QF620 Stochastic Modelling in Finance

Group Project

ANKIT RAWAT

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Part I (Analytical Option Formulae)

1. Black-Scholes Model

The Black-Scholes model for the stock price process is defined under the risk-neutral measure Q∗ as:

$$dS_t = rS_t dt + \sigma S_t dW_t^*$$

Applying Itoˆs formula to the function Xt = f (St), where f (x) = log(x), we have:

$$S_T = S_0 e^{(r-\frac{\sigma^2}{2})T+\sigma W_T^*}, W_T^* \sim N(0,T)$$

This model is a more accurate as a reflection of non-zero equity price.
Evaluate the below formula with a discount factor:

$$\mathbb{E}[(S_T - K)^+]$$

We can obtain the Vanilla European call option price:

$$V_0^c = S_0\Phi\left(\frac{\log\frac{S_0}{K}+\left(r+\frac{\sigma^2}{2}\right)T}{\sigma\sqrt{T}}\right) - Ke^{-rT}\Phi\left(\frac{\log\frac{S_0}{K}+\left(r-\frac{\sigma^2}{2}\right)T}{\sigma\sqrt{T}}\right)$$

Let:

$$d1 = \frac{log(\frac{S_0}{K})+(r+\frac{1}{2}\sigma^2)T}{\sigma\sqrt{T}} \quad d2 = \frac{log(\frac{S_0}{K})+(r-\frac{1}{2}\sigma^2)T}{\sigma\sqrt{T}}$$

Vanilla Call:

$$V_0 = S_0\Phi(d1) - Ke^{-rT}\Phi(d2)$$

By Put Call parity get the vanilla put:

$$V_0 = S_0\Phi(-d2) - Ke^{-rT}\Phi(-d1)$$

Digital Cash-or-nothing Call:

$$\begin{aligned} V_{\text{Cash Digital}}(0) &= e^{-rT}\mathbb{E}^{\mathbb{Q}^*}\left[\mathbb{1}_{S_T>K}\right] \\ &= \frac{e^{-rT}}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\mathbb{1}_{S_T>K}e^{-\frac{x^2}{2}}dx \\ &= \frac{e^{-rT}}{\sqrt{2\pi}}\int_{x^*}^{\infty}e^{-\frac{x^2}{2}}dx \\ &= e^{-rT}\Phi\left(\frac{\log\left(\frac{S_0}{K}\right)+\left(r-\frac{1}{2}\sigma^2\right)T}{\sigma\sqrt{T}}\right) \end{aligned}$$

$$V_0 = e^{-rT}\Phi(d2)$$

Then Digital Cash-or-nothing Put:

$$V_0 = e^{-rT}\Phi(-d2)$$

Digital Asset-or-nothing Call:

$$\begin{aligned}V_{\text{Asset Digital}}(0) &= e^{-rT}\mathbb{E}^{\mathbb{Q}^*}\left[S_T \mathbb{1}_{S_T>K}\right] \\ &= \frac{e^{-rT}}{\sqrt{2\pi}}\int_{-\infty}^{\infty} S_T \mathbb{1}_{S_T>K} e^{-\frac{x^2}{2}}\,dx \\ &= \frac{e^{-rT}}{\sqrt{2\pi}}\int_{x^*}^{\infty} S_0 e^{\left(r-\frac{1}{2}\sigma^2\right)T+\sigma\sqrt{T}x} e^{-\frac{x^2}{2}}\,dx \\ &= S_0\Phi\left(\frac{\log\left(\frac{S_0}{K}\right)+\left(r+\frac{1}{2}\sigma^2\right)T}{\sigma\sqrt{T}}\right).\end{aligned}$$

$$V_0 = S_0\Phi(d1)$$

Digital Asset-or-nothing Put:

$$V_0 = S_0\Phi(-d1)$$

2. Bachelier Model

The Bachelier model for the stock price process is defined as:

$$dS_t = \sigma S_0 dW_t$$

Bachelier modelled stock price as normally distributed. However, a shortcoming is that the lack of a lower bound at 0. In other words, while this is a good model for interest rates, it leads to non-zero probability for negative stock prices. By directly integrating it, we obtain:

$$\begin{aligned}\int_0^T dS_u &= \sigma S_0 \int_0^T dW_u \\ S_T - S_0 &= \sigma S_0 W_T \\ S_T &= S_0 + \sigma S_0 W_T.\end{aligned}$$

