Lecture-14

- ▷ Linear Mapping & Properties (2.7 in book)
- ▷ Norms
- ▷ Bilinear mapping & properties

→ Let V and W are two vector space and

$$\phi : V \rightarrow W \quad \phi(\vec{v}) = \vec{\omega}. \quad \vec{v} \in V, \vec{\omega} \in W$$

→ ϕ is structure preserving (homomorphism) / ~~linear~~

① $\phi(\vec{x} + \vec{y}) = \phi(\vec{x}) + \phi(\vec{y})$

and ② $\phi(\alpha \vec{x}) = \alpha \cdot \phi(\vec{x}) \ \forall \alpha \in R$

→ ϕ is linear mapping if it is structure preserving

and $\phi(\alpha\vec{x} + \beta\vec{y}) = \alpha\phi(\vec{x}) + \beta\phi(\vec{y})$

$\phi(\alpha\vec{x}) + \phi(\beta\vec{y})$

where $\vec{x}$ and $\vec{y} \in V$ and $\alpha, \beta \in R$

$\phi(\vec{x}), \phi(\vec{y}) \in W$

<u>Properties</u>

(1:1) 1. Injective → Unique mapping (If $\phi(\vec{x}) = \phi(\vec{y})$ then $\vec{x} = \vec{y}$)

(onto) 2. Surjective $\forall \vec{\omega} \in W \ \exists \vec{v} \ \boxed{\phi(\vec{v}) = \vec{\omega}}$

3. Bijective → Both Injective + Surjective

4. ~~Into~~ Isomorphism

5. Endomorphism

6. Automorphism

▷ Isomorphism : If ϕ is linear and ϕ is bijective then ϕ is isomorphism

▷ Endomorphism : If $\phi : \vec{V} \to \vec{V}$ and ϕ is linear

▷ Automorphism : If $\phi : \vec{V} \to \vec{V}$ and ϕ is bijective

(Isomorphism
Endo —"—
Auto —"—)

Examples	Linear	Injective	S	Bijective
$\phi\begin{pmatrix}x_1\\x_2\end{pmatrix} = \begin{pmatrix}x_1\\x_2\\0\end{pmatrix}$	YES	YES	✗	NO
$\phi\begin{pmatrix}x_1\\x_2\\x_3\end{pmatrix} = \begin{pmatrix}x_1\\x_2\end{pmatrix}$	YES	NO	✓	NO
$\phi\begin{pmatrix}x_1\\x_2\\x_3\end{pmatrix} = \begin{pmatrix}x_1+x_2+x_3\\x_2+x_3\\x_3\end{pmatrix}$ ← Automorphism	YES	YES	✓	YES

Ex 1 : $\phi : \mathbb{R}^2 \longrightarrow \mathbb{R}^3$ $\quad \vec{x} = \begin{pmatrix}x_1\\x_2\end{pmatrix}$

where $\phi\begin{pmatrix}x_1\\x_2\end{pmatrix} = \begin{pmatrix}x_1\\x_2\\0\end{pmatrix}$ $\quad \vec{y} = \begin{pmatrix}y_1\\y_2\end{pmatrix}$

✓ Injective

$$\phi(\vec{x}) = \phi(\vec{y})$$

$$\Rightarrow \begin{pmatrix}x_1\\x_2\\0\end{pmatrix} = \begin{pmatrix}y_1\\y_2\\0\end{pmatrix}$$

$$\Rightarrow x_1 = y_1 ; x_2 = y_2 ; 0 = 0$$

× Surjective:

~~$\forall \vec{\omega} \in \mathbb{R}^3, \exists$~~.

Let $\vec{\omega}$ be $\begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} \in \mathbb{R}^3$ where $y_3 \neq 0$

there exists no $\vec{v} \in V$ such that $\phi(\vec{v}) = \vec{\omega}$

× Bijective: NOT surjective so not bijective

Ex. 2

$$\phi : \mathbb{R}^3 \longrightarrow \mathbb{R}^2$$

$$\phi \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

× Injective:

Definition: $\phi(\vec{x}) = \phi(\vec{y}) \Rightarrow \vec{x} = \vec{y}$

$$x' = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \quad x'' = \begin{pmatrix} x_1 \\ x_2 \\ x_4 \end{pmatrix}$$

Hence $\phi(\vec{x'}) = \phi(\vec{x''})$

but $\vec{x'} \neq \vec{x''}$

It means it is not injective.

