Lecture - 21 (Saturday)

~~For ne two~~

We take two wild guesses (a, b)

where $f(a) =$ positive

and $f(b) =$ negative

then we take $f\left(\frac{a+b}{2}\right)$

Depending on this we change the direction and reach to some good approximation

$$e_{k+1} \simeq e_k^2 \left(\frac{f''(x^*)}{2f'(x^*)}\right)$$

η

$$\boxed{e_k \simeq (e_0 \eta)^{(2^k)} \cdot \eta^{-1}}$$

$$|e_0| = |x_0 - x^*| \leq |x_0 - x^*_{LB/UB}|$$

(For convex function)

▷ Why zeroes ? ⟶ Local optimization

$f'(x) = 0$ with quadratic covergence rate $\frac{f'''(x^*)}{2f''(x^*)}$.

Gradient Descent Method

$$|f''(x)| \leq L \text{ for } x \in [a,b]$$

$$f'(x^*) = 0$$

1. Start with an initial guess.
   Define $\gamma \leftarrow L^{-1}$

   known bound
2. while $f'(x) \neq 0$
   and do $x \leftarrow x - \gamma f'(x)$ endwhile
3. return x

we go the opposite direction for minimum

(Analysis has not been written!!!)

Scalar and vector functions,

$f: R^n \longrightarrow R^m,\ n, m \geq 1$

1. If $m = 1$ (real valued function) Scalar
2. If $m > 1$ (vector ——"——)
3. If $n = 1$ and $m > 1$ (trajectories | projectile)

$\vec{x} \in \mathbb{R}^d,\ \|\vec{x}\|_2 = \sqrt{x_1^2 + \dots + x_d^2}$ (L_2- Norm)

$\vec{x}, \vec{y} \in R^d,\ d_2(\vec{x}, \vec{y}) = \|\vec{x} - \vec{y}\|_2$ (L_2-Distance)

$f: R^n \longrightarrow R^m \quad \vec{a} \in R^n,\ \vec{l} \in R^m$

$\lim_{\vec{x} \to \vec{a}} f(x) = \vec{l}$. if $\forall \epsilon > 0, \exists \delta > 0$

$d_2(f(\vec{x}), \vec{l}) \leq \epsilon$

whenever

$0 < d_2(\vec{x}, \vec{a}) \leq \delta$

f is continuous at $\vec{a}$ if $\lim_{\vec{x} \to \vec{a}} f(\vec{x}) = f(\vec{a})$

Norm function is continuous

$f: R^3 \to R$

$(x, y, z) \to \sqrt{x^2 + y^2 + z^2}$

$$\left.\begin{array}{l} f(\vec{x}) \to \vec{f} \\ g(\vec{x}) \to \vec{g} \end{array}\right\} \text{ when } \vec{x} \to \vec{a}$$

$$f(\vec{x}) \pm g(\vec{x}) \to \vec{f} \pm \vec{g} \text{ as } \vec{x} \to \vec{a}$$

① $\alpha f(\vec{x}) \to \alpha \vec{f} \quad \forall \alpha \in \mathbb{R}$

② $f(\vec{x}) \cdot g(\vec{x}) \longrightarrow \vec{f} \cdot \vec{g}$

$$\boxed{\vec{x} \cdot \vec{y} = \sum_{i}^{n} x_i y_i}$$

$\vec{x}, \vec{y} \in \mathbb{R}^n$

③

$f : R^n \to R^m$

$g : R^m \to R^p$

$h = g \cdot f : R^n \to R^p$ by $h(\vec{x}) = g(f(\vec{x}))$ for $x \in \mathbb{R}^n$

f is continuous at $\vec{a} \in \mathbb{R}^n$

g is continuous at $f(\vec{a}) \in \mathbb{R}^m$

Then h is continuous at $\vec{a}$.

Ex1.

$f_1(x,y) = \sin(x^2 y)$ → Continuous everywhere

$f_2(x,y) = \log_e(x^2+y^2)$ → Not continuous at $(0,0)$

$f_3(x,y) = \dfrac{e^{x+y}}{x+y}$ → Not continuous where $x+y=0$

$$f(x,y) = \begin{cases} \dfrac{xy}{x^2+y^2} & ; (x,y) \neq (0,0) \\ 0 & ; (x,y) = (0,0) \end{cases}$$

