}(K)}{\partial K^2}dK \\ &= h'(L)\partial^2 V^{pay}(L) + \int_L^\infty h''(K)V^{pay}(K)dK \end{aligned} \tag{4.6}$$

This payoff function is similar to the payoff in part 1, except a floor value is incorporated.
Again, using the defined variables, the value of the PV of this decompounded option payoff can be computed using the formulas above.
PV of payoff V_0^+ = **0.246552**