Ex 3:

$\phi: \mathbb{R}^3 \to \mathbb{R}^3$

For $\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$

$$\phi(\vec{x}) = \vec{y} = \begin{bmatrix} x_1 + x_2 + x_3 \\ x_2 + x_3 \\ x_3 \end{bmatrix}$$

Linear:

$$\phi(\alpha\vec{x} + \beta\vec{y}), \vec{z} = (\alpha\vec{x} + \beta\vec{y})$$

$$\phi(\vec{z}) = \begin{pmatrix} z_1 + z_2 + z_3 \\ z_2 + z_3 \\ z_3 \end{pmatrix}$$

$$\phi(\alpha\vec{x} + \beta\vec{y}) = \begin{pmatrix} (\alpha x_1 + \beta y_1) + (\alpha x_2 + \beta y_2) + (\alpha x_3 + \beta y_3) \\ (\alpha x_2 + \beta y_2) + (\alpha x_3 + \beta y_3) \\ \alpha x_3 + \beta y_3 \end{pmatrix}$$

$$= \begin{pmatrix} \alpha(x_1 + x_2 + x_3) + \beta(y_1 + y_2 + y_3) \\ \alpha(x_2 + x_3) + \beta(y_2 + y_3) \\ \alpha x_3 + \beta y_3 \end{pmatrix}$$

$$= \alpha \begin{pmatrix} x_1 + x_2 + x_3 \\ x_2 + x_3 \\ x_3 \end{pmatrix} + \beta \begin{pmatrix} y_1 + y_2 + y_3 \\ y_2 + y_3 \\ y_3 \end{pmatrix}$$

$$\Rightarrow \phi(\alpha \vec{x} + \beta \vec{y}) = \alpha \phi(\vec{x}) + \beta \phi(\vec{y})$$

✓ Injective:

Let $\phi(\vec{x}) = \phi(\vec{y})$

$$\begin{pmatrix} x_1 + x_2 + x_3 \\ x_2 + x_3 \\ x_3 \end{pmatrix} = \begin{pmatrix} y_1 + y_2 + y_3 \\ y_2 + y_3 \\ y_3 \end{pmatrix}$$

$\Rightarrow x_3 = y_3$ (3rd position)

$x_2 = y_2$ (2nd position)

$x_1 = y_1$ (1st position)

✓ Surjective:

Let, $\phi(\vec{z}) = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$

$$\Rightarrow 6 \begin{pmatrix} z_1 + z_2 + z_3 \\ z_2 + z_3 \\ z_3 \end{pmatrix} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$$

$$\boxed{z_3 = x_3}$$ ✓

$$z_2 + z_3 = x_2 \Rightarrow z_2 = x_2 - x_3$$

$$z_1 + z_2 + z_3 = x_1 \Rightarrow z_1 = x_1 - x_2$$

Lecture 15 (A)

- Linear mapping in vector space — Ch 2
- Norm
- ~~the~~ Bilinear mapping

} Ch 3

Norm: A 'norm' in a vector space V is a mapping V to ℝ denoted by and satisfies:

$\| \; \| : V \to \mathbb{R}$

For every $\vec{x}$ and $\vec{y} \in V$ and $\alpha \in \mathbb{R}$

(i) Absolute Homogenuous

$\cancel{\|\vec{x}+\vec{y}}$ $\|\alpha \vec{x}\| = |\alpha| \, \|\vec{x}\|$

Absolute value

(ii) Triangle property

$$\|\vec{x}+\vec{y}\| \leq \|\vec{x}\| + \|\vec{y}\|$$

(iii) Positive Definite

$$\|\vec{x}\| > 0, \vec{x} \neq 0$$

$$\|\vec{x}\| = 0 \Leftrightarrow \vec{x} = 0$$

if and only if (iff)

