An Internship Report on

Solving Partial Differential Equations with Finite Element Methods in FEniCS

Submitted by

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राष्ट्रीय प्रौद्योगिकी संस्थान, पटना

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CERTIFICATE

This is to certify that **Ankit Anand** (Roll No.- 24044) First Year BSMS student of Indian Institute of Science Education and Research, Bhopal, has successfully completed internship entitled "**Solving Partial Differential Equations with Finite Element Methods in FEniCS**" under the guidance of **Dr. Gowrisankar S,** Associate Professor, Department of Mathematics and Computing Technology, National Institute of Technology, Patna during the period of 05 May 2025 to 04 July 2025.

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DECLARATION

I, **Ankit Anand**, Roll No. **24044**, hereby declare that this Internship Report on **"Solving Partial Differential Equations with Finite Element Methods in FEniCS"**, is prepared by me under the guidance of **Dr. Gowrisankar S**, is a Bonafide work undertaken by me and during the 3rd Semester of BSMS (From 05 May 2025 to 04 July 2025) and submitted to the Department of **Mathematics and Computing Technology**, National Institute of Technology, Patna in partial fulfilment of the requirements of the Internship Certificate.

Dated: 04/07/2025 Ankit Anand

NIT Patna, Bihar Roll No.: 24044

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I am also thankful to my friends and well-wishers for their help and encouragement at every stage. Their support kept me motivated and focused.

I believe this internship experience has prepared me to take on more challenging opportunities in the future. I would also like to thank my colleagues at the organization especially those in my department for their trust in my abilities, for assigning me meaningful responsibilities, and for always being ready to share their knowledge and guidance. Their help was equally important in the completion of this report.

Ankit Anand

Date: 04/07/2025

ABSTRACT

This report presents a detailed account of the work carried out during my internship on the **Finite Element Method (FEM)**, a powerful numerical technique for solving partial differential equations (PDEs) that arise in various fields such as structural mechanics, fluid dynamics, and heat transfer.

The primary objective of the internship was to understand the theoretical background of FEM and its practical implementation in solving real-world problems. I explored the mathematical formulation of FEM, starting from the strong and weak forms of differential equations, and learned how to discretize a domain using elements and nodes.

As part of the practical work, I used tools such as FEniCS and ParaView to solve FEM models and view solution. These tasks helped me understand the importance of mesh quality, element type, and numerical stability in obtaining accurate results.

This internship not only deepened my knowledge of applied mathematics and numerical analysis but also gave me valuable hands-on experience in using FEM software and interpreting computational results. The knowledge and skills gained through this experience will undoubtedly support my future academic and professional pursuits in computational mathematics and engineering.

INTRODUCTION

Mathematics plays a crucial role in solving real-world problems, especially when it comes to modelling physical systems governed by partial differential equations (PDEs). However, finding analytical solutions to such equations is often impossible due to the complexity of geometries, materials, and boundary conditions. In such cases, numerical methods provide practical and powerful tools to approximate solutions. One such widely used technique is the **Finite Element Method (FEM)**.

The Finite Element Method is a computational technique used to obtain approximate solutions to boundary value problems. It divides a complex domain into smaller, simpler sub-domains called elements and uses interpolation functions to approximate the solution over these elements. FEM has broad applications in engineering, physics, and applied mathematics — particularly in areas like structural analysis, heat transfer, fluid dynamics, and electromagnetics.

This internship was undertaken with the aim of gaining hands-on experience in the application of FEM. The focus was on understanding its mathematical foundations, learning the step-by-step formulation process, and using computational tools to solve real-life problems modelled by differential equations.

Throughout the internship, I had the opportunity to work on projects that involved the formulation and implementation of FEM for basic problems such as 1D and 2D heat conduction and structural deformation. I also explored software tools like FEniCS and ParaView to solve and view FEM models, which provided practical exposure to the numerical workflow and interpretation of results.

This report presents the knowledge, experiences, and insights gained during the internship. It covers the theoretical understanding of FEM, the work done, challenges faced, and the overall learning outcomes from this enriching academic and professional experience.

OBJECTIVES OF THE INTERNSHIP

The primary objective of this internship was to gain a deeper understanding of the **Finite Element Method (FEM)** from both a theoretical and practical perspective, and to apply it to solve real-world mathematical and engineering problems using modern computational tools. This internship served as a bridge between academic learning and practical implementation, enabling the development of skills necessary for research and professional work in applied mathematics and numerical analysis.