We want to evaluate:

$$\mathbb{E}[(S_T-K)^+] = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\left(S_0+\sigma S_0\sqrt{T}x-K\right)^+ e^{-\frac{x^2}{2}}\,dx$$

So:

$$\begin{aligned}\mathbb{E}[(S_T-K)^+] &= \frac{1}{\sqrt{2\pi}}\int_{x^*}^{\infty}(S_0+\sigma S_0\sqrt{T}x-K)e^{-\frac{x^2}{2}}\,dx \\ &= \frac{1}{\sqrt{2\pi}}\int_{x^*}^{\infty}(S_0-K)e^{-\frac{x^2}{2}}\,dx + \frac{1}{\sqrt{2\pi}}\int_{x^*}^{\infty}\sigma S_0\sqrt{T}xe^{-\frac{x^2}{2}}\,dx \\ &= (S_0-K)\Phi(-x^*) + \frac{\sigma S_0\sqrt{T}}{\sqrt{2\pi}}e^{\frac{-x^{*2}}{2}} \\ &= (S_0-K)\Phi\left(\frac{S_0-K}{\sigma S_0\sqrt{T}}\right) + \sigma S_0\sqrt{T}\phi\left(\frac{S_0-K}{\sigma S_0\sqrt{T}}\right)\end{aligned}$$

Let:

$$c = \frac{S_0-K}{\sigma S_0\sqrt{T}}$$

Vanilla Call:

$$V_0 = (S_0 - K)\Phi(c) + \sigma S_0 \sqrt{T}\phi(c)$$

By Put-Call parity, we can get Vanilla Put:

$$V_0 = (K - S_0)\Phi(-c) + \sigma S_0 \sqrt{T}\phi(-c)$$

Digital Cash-or-nothing Call:

$$\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \mathbb{1}_{S_T > K} e^{-\frac{x^2}{2}} dx = \frac{1}{\sqrt{2\pi}} \int_{x^*}^{\infty} e^{-\frac{x^2}{2}} dx$$
$$= \Phi(-x^*) = \Phi\left(\frac{S_0 - K}{S_0 \sigma \sqrt{T}}\right)$$

$$V_0 = \Phi(c)$$

Then Digital Cash-or-nothing Put:

$$V_0 = \Phi(-c)$$

The asset digital call option can be priced as:

$$V_0 = S_0\Phi(c) + \sigma S_0 \sqrt{T}\phi(c)$$

Digital Asset-or-nothing Put:

$$V_0 = S_0\Phi(-c) - \sigma S_0 \sqrt{T}\phi(-c)$$

3. Black76 Model

The Black76 model is defined on the forward price and is given by:

$$dF_t = \sigma F_t dW_t$$

Applying Ito's formula to the function Xt = f (Ft), where f (x) = log(x), we have:

$$F_T = F_0 e^{-\frac{\sigma^2 T}{2} + \sigma W_T}$$

Then follow same process as above to price the option, and take a discount factor, we can get the vanilla European Call option:

$$V_0^c = D(0,T)\left[F\Phi\left(\frac{\log\frac{F}{K} + \frac{1}{2}\sigma^2 T}{\sigma\sqrt{T}}\right) - K\Phi\left(\frac{\log\frac{F}{K} - \frac{1}{2}\sigma^2 T}{\sigma\sqrt{T}}\right)\right]$$

Let:

$$e1 = \frac{log(\frac{F_0}{K}) + \frac{1}{2}\sigma^2 T}{\sigma\sqrt{T}} \qquad e2 = \frac{log(\frac{F_0}{K}) - \frac{1}{2}\sigma^2 T}{\sigma\sqrt{T}}$$