- For every $\vec{x} \in V$, $\|\vec{x}\|$ is called the length of $\vec{x}$
- There are three popular "norms" in a Euclidean space ($\mathbb{R}^n$)

1. Manhattan Norm ($\| \|_1$)

$$\|\vec{x}\|_1 = \sum_{i=1}^{n} |x_i| \quad \vec{x} \in \mathbb{R}^n$$

$$[x_1, x_2, x_3]\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = x_1^2 + x_2^2 + x_3^2$$

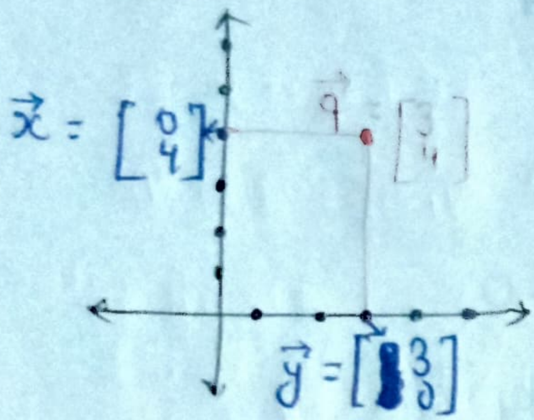
2. Euclidean Norm ($\| \|_2$)

$$\|\vec{x}\|_2 = \sum_{i=1}^{n} \sqrt{x_i^2} = \sqrt{\vec{x}^T x} \quad \vec{x} \in \mathbb{R}^n$$

3. Max Norm ($\| \|_{max}$)

$$\|\vec{x}\|_{max} = \max |x_i|_{i: 1 \text{ to } n}$$

Example:



$\|\vec{x}\|_1 = 3$ $\quad \|\vec{x}\|_{max} = 3$

$\|\vec{x}\|_2 = 3$

$\vec{x} = \begin{bmatrix} 3 \\ 0 \end{bmatrix} \quad \vec{y} = \begin{bmatrix} 0 \\ 4 \end{bmatrix}$

$\|\vec{y}\|_1 = 4$

$\|\vec{y}\|_2 = 4$

$\|\vec{y}\|_{max} = 4$

$\vec{q} = \begin{bmatrix} 3 \\ 4 \end{bmatrix}$

$\boxed{\|\vec{q}\|_1 = 7}$ Total

$\boxed{\|\vec{q}\|_2 = 5}$

$\boxed{\|\vec{q}\|_{max} = 4}$ Maximum

Bi-linear Map, Inner Product & Inner Product Space

- A function $\Omega : V \times V \rightarrow \mathbb{R}$ is a Bi-linear mapping if it is structure preserving in both of its arguments.

▷ for $\vec{x}, \vec{y} \in V$ $\quad$ for every $\vec{x}, \vec{y}, \vec{z} \in V$

$\Omega(\vec{x}, \vec{y}) \in \mathbb{R}$ $\boxed{\text{and}}$ $\quad \alpha, \beta, \gamma \in \mathbb{R}$

$$\Omega(\alpha\vec{x} + \beta\vec{y}, \vec{z}) = \alpha\,\Omega(\vec{x}, \vec{z}) + \beta\,\Omega(\vec{y}, \vec{z})$$

and

$$\Omega(\vec{x}, \beta\vec{y} + \gamma\vec{z}) = \beta\,\Omega(\vec{x}, \vec{y}) + \gamma\,\Omega(\vec{x}, \vec{z})$$

→ A Bilinear map is $\boxed{\text{Symmetric}}$ if $\Omega(\vec{x}, \vec{y}) = \Omega(\vec{y}, \vec{x})$ $\forall\, \vec{x}, \vec{y} \in V$

→ A Bi-linear map is $\boxed{\text{positive definite}}$ if $\Omega(\vec{x}, \vec{x}) > 0$ and $\Omega(\vec{x}, \vec{x}) = 0$ only if $\vec{x} = 0$

Inner Product + Inner Product Space

▷ A mapping $\Omega : V \times V \to R$ is Inner product if Ω is Bilinear, Symmetric. and Positive definite

▷ A inner product $\Omega(\vec{x}, \vec{y})$ is denoted as $\langle \vec{x}, \vec{y} \rangle$

▷ And vector space $(V, \langle \cdot, \cdot \rangle)$ is called inner product space.