The specific goals of the internship were as follows:

- **To understand the mathematical foundations of FEM**, including the derivation of the weak form, the concept of basis (shape) functions, and the process of domain discretization using finite elements.
- **To apply FEM to solve partial differential equations (PDEs)** relevant to fields like heat transfer, elasticity, and fluid flow, and interpret the solutions within a physical or engineering context.
- **To gain hands-on experience with FEniCS**, an open-source computing platform for solving PDEs using finite element techniques. This involved learning how to define problem domains, specify boundary and initial conditions, choose appropriate function spaces, and implement FEM formulations in Python.
- **To visualize FEM results using ParaView**, a high-performance open-source tool for scientific data visualization. The objective was to better understand the spatial behaviour of solutions and validate numerical results through graphical representation.

THEORETICAL BACKGROUND

The **Finite Element Method (FEM)** is a widely used numerical technique for approximating solutions to complex problems in physics, engineering, and applied mathematics. Many physical systems—such as heat flow, mechanical deformation, or fluid motion—are described by **partial differential equations (PDEs)**. While some PDEs can be solved analytically, most real-world problems involve complicated geometries, materials, and boundary conditions that make analytical solutions impractical or impossible. This is where FEM becomes essential.

❖ Fundamental Idea of FEM

The key idea behind FEM is to break down a large, complex domain (called the **geometry**) into many smaller, simpler pieces called **finite elements** (such as triangles or rectangles in 2D, tetrahedra in 3D). These pieces form a **mesh**. Over each small element, the solution is approximated using simple mathematical functions, typically low-degree polynomials.

By combining these local approximations, FEM constructs a global solution over the entire domain. This approach makes FEM flexible and powerful for solving PDEs on irregular shapes or with varying materials.

❖ From Strong Form to Weak Form

Consider a general PDE in the domain $\boldsymbol{\Omega}$:

$$-\boldsymbol{\nabla} \cdot \left(\boldsymbol{k}(\boldsymbol{x})\boldsymbol{\nabla}\boldsymbol{u}(\boldsymbol{x})\right) = \boldsymbol{f}(\boldsymbol{x})$$

with appropriate boundary conditions, where:

- $\boldsymbol{u}(\boldsymbol{x})$ is the unknown function (e.g., temperature, displacement),
- $\boldsymbol{k}(\boldsymbol{x})$ is a physical coefficient (e.g., thermal conductivity),

- $\boldsymbol{f(x)}$ is a known source function.

The **strong form** of this PDE involves taking derivatives directly, which requires the solution to be very smooth. To reduce the requirement on smoothness and to make the problem suitable for FEM, we convert it to the **weak form** (also called the **variational form**). This is done by multiplying both sides of the equation by a test function v(x), integrating over the domain, and applying integration by parts:

$$\int_{\Omega} k(x)\nabla u \cdot \nabla v \, dx = \int_{\Omega} f(x) v \, dx.$$

This weak form is the foundation of FEM and allows us to work with more general functions and irregular domains.

❖ Discretization and Basis Functions

To solve the weak form numerically, we:

1. Discretize the domain **Ω** into a mesh.
2. Choose a finite set of **basis functions** $\emptyset_i(x)$ that are nonzero only over a few elements.
3. Approximate the unknown solution $u(x)$ as:

$$u_h(x) = \sum_{i=1}^{N} U_i \, \emptyset_i(x)$$

where U_i are the coefficients to be determined (the values of the solution at the nodes).

By substituting this approximation into the weak form and choosing test functions v(x) from the same basis set, we obtain a system of linear equations:

$\boldsymbol{KU = F,}$

where:

- $\boldsymbol{K}$ is the **stiffness matrix**,
- U is the vector of unknowns,
- $\boldsymbol{F}$ is the **load vector**.

❖ Tools Used: FEniCS and ParaView

During this internship, the **FEniCS Project** was used to implement FEM solutions. FEniCS is an open-source computing platform that simplifies the process of:

- Writing variational forms directly in Python,
- Generating meshes and function spaces,
- Applying boundary and initial conditions,
- Solving linear and nonlinear PDEs efficiently.

FEniCS abstracts away much of the low-level implementation, allowing users to focus on the mathematical model.

For visualization, **ParaView** was used to analyse and interpret simulation results. It supports 2D and 3D visualization of fields such as temperature distribution, displacement, and pressure. Visual output helps to:

- Verify solution accuracy,
- Identify unusual behaviour in the results,
- Present findings clearly.