Continuous for 'x' alone

Continuous for 'y' alone

→ Not continuous for both 'x' and 'y'
for $(x=y)$ it is not continuous

Differentiability of Scalar functions

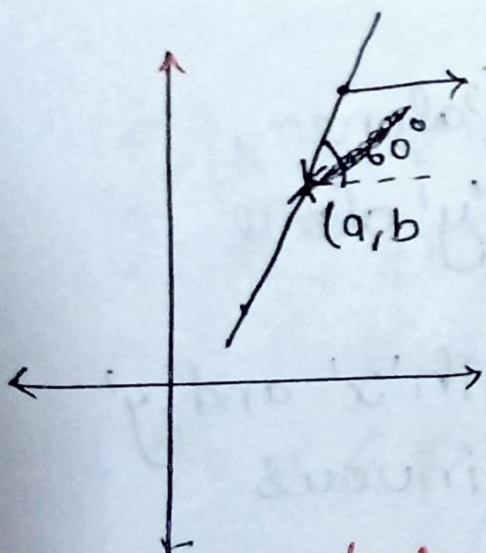
$f: R^n \to R$

$f(x) = 4x^3 - 5y^3$

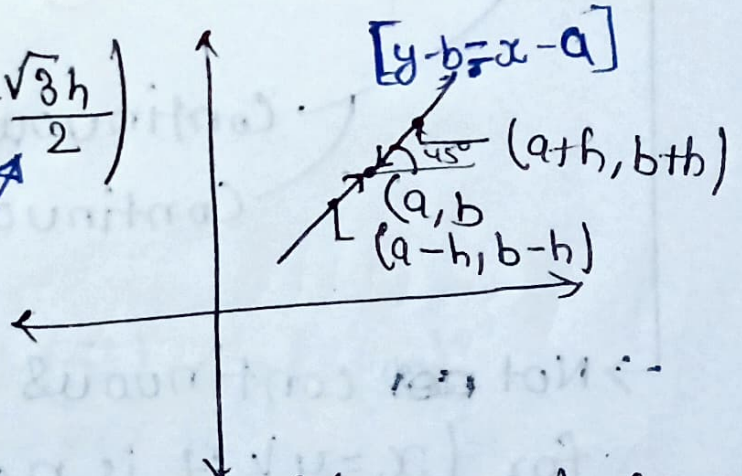
$\frac{\partial f}{\partial x} = 12x^2$ (Positive except 0) ≥ 0 | $\frac{\partial f}{\partial y} = -15y^2$ (Negative except 0) ≤ 0

cos θ

$\left(a + \frac{h}{2}, b + \frac{\sqrt{3}h}{2}\right)$



[y-b = x-a]



sin θ

$$\lim_{h \to 0} \frac{f\left(a+\frac{h}{2}, b+\frac{\sqrt{3}h}{2}\right) - f(a,b)}{h}$$

$$\frac{f(a+h, b+h) - f(a,b)}{h}$$

Gradient of f at $\vec{a}$ is the vector

$$\nabla f(\vec{a}) = \left(\frac{\partial f(\vec{a})}{\partial x_1}, \ldots, \frac{\partial f(\vec{a})}{\partial x_n}\right)$$

Directional derivatives

$f(\vec{a}, \vec{r}) = 0$

$\|\vec{y}\|_2 = 1$

$f'(\vec{a}, \vec{y})$ is directional ~~directive~~ derivative of f at $\vec{x}$

$\vec{y} = e_i$ along x_i

$f(\vec{a}, e_i) = \dfrac{\partial f(\vec{a})}{\partial x_i}$

⚠

▷ Existence of directional derivative $f'(\vec{a}, \vec{y})$ for each $\vec{y}$ does not guarantee f is continuous at $\vec{a}$

$$f(x, y) = \begin{cases} \dfrac{xy^2}{x^2 + y^4} & ; x \neq 0 \\ 0 & ; x = 0 \end{cases}$$

$\boxed{x = y^2}$ Not continuous

$$f(y^2, y) = \frac{y^4}{y^4 + y^4} = \frac{y^4}{2y^4} = \frac{1}{2}$$

f is not continuous at $(0,0)$

f is differentiable at $\vec{a}$ if for some $\delta > 0$ there exist $T_{\vec{a}} : \mathbb{R}^n \to \mathbb{R}$ and $E_{\vec{a}}(\vec{y})$ such that

$$f(\vec{a}+\vec{y}) = f(\vec{a}) + T_{\vec{a}}(\vec{y}) + \|\vec{y}\|_2 \cdot E_{\vec{a}}(\vec{y})$$

$$\forall \|\vec{y}\| < \delta$$

and $E_{\vec{a}}(\vec{y}) \to 0$ as $\|\vec{y}\| \to 0$

$T_{\vec{a}}$ is total derivative of f $\boxed{f'(\vec{a})}$

f is differentiable at $\vec{a}$

$\Rightarrow T_{\vec{a}}(\vec{y}) = f'(\vec{a}, \vec{y}) \;\forall \vec{y}$

$$T_{\vec{a}}(\vec{y}) = \nabla f(\vec{a}) \cdot \vec{y} = \sum_{i=1}^{n} \frac{\partial f(\vec{a})}{\partial x_i} y_i \;\forall \vec{y}$$