Vanilla Call:

$$V_0 = e^{-rT}\left[F_0\Phi(e1) - K\Phi(e2)\right]$$

Vanilla Put:

$$V_0 = e^{-rT}\left[K\Phi(-e2) - F_0\Phi(-e1)\right]$$

Digital Cash-or-nothing Call:

$$V_0 = e^{-rT}\Phi(e2)$$

Digital Cash-or-nothing Put:

$$V_0 = e^{-rT}\Phi(-e2)$$

Digital Asset-or-nothing Call:

$$V_0 = F_0 e^{-rT}\Phi(e1)$$

Digital Asset-or-nothing Put:

$$V_0 = F_0 e^{-rT}\Phi(-e1)$$

4. Displaced Diffusion Model

We say that Ft follows a lognormal distribution. Based on this definition, we call the following a displaced-diffusion, or shifted lognormal, process:

$$d(F_t + \alpha) = \sigma(F_t + \alpha)dW_t, \qquad \alpha \in \mathbb{R}.$$

The following stochastic differential equation is the most commonly used form for displaced-diffusion process:

$$dF_t = \sigma[\beta F_t + (1-\beta)F_0]dW_t, \qquad \beta \in [0,1].$$

apply *Ito^s* formula to the function f:

$$X_t = f(F_t), \qquad \text{where } f(x) = \log[\beta x + (1-\beta)F_0]$$

And obtain:

$$F_T = \frac{F_0}{\beta}\exp\left[-\frac{\beta^2\sigma^2 T}{2} + \beta\sigma_T W_T\right] - \frac{1-\beta}{\beta}F_0$$

Let:

$$f1 = \frac{log(\frac{F_0}{F_0+\beta(K-F_0)}) + \frac{1}{2}\beta^2\sigma^2 T}{\sigma\sqrt{T}} \quad f2 = \frac{log(\frac{F_0}{F_0+\beta(K-F_0)}) - \frac{1}{2}\beta^2\sigma^2 T}{\sigma\sqrt{T}}$$

Vanilla Call:

$$V_0 = e^{-rT}\left[\frac{F_0}{\beta}\Phi(f1) - (\frac{1-\beta}{\beta}F_0 + K)\Phi(f2)\right]$$

Vanilla Put:

$$V_0 = e^{-rT}\left[(\frac{1-\beta}{\beta}F_0 + K)\Phi(-f2) - \frac{F_0}{\beta}\Phi(-f1)\right]$$

Digital Cash-or-nothing Call:

$$V_0 = e^{-rT}\Phi(f2)$$

Digital Cash-or-nothing Put:

$$V_0 = e^{-rT}\Phi(-f2)$$

Digital Asset-or-nothing Call:

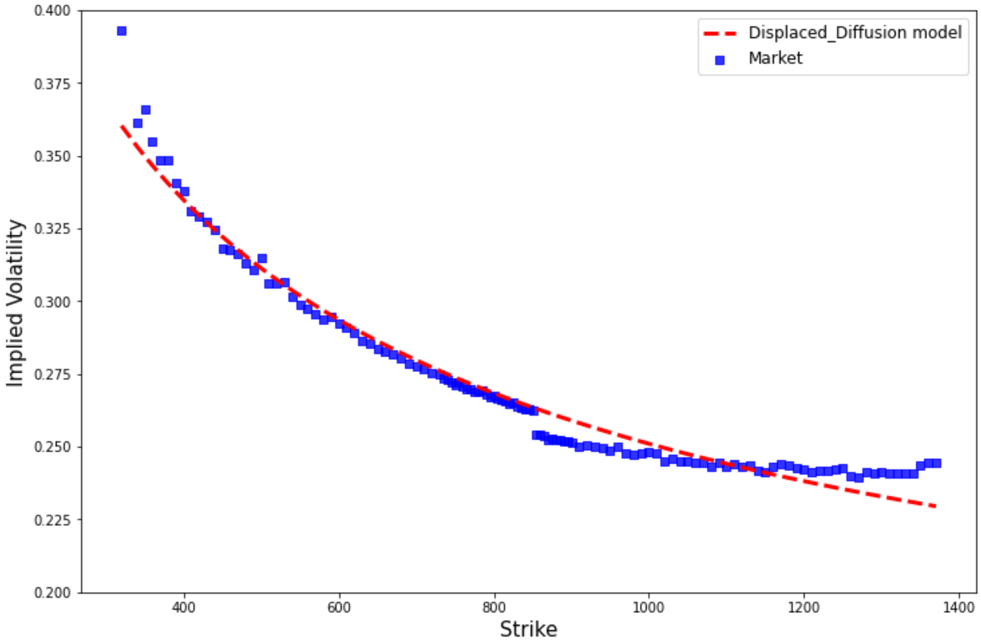
$$V_0 = e^{-rT}\left[\frac{F_0}{\beta}\Phi(f1) - (\frac{1-\beta}{\beta}F_0)\Phi(f2)\right]$$