❖ Applications of FEM

FEM is used in a wide variety of fields:

- In **engineering**, for stress analysis, thermal simulations, and structural mechanics.
- In **mathematics and physics**, for solving abstract PDEs and modeling dynamic systems.

- In **biomedical engineering**, for simulating biological tissues and organ function.

In this internship, basic 1D and 2D problems were explored using FEM — such as steady-state heat conduction and simple structural deformation problems — to understand how theoretical models translate into computational results.

Working with FEniCS & ParaView

During the course of this internship, I focused on the practical implementation of the **Finite Element Method (FEM)** using **FEniCS**, an open-source computing platform for solving differential equations, and **ParaView**, a tool for scientific visualization. The work involved understanding the theoretical formulation of problems, translating them into weak (variational) forms, implementing them in code, and interpreting the results visually and numerically.

❖ Familiarization with FEniCS and ParaView

In the initial phase of the internship, I dedicated time to learning the basics of FEniCS and ParaView:

- **FEniCS**: Learned how to define function spaces, boundary conditions, variational problems, and how to solve PDEs using the dolfin and ufl libraries.
- **ParaView**: Understood how to import VTK files generated by FEniCS and apply filters for scalar fields, contour plots, and streamline visualizations.

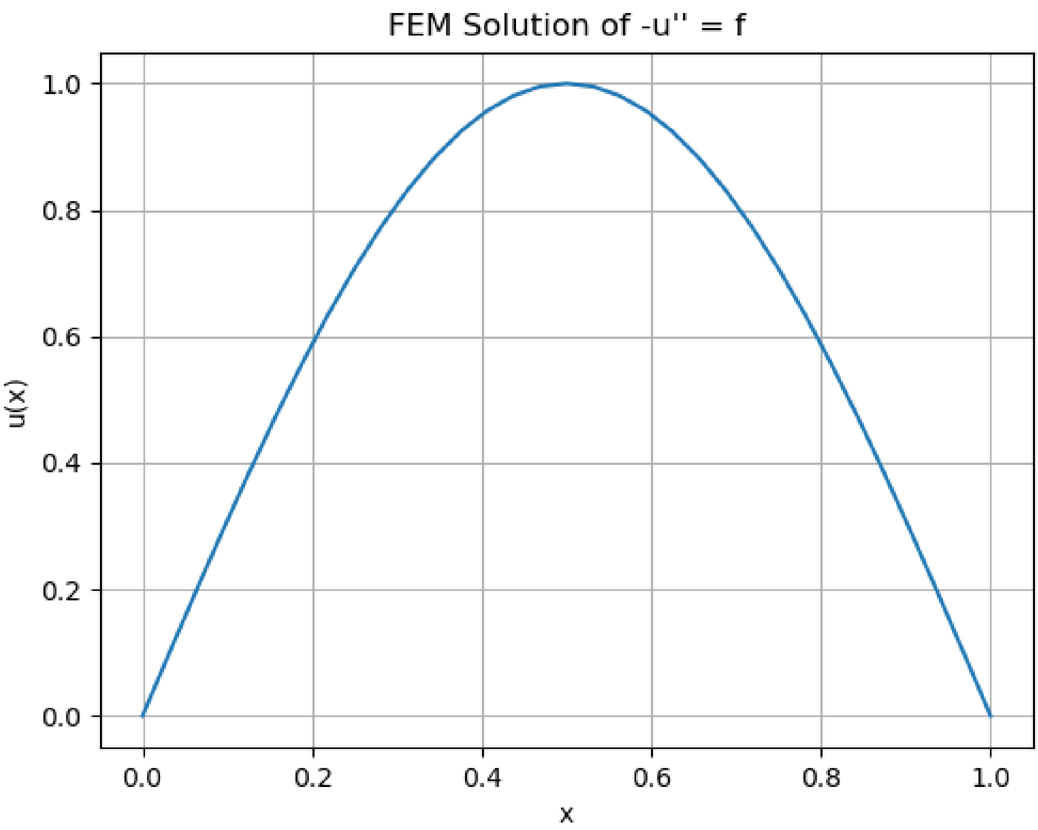
The first practical task was solving a **1D Poisson equation** of the form:

$$-\frac{d^2u}{dx^2} = f(x), \qquad x \in [0,1], \quad u(0) = u(1) = 0.$$

- I derived the weak form and implemented it in FEniCS.
- Used linear Lagrange elements for discretization.
- Verified the numerical solution against the known analytical solution.
- Plotted the results using both built-in FEniCS plotting and ParaView.