Taylor's first order formula

$$f(\vec{a}+\vec{y}) = f(\vec{a}) + \nabla f(\vec{a}) \cdot \vec{y} + \|y\| E_{\vec{a}}(\vec{y}),$$

$$\|y\| < \delta$$

$\|y\| = 1, \; f'(\vec{a}, \vec{y}) = \|\nabla f(\vec{a})\| \cdot \cos\theta$

θ angle between $f(\vec{a})$

$\vdots$

⚠ {Iske Baad Kya padhya No IDEA}

Lecture-22 (21st October '24)

- If A is invertible then $\det(A)$ can be written as Laplace Expansion either assume a row or a column

$$A = (a_{ij}) \in \mathbb{R}^{n\times n}$$

$$\det(A) = \sum_{i=1}^{n} (-1)^{i+j} \det(A_{ij}) \text{ for any row 'i'}$$

or

$$\det(A) = \sum_{j=1}^{n} (-1)^{i+j} \det(A_{ij}) \text{ for any column 'j'}$$

where the matrix $A_{ij} \in \mathbb{R}^{(n-1)\times(n-1)}$ and obtained by eliminating the 'ith' row and 'jth' column of A

A_{ij} is called 'MINOR' of A and $C_{ij} = (-1)^{i+j} A_{ij}$ is called the 'COFACTOR' of A.

Definitions:

(I) Two matrices $A, B \in \mathbb{R}^{n\times n}$ are 'SIMILAR' iff there exists an invertible matrix $S \in \mathbb{R}^{n\times n}$ where $B = S^{-1} A S$

(II) $\text{Trace}(A) = \sum_{i=1}^{n} a_{ii}$

Prove:

$\boxed{(a)}$ If $A, B \in \mathbb{R}^{n\times n}$ then $\det(AB) = \det(A)\cdot\det(B)$
$= \det(B)\cdot\det(A) = \det(BA)$

(b) A is invertible iff $\det(A) \neq 0$

(c) A is invertible then $\det(A^{-1}) = \dfrac{1}{\det(A)}$

(d) If A and B are similar.
then $\det(A) = \det(B)$

(e) $\det(A) = \det(A^T)\ \forall\ A \in \mathbb{R}^{n\times n}$

(f) $\det(\lambda A) = \lambda^n \cdot \det(A)$, $A \in \mathbb{R}^{n\times n}$

(g) $\det(A) \neq 0$ then rank of $(A) = n$

(h) $\text{Trace}(A+B) = \text{Trace}(A) + \text{trace}(B)$

(i) $\text{Trace}(\alpha A) = \alpha \cdot [\text{Trace}(A)]$

(j) $\text{Trace}(I) = n$, $I \in \mathbb{R}^{n\times n}$

$\boxed{(k)}$ $\text{Trace}(AB) = [\text{Trace}(A)] \cdot [\text{Trace}(B)]$

(l) If T is a triangular matrix $\in \mathbb{R}^{n\times n}$ (lower/upper)
then $\det(T) =$

- Characteristic polynomial
Eigenvector and Eigen value

Polynomial of degree 'n'

$$\sum_{i=0}^{n} a_i x^i = a_0 + a_1 x + a_2 x^2 \cdots a_n x^n = P_n(x)$$

Polynomial Equation
$P_n(x) = 0$ (when it is true → next page)

$P_n(x) = 0$ iff ~~q(n~~ $q_n(x) = 0$; where $q_n(x)$
=

Chacteristics polynomial

If $A \in \mathbb{R}^{n \times n}$
then $\det(A - \lambda I)$ is called as characteristics polynomial of degree 'n'.

$$\det(A - \lambda I) = \boxed{C_0} + C_1 \lambda + C_2 \lambda^2 + \dots \boxed{C_{n-1}} \lambda^{n-1} + \boxed{C_n} \lambda^n$$

$C_0 \rightarrow$ Determinant of A

$C_n \rightarrow (-1)^n$

$C_{n-1} \rightarrow (-1)^{n-1}$ Trace (A)