Digital Asset-or-nothing Put:

$$V_0 = e^{-rT}\left[\frac{F_0}{\beta}\Phi(-f1) - (\frac{1-\beta}{\beta}F_0)\Phi(-f2)\right]$$

Part II (Model Calibration)

1. Displaced-diffusion model



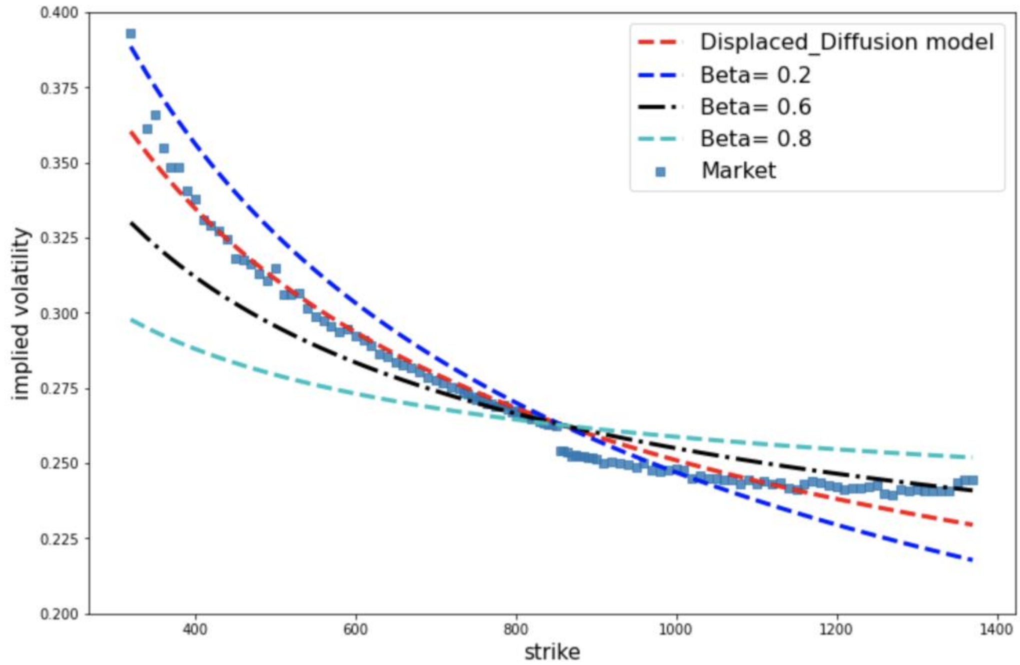
By fitting the observed option prices in the market to the Displaced Diffusion model, following model parameters have been derived.