Example:

$\Omega : \mathbb{R}^n \times \mathbb{R}^n \longrightarrow \mathbb{R}$

is defined as $\Omega(\vec{x}, \vec{y}) = \vec{x}^T \vec{y}$

Q: Prove that Ω is Inner product.

Sol: Bilinear

Let $\vec{x}, \vec{y}$ and $\vec{z} \in \mathbb{R}^n$, $\alpha, \beta, \gamma \in \mathbb{R}$

$$\Omega(\alpha\vec{x} + \beta\vec{y}, \vec{z}) = (\alpha\vec{x} + \beta\vec{y})^T \cdot \vec{z}$$

$$= [\alpha(\vec{x}^T) + \beta(\vec{y}^T)] \cdot \vec{z}$$

$$= \alpha(\vec{x}^T\vec{z}) + \beta(\vec{y}\vec{z})$$

$$\Omega(\vec{x}, \beta\vec{y} + \gamma\vec{z}) = \vec{x}^T \cdot (\beta\vec{y} + \gamma\vec{z})$$

$$= \beta(\vec{x}^T y) + \gamma(\vec{x}^T\vec{z})$$

Hence, Ω is Bilinear mapping,

Symmetric

$$\Omega(\vec{x}\vec{y}) = \vec{x}^T\vec{y} = \sum_{i=1}^{n} x_i y_i$$

$$= \sum_{i=1}^{n} y_i x_i = y^T\vec{x}$$

$$= \Omega(\vec{y}\vec{x})$$

Hence, Ω is symmetric

Positive Definite

$$\Omega(\vec{x},\vec{x}) = \vec{x}^T\vec{x} = \sum_{i=1}^{n} x_i^2$$

$$\begin{matrix} > 0 \\ = 0 \end{matrix} \begin{cases} \text{if } \vec{x} = \vec{0} \\ \text{if } \vec{x} = \vec{0} \end{cases}$$

Hence Ω. is positive Definite.

~~Lecture 15 (15)~~

~~X X X XXX XX~~

~~Convex Optimizations~~

Lecture 15(B)

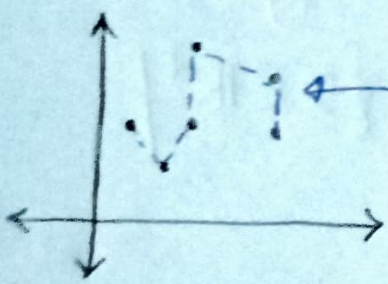
Convex optimizations

Curve fitting

n poits

① Given : $P = \{ P_i = (x_i, y_i) \}$ $x_i, y_i \in \mathbb{R}$

Q : $\exists$? a function $f : \mathbb{R} \to \mathbb{R}$, $y = f(x)$ which fits all points of P, that is $y_i = f(x_i)$, $\forall i = 1 \dots n$?



Piecewise function

$$y = ax + b$$

① $\exists$? $a, b \in \mathbb{R}$ such that

$$y_i = a x_i + b \quad \forall i = 1 \dots n \ ?$$

② $\exists$? a 'd' degree polynomial fitting P

$$P(x) = a_d x^d + a_{d-1} x^{d-1} + \dots + a_1 x + a_0$$

$$y_i = a_d x_i^d + \dots a_1 x_i + a_0 \quad \forall i = 1 \dots n \ ?$$

$\exists\, ?\ \bar{a} \in \mathbb{R}^{d+1}\quad \bar{y} = X\bar{a}\ ?$

$$\begin{pmatrix} y_1 \\ \vdots \\ \vdots \\ y_n \end{pmatrix} = \begin{pmatrix} x_1^d & x_1^{d-1} & \cdots & x_1 & 1 \\ x_2^d & x_2^{d-1} & \cdots & x_2 & 1 \\ \vdots & & \ddots & \vdots & \vdots \\ x_n^d & x_n^{d-1} & \cdots & x_n & 1 \end{pmatrix} \begin{pmatrix} a_d \\ a_{d-1} \\ \vdots \\ \vdots \\ a_1 \\ a_0 \end{pmatrix}$$