→ If $A \in \mathbb{R}^{n \times n}$, λ is an eigen ~~vect~~ value of A with an eigenvector $\vec{x} \mid \vec{x} \neq 0$ iff $A\vec{x} = \lambda \vec{x}$

Algebraic multiplicity (Algebra)
$\neq$
Geometric multiplicity
(Graph)

Proof: λ is an eigenvalue of A iff λ is a root of $\det(A - \lambda I^{n})= 0$

Note:

As polynomial λ can be =
① real
② real with multiplicity (A.M) ≥ 1
③ complex number along with its conjugate as a summation

$$A = \begin{bmatrix} 4 & 2 \\ 1 & 3 \end{bmatrix}$$

$$\det(A-\lambda I) = \det\left(\begin{bmatrix} 4-\lambda & 2 \\ 1 & 3-\lambda \end{bmatrix}\right)$$

$$= (4-\lambda)(3-\lambda) - 2$$

$$= 12 - 7\lambda + \lambda^2 - 2$$

$$= (\lambda-2)(\lambda-5)$$

The roots of $\det(A-\lambda I) = 0$

$$\Rightarrow \lambda = 2 \text{ and } \lambda = 5$$

Eigen vector

$$(A-\lambda I)\vec{x} = \vec{0}$$

$$\begin{bmatrix} 4-2 & 2 \\ 1 & 3-1 \end{bmatrix}\vec{x} = \vec{0} \Rightarrow \begin{bmatrix} 2 & 2 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

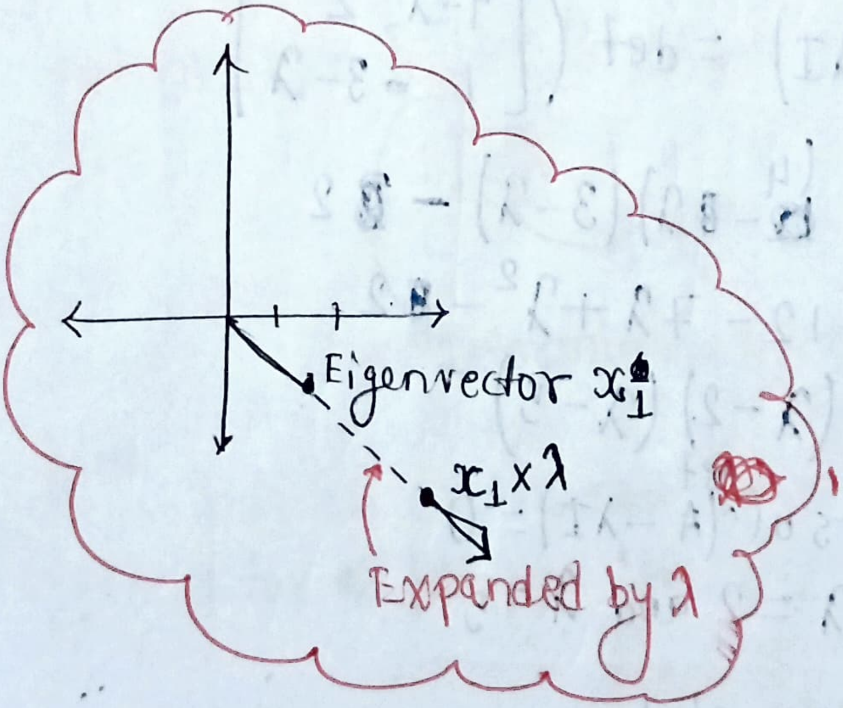
$$\Rightarrow \vec{x} = \begin{bmatrix} x_1 \\ -x_1 \end{bmatrix} \quad \vec{x}_1 = \begin{bmatrix} 1/\sqrt{2} \\ -1/\sqrt{2} \end{bmatrix}$$

$\lambda = 5$

$$\begin{bmatrix}4-5 & 2\\ 1 & 3-5\end{bmatrix}\begin{bmatrix}x_1\\ x_2\end{bmatrix} = \begin{bmatrix}0\\ 0\end{bmatrix}$$

$$\Rightarrow \begin{bmatrix}-1 & 2\\ 1 & -2\end{bmatrix}\begin{bmatrix}x_1\\ x_2\end{bmatrix} = \vec{0}$$

$$\Rightarrow \vec{x} = \begin{bmatrix}2x_2\\ x_2\end{bmatrix} \quad \hat{\vec{x}} = \begin{bmatrix}2/\sqrt{5}\\ 1/\sqrt{5}\end{bmatrix}$$