Model Parameters	
σ	0.2625
β	0.3989

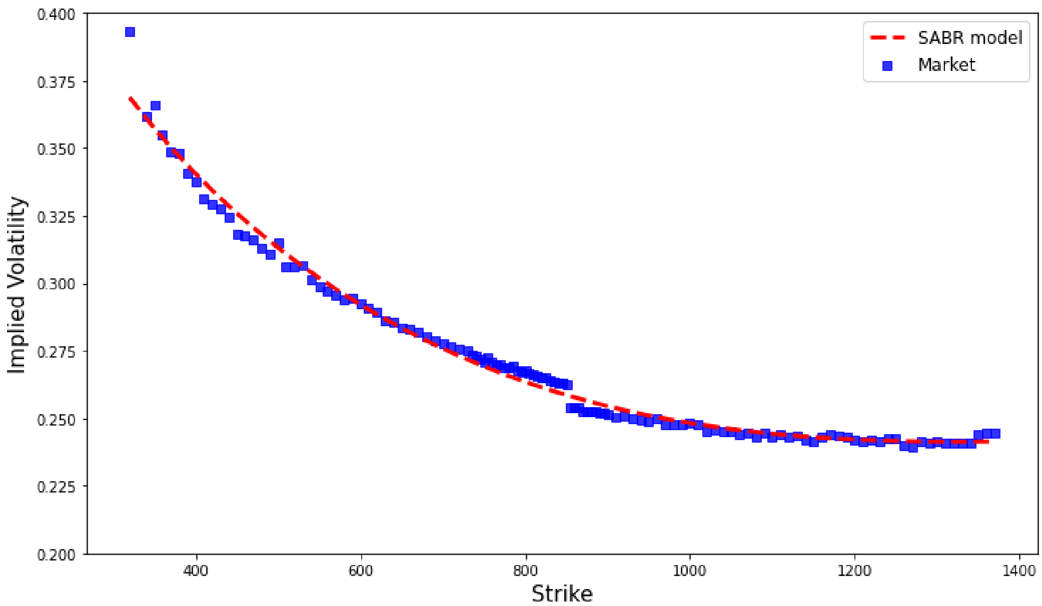
Considering the Beta value being less than 0.5, we infer that the price process has a slight tilt towards normal (arithmetic) Brownian motion. Increasing the Beta value will add more weightage to the log normal process and can capture volatility (implied vol) at higher strike prices, however at the same time it will move away from volatility at lower strike prices. In other words, the model is effective in capturing implied volatility skew however is unable to capture the 'smile' characteristic.

As can be observed from the above, β close to 0.4 captures the volatility changes at lower strike prices while the volatility changes at higher strike prices are not adequately captured by the model.

Then we keep the volatility constant, keep changing β to see how the curve is shaping. When β is close to 0, means that the model is close to normal model, the implied vol is showing a curved line. When β is close to 1, the curve is becoming more and more flat close to the performance of a lognormal model. And importantly, with β changing, the curve only skew to some extent, does not show the property of 'smile'. Shown as below:



2. SABR model (fixed β= 0.8)

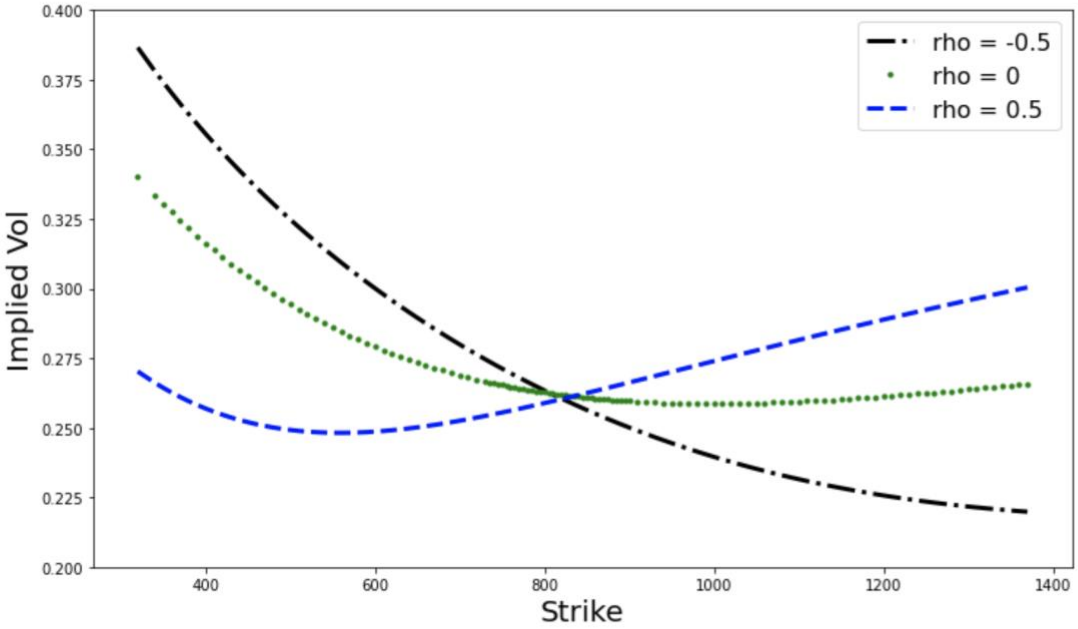


By fitting the observed option prices in the market to the SABR model, following model parameters have been derived.