$\exists$ Solution iff $\operatorname{rank}(X) = \operatorname{rank}(X, \bar{y})$

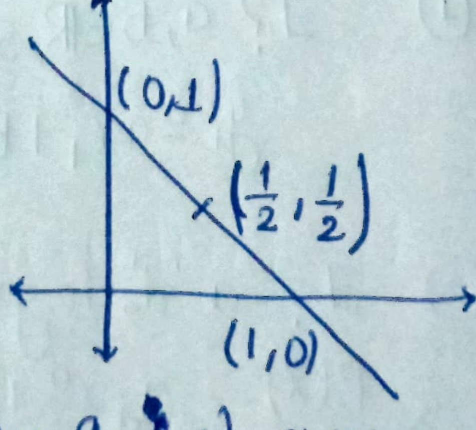
② Given

$x_i = (x_{i1}, x_{i2}, x_{i3} \cdots x_{id})$

$P = \{ P_i = (x_i, y_i) \}, \quad x_i \in \mathbb{R}^d, y \in \mathbb{R}$

→ A Hyperplane in $\mathbb{R}^d$ is a set $\{ x \in \mathbb{R}^d \mid a_1 x_1 + a_2 x_2 + \cdots a_d x_d = c \}$

for some $(a_1, a_2 \ldots, a_d, c)$



Q: $\exists$? Hyperplane $(a_1, a_2 \cdots a_d, c)$ fitting all points p_i ?

Q: $\exists$? Hyperplane $(a_1, a_2 \cdots a_d, c)$ which minimize total error

$$y_i = a_1 x_{i_1} + a_2 x_{i_2} + \cdots + a_d x_{i_d}$$

$$\sum_{i=1}^{n} (y_i - a_1 x_{i_1} - a_2 x_{i_2} - \cdots - a_d x_{i_d})^2$$

Minimize

▷ Find a vector $\vec{a} = (a_1, a_2 \cdots a_d) \in \mathbb{R}^d$ which minimizes

$$f(\bar{a}) = \sum_{i=1}^{n} (y_i - (a_1 x_{i_1} + a_2 x_{i_2} \cdots a_d x_{i_d}))^2$$

Calculus

Consider a function $f: \mathbb{R} \rightarrow \mathbb{R}$

Fix $a \in \mathbb{R}$

$$\lim_{x \to a} f(x) = L$$

$f(x) = x^2,\ x \neq 0$

$\boxed{f(0) \leftarrow \text{undefined}}$

or $\boxed{f(0) = 1000}$

different examples

$$\lim_{x \to a} \frac{f(x) - f(a)}{x - a} = f'(a) = \frac{d(f(a))}{dx}$$

$\forall \epsilon > 0 \;\; \exists \delta > 0$ | $|f(x) - L| \leq \epsilon$ if $0 < |x - a| \leq \delta$

Closeness for 'L'

Closeness for 'a'

▷ f is continuous at a

→ if $f(a)$ is defined and

→ $\lim_{x \to a} f(x) = f(a)$

Ex.

$$f(x) = \begin{cases} x^2, & x \neq 0 \\ 1000, & x = 0 \end{cases}$$

$\lim_{x \to 0} f(x) = 0$. f is not continuous

▷ A function $f: \mathbb{R} \to \mathbb{R}$ is differentiable at $a \in \mathbb{R}$ if $\lim_{x \to a} \frac{f(x) - f(a)}{x - a}$ exists

change of rate

▷ This limit, if it exists, is called the derivative of f at a and is denoted by $f'(a)$, $f^{(1)}(a)$, $\frac{d(f(a))}{dx}$

Lecture 16 - Saturday Class (Missed)