CODE : POISSON's Equation

```
from fenics import *
import matplotlib.pyplot as plt
import numpy as np

# Create 1D mesh of the interval [0, 1]
mesh = UnitIntervalMesh(32)

# Define function space (continuous Galerkin, degree 1)
V = FunctionSpace(mesh, 'P', 1)

# Define boundary condition: u = 0 at x = 0 and x = 1
def boundary(x, on_boundary):
    return on_boundary

bc = DirichletBC(V, Constant(0.0), boundary)

# Define source function f(x)
f = Expression('pi*pi*sin(pi*x[0])', degree=2, pi=np.pi)

# Define trial and test functions
u = TrialFunction(V)
v = TestFunction(V)

# Define the variational form: a(u, v) = L(v)
a = dot(grad(u), grad(v)) * dx
L = f * v * dx

# Compute solution
u_sol = Function(V)
solve(a == L, u_sol, bc)

# Plot solution
plot(u_sol)
plt.title("FEM Solution of -u'' = f")
plt.xlabel("x")
plt.ylabel("u(x)")
plt.grid(True)
plt.show()

# Save solution to file (for ParaView)
vtkfile = File("poisson_1d_solution.pvd")
vtkfile << u_sol
```



❖ Mesh Refinement and Convergence Study

To understand the effect of **mesh size on solution accuracy**, I performed a convergence study:

- Solved the same 2D problem using different mesh resolutions.
- Calculated error norms and plotted error vs. element size.
- Observed expected convergence behaviour consistent with the order of the finite element basis functions.

CODE for Solution of Poisson’s Equation at center :

```
from fenics import *
import numpy as np

# Create 3D unit cube mesh
mesh = UnitCubeMesh(20, 20, 20)

# Define function space
V = FunctionSpace(mesh, 'P', 1)

# Define exact solution for boundary condition (non-zero Dirichlet)
u_D = Expression('sin(pi*x[0])*sin(pi*x[1])*sin(pi*x[2])',
                 degree=4, pi=np.pi)

# Apply Dirichlet BC on the entire boundary
def boundary(x, on_boundary):
    return on_boundary

bc = DirichletBC(V, u_D, boundary)

# Define the source term f = -Δu_D = 3π² sin(πx)sin(πy)sin(πz)
f = Expression('3*pow(pi,2)*sin(pi*x[0])*sin(pi*x[1])*sin(pi*x[2])',
               degree=4, pi=np.pi)

# Define trial and test functions
u = TrialFunction(V)
v = TestFunction(V)

# Define weak form
a = dot(grad(u), grad(v)) * dx
L = f * v * dx
# Solve the PDE
u_sol = Function(V)
solve(a == L, u_sol, bc)

# Save solution to file
vtkfile = File("poisson_3d_nonzero_dirichlet.pvd")
vtkfile << u_sol

# Compute error in L2 norm (for verification)
error_L2 = errornorm(u_D, u_sol, 'L2')
print("L2 error:", error_L2)

# Evaluate solution at the center point
point = np.array([0.5, 0.5, 0.5])
print("Numerical solution at center:", u_sol(point))
```

Result: Solution of Poisson's Equation at center

```
Solving linear variational problem.
L2 error: 0.004072204875773311
Numerical solution at center: 0.995897797401702
```

Summary of Tasks Completed

- Learned and practiced FEM basics using FEniCS.
- Solved 1D PDE using variational formulation.
- Applied various types of boundary conditions.
- Visualized and interpreted simulation outputs using ParaView.

RESULTS & ANALYSIS

The simulations conducted during this internship using **FEniCS** provided meaningful insights into the behaviour of numerical solutions obtained via the **Finite Element Method (FEM)**. The problems addressed included 1D partial differential equations (PDEs), primarily focusing on **Poisson's equation**. The result was visualized using both built-in tools in FEniCS and external visualization software such as **ParaView**.