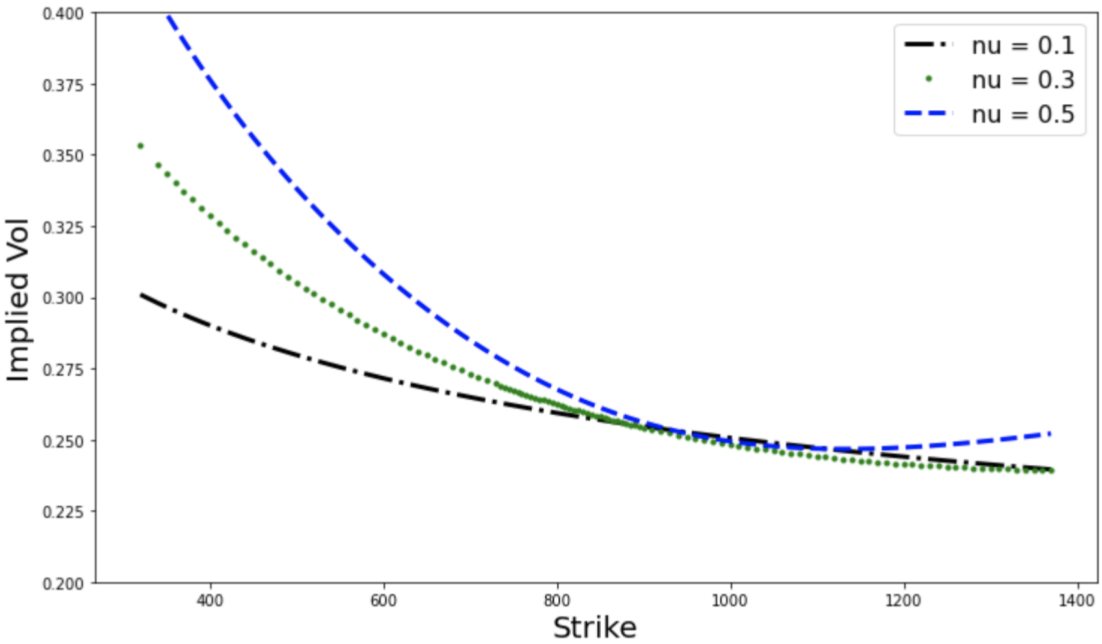
Model Parameters	
α	0.991
ρ	-0.285
v	0.352

ρ is the correlation parameter, and it moves in line with the skewness of stock returns. Negative correlation increases the price of out-of-the-money put options as there will be high volatility when the stock price drops and decreases the price of out-of-the-money call options.

However as ρ increases the implied vol smile becomes <u>more pronounced</u> in the curve, while the curve also becoming more skewed.



ν, a measure of volatility of volatility moves in line with the 'Kurtosis' the stock returns. Increase in volatility of volatility creates fat tails at both ends of the return distribution resulting in increase in prices of far from the money options relative to near the money options. This translates into a <u>more pronounced</u> 'smile' characteristic in the implied volatility curve plotted against strike.



Part III

The price of the European derivatives contract calculated using the given payoff function is given below for the respective models.

Model	Price of Derivative
Black Scholes	21.79
Bachelier	21.38

Choice of Sigma (Volatility) – The Volatility input (close to 0.26) for the price calculation is the implied volatility measured through the Black Scholes model at the strike equalling the stock price as on 30/08/2013.

Part IV (Dynamic Hedging)

Monte Carlo simulation of dynamic hedging strategy to calculate the hedging error which results from discrete, rather than continuous hedging.

