Mathematical Foundations of AI (Aug'24-Dec'24)

Problem Sheet-2

1. Let $f : R^n \to R$ be a function. Prove that f is both convex and concave if and only if $f(x) = a^T x + b$ for some $c \in R^n$ and $b \in R$.

2. If $f_1, \ldots, f_d$ are convex functions defined over R^n, then any convex combination $h = \sum_{j=1}^{d} \lambda_j f_j$ (where $\lambda_j \geq 0,\ \forall j$ and $\sum_j \lambda_j = 1$) defined as $h(x) = \sum_j \lambda_j f_j(x)$ for each x, is also a convex function.

3. Let $f_1, \ldots, f_m$ be convex functions on R^n. Prove that the set $S = \{x \in R^n : f_j(x) \leq 0,\ \forall j\}$ is convex.

4. Determine if $f(x, y) = 2x^2 - 3xy + 5y^2 - 2x + 6y$ is convex, concave, both or neither for $(x, y) \in R^2$.

5. Determine if $f(x, y) = 4x^2 + 12xy + 9y^2$ is convex, concave, both or neither for $(x, y) \in R^2$.

6. Consider $f(x) = 15 - 2x - 25x^2 + 2x^3$. Determine all local minima, local maxima of f. Does f have any global minimum or global maximum ? If so, find them.

7. Consider $f(x, y) = 8x^2 + 3xy + 7y^2 - 25x + 31y - 29$. Determine all local minima and local maxima (if any). Does $f(x, y)$ have any global minimum or global maximum ?

8. Consider : Minimize $f(x, y) = (x - 2y)^2 + x^4$. Find a minimizer of f. Is it a strict local minimizer ? Is it a global minimizer ?

9. Consider the problem : Minimize $f(x) = x^4 - 1$. Solve this problem by Newton's method. Start from $x_0 = 4$ and perform three iterations. Prove that the iterates converge to the solution. What is the rate of convergence ?

10. Consider Minimize $f(x, y) = x^2 + 2y^2$. Starting with initial guess $\vec{x}_0 = (2, 1)$, show that the sequence of points generated by the steepest-descent algorithm (with exact line search $\min_\alpha f(\vec{x}_k + \alpha\vec{p})$ to determine step-size α, $\vec{p}$ is the direction of steepest descent), is given by $\vec{x}_k = \frac{1}{3}^k\, (2, (-1)^k)$. Show that $f(x_{k+1}) = f(x_k)/9$.