Lecture - 23 (23rd October '24)

$$P_n(\lambda) = C_0 + C_1\lambda + \dots C_{n-1}\lambda^{n-1} + C_n\lambda^n$$

$$C_0 = \det(A), \quad C_{n-1} = (-1)^{n-1}\, \text{trace}(A)$$

Prove:

(a) $\det(A) = \prod_{i=1}^{n} \lambda_i$

(b) $\text{trace}(A) = \sum_{i=1}^{n} \lambda_i$

Always real

Definition:

- If $A \in \mathbb{R}^{n\times n}$ is a symmetric matrix then
  (a) the eigen values are real
  (b) there exists $\boxed{\text{an}}$ orthonormal basis ~~will be~~ ~~real~~ (eigen vector)

- Moreover if A is positive definite
  [In Addition to Symmetric ($\vec{x}^T A \vec{x} > 0, \vec{x} \neq 0$)]
  then eigenvalues are positive

Example:

$$A = \begin{bmatrix} 3 & 2 & 2 \\ 2 & 3 & 2 \\ 2 & 2 & 3 \end{bmatrix}$$ find eigenvalues and eigen vectors.

Sol:

$$\det(A-\lambda I) = \begin{vmatrix} 3-\lambda & 2 & 2 \\ 2 & 3-\lambda & 2 \\ 2 & 2 & 3-\lambda \end{vmatrix}$$

$$= (-1)^{1+1}(3-\lambda)\begin{vmatrix} 3-\lambda & 2 \\ 2 & 2-\lambda \end{vmatrix} + (-1)^{1+2}\, 2\begin{vmatrix} 2 & 2 \\ 2 & 3-\lambda \end{vmatrix}$$

$$+ (-1)^{1+3}\, 2\begin{vmatrix} 2 & 3-\lambda \\ 2 & 2 \end{vmatrix}$$

$$= (3-\lambda)(\lambda^2-5\lambda+6) - (2)(6-2\lambda-4)$$

$$+ (2)(4-6-2\lambda)$$

$=$

$\Rightarrow \boxed{\lambda = 1, 1, 7}$

$[A - \lambda I]\vec{x} = \vec{0}, \vec{x} \neq \vec{0}$

$\lambda = 1$

$$\begin{bmatrix} 2 & 2 & 2 \\ 2 & 2 & 2 \\ 2 & 2 & 2 \end{bmatrix} \begin{bmatrix} x_0 \\ x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow \vec{x} = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$$

$$\hat{\vec{x}} = \begin{bmatrix} 1/\sqrt{2} \\ -1/\sqrt{2} \\ 0 \end{bmatrix}, \begin{bmatrix} -1/\sqrt{2} \\ 1/\sqrt{2} \\ 0 \end{bmatrix}, \begin{bmatrix} 1/\sqrt{2} \\ 0 \\ -1/\sqrt{2} \end{bmatrix}, \begin{bmatrix} -1/\sqrt{2} \\ 0 \\ 1/\sqrt{2} \end{bmatrix}$$

$\lambda = 7$

$$\begin{bmatrix} -4 & 2 & 2 \\ 2 & -4 & 2 \\ 2 & 2 & -4 \end{bmatrix} \begin{bmatrix} y_0 \\ y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

+

$$\Rightarrow \vec{y} = \begin{bmatrix} 1 \\ 1 \\ 4 \end{bmatrix} \quad \hat{\vec{y}} = \begin{bmatrix} 1/\sqrt{3} \\ 1/\sqrt{3} \\ 1/\sqrt{3} \end{bmatrix}$$

+

$$\text{Basis 1} = \begin{bmatrix} \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} \\ 0 \end{bmatrix} \begin{bmatrix} -\frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} \\ 0 \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{3}} \end{bmatrix}$$

$$\text{Basis 2} = \begin{bmatrix} \frac{1}{\sqrt{2}} \\ 0 \\ -\frac{1}{\sqrt{2}} \end{bmatrix} \begin{bmatrix} -\frac{1}{\sqrt{2}} \\ 0 \\ \frac{1}{\sqrt{2}} \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{3}} \end{bmatrix}$$

Diagonalizable:

A matrix is diagonalizable if there exists a diagonal matrix D ~~such that A=~~ and an invertible matrix S such that $\boxed{A = S^{-1} D S}$

Prove: A symmetric matric is diagonalizable

Proof: Let $P = [\vec{x}_1, \vec{x}_2 \dots \vec{x}_n]$ ← Also the orthonormal basis, be the eigenvectors of A

and let $D = \begin{bmatrix} \lambda_1 & & \\ & \lambda_2 & \\ & & \ddots \\ & & & \lambda