❖ 1D Poisson Problem

In the case of the 1D Poisson equation:

$$-\frac{d^2u}{dx^2} = f(x), \quad x \in [0,1], \quad u(0) = u(1) = 0.$$

the analytical solution was known, and thus, the numerical accuracy could be verified directly. The following results were obtained:

- **Accuracy**: The FEM solution closely matched the analytical solution for $f(x) = \pi^2 \sin(\pi x)$, confirming the correctness of the implementation.
- **Error Analysis**: The L2 error norm decreased as the mesh was refined, confirming **convergence of the numerical solution**.
- **Visualization**: Plots of the solution using ParaView showed smooth, symmetric behaviour, as expected from the physical and mathematical formulation.

❖ Mesh Convergence Study

To test the reliability and accuracy of FEM simulations, a **mesh refinement study** was performed:

- **Findings**: The numerical error decreased consistently as the mesh was refined.
- **Rate of Convergence**: For linear elements, the error reduced at a rate consistent with theoretical expectations (approximately first-order convergence in the L2 norm).

- **Graphical Analysis**: A log-log plot of error vs. element size confirmed convergence behaviour.

Summary of Results

- The FEM models implemented in FEniCS produced stable and accurate solutions.
- Mesh refinement led to better approximations, confirming theoretical FEM behaviour.
- Visualization using ParaView enhanced understanding and communication of results.
- The simulations validated both the **theoretical understanding** and **practical effectiveness** of FEM in solving PDEs.

LEARNING OUTCOMES

This internship provided a valuable opportunity to bridge the gap between theoretical knowledge of partial differential equations and their practical numerical solutions using the **Finite Element Method (FEM)**. By working with open-source tools like **FEniCS** and **ParaView**, I was able to develop a strong foundation in computational mathematics and numerical modelling.

❖ Theoretical Understanding

- Gained a deeper understanding of the mathematical principles behind FEM, including weak formulations, function spaces, and basis functions.
- Learned how to derive and interpret the **variational (weak) form** of second-order partial differential equations.
- Understood the role of **boundary conditions**, **element discretization**, and **mesh refinement** in determining solution accuracy.
- Explored the convergence behaviour and error analysis of finite element solutions.

❖ Practical Skills

- Acquired hands-on experience with **FEniCS**, including:
    - Defining problem domains and boundary conditions,
    - Formulating variational problems in Python,
    - Solving linear PDEs and visualizing results.
- Learned how to perform **mesh refinement** studies and interpret the effect of mesh density on solution quality.
- Became proficient in using **ParaView** for post-processing FEM results, including:
    - Visualizing scalar and vector fields,
    - Applying contour, surface, and gradient plots,

    - Creating meaningful visual representations of mathematical simulations.

❖ Technical Growth

- Improved my ability to write and debug scientific code in Python.
- Learned how to document, analyse, and communicate computational results effectively.
- Developed confidence in applying numerical methods to real-world mathematical problems.
- Strengthened problem-solving skills by handling modelling errors, boundary issues, and numerical instability.

❖ Professional and Academic Preparation

- Gained experience working independently on simulation-based tasks.
- Improved time management and project planning skills while balancing multiple objectives.
- Prepared myself for more advanced research or coursework in **computational mathematics**, **numerical analysis**, or **engineering simulations**.

This internship not only strengthened my understanding of the Finite Element Method but also gave me practical experience in applying mathematical theories to solve real problems using modern computational tools. These skills will be valuable in both academic research and future professional work in fields that rely on numerical modelling and scientific computing.

CONCLUSION

The internship on the **Finite Element Method (FEM)** has been a deeply enriching academic and practical experience. It allowed me to connect theoretical mathematical concepts with real-world problem-solving through numerical simulations. By working on problems involving partial differential equations, I gained hands-on experience in applying FEM to various scenarios such as 1D Poisson equations.

Using **FEniCS**, I was able to implement variational formulations, define complex geometries, and apply boundary conditions effectively in a programming environment. The automation and flexibility provided by FEniCS helped me focus more on the mathematical modelling rather than the low-level computational details. Additionally, using **ParaView** allowed me to visualize and interpret the simulation results, which significantly enhanced my understanding of the physical behaviour of the systems being studied.

Through this internship, I not only deepened my knowledge of FEM but also developed strong computational and problem-solving skills. I learned how mesh refinement affects accuracy, how to analyse numerical results, and how to communicate findings in a clear and meaningful way.

Overall, this internship has strengthened my foundation in **computational mathematics** and has motivated me to explore more advanced topics in numerical analysis and scientific computing. The knowledge and skills I have gained will be highly beneficial for future academic research and professional opportunities in mathematics, engineering, and applied sciences.

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