The Black-Scholes formula is given by:

$$\text{European Call} = C(S_0, K, r, \sigma, T) = S_0\Phi(d_1) - Ke^{-rT}\Phi(d_2)$$

where

$$d_1 = \frac{\log\left(\frac{S_0}{K}\right) + \left(r + \frac{\sigma^2}{2}\right)T}{\sigma\sqrt{T}}, \qquad d_2 = d_1 - \sigma\sqrt{T} = \frac{\log\left(\frac{S_0}{K}\right) + \left(r - \frac{\sigma^2}{2}\right)T}{\sigma\sqrt{T}}.$$

Let S_t, B_t and V_t denote the value of the stock price, the bond price and the value of the portfolio (ϕ_t, ψ_t), respectively.

At time t, the worth of the portfolio is given by

$$V_t = \phi_t S_t + \psi_t B_t.$$

The pair (ϕ_t, ψ_t) is a dynamic trading strategy detailing the amount of each component to be held at each instant.

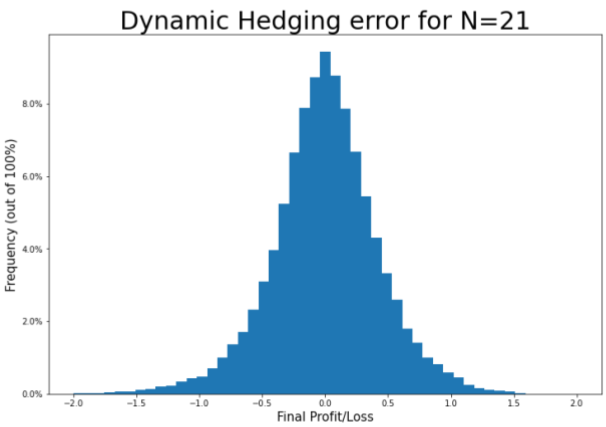
$$
(\phi_t, \psi_t) :\rightarrow \begin{cases} \phi_t = \Delta_t = \dfrac{\partial C}{\partial S} = \Phi\left(\dfrac{\log \frac{S}{K} + \left(r + \frac{\sigma^2}{2}\right) T}{\sigma\sqrt{T}}\right) \\ \psi_t B_t = -Ke^{-rT}\Phi\left(\dfrac{\log \frac{S}{K} + \left(r - \frac{\sigma^2}{2}\right) T}{\sigma\sqrt{T}}\right) \end{cases}
$$

The final P&L of this hedging strategy i.e. the hedging error is defined as

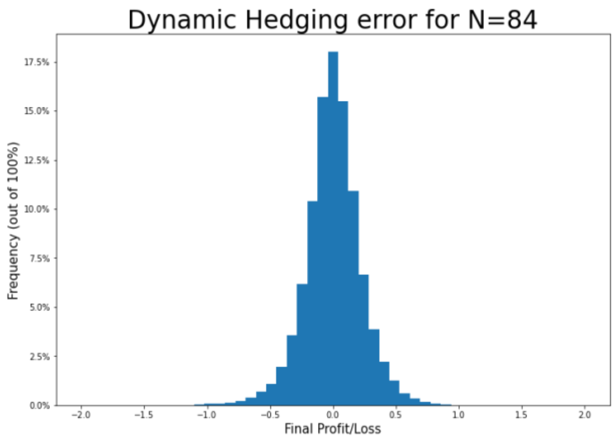
$$
\text{Hedging Error} = \Big(\phi_T S_T - \psi_T B_T\Big) - \max\{S_T - K, 0\}.
$$

Histograms for the final P&L for a one-month at-the-money call hedged, at discrete times, to expiration for (a) 21 and (b) 84 rebalancing trades.

a)



b)



Statistical summary of the simulated profit/loss.

	Mean P&L	Standard Dev. of P&L	StDev of P&L as a % of option premium
N=21	-0.002	0.43	16.98
N=84	0.002	0.22	8.76

Observations from the result

1) The average profit & loss within the statistical range of the simulated error is close to zero
2) Hedging more frequently reduces the volatility of the profit & loss
3) The distribution of replication error resembles a normal distribution, hence allowing risk to be measured in standard deviation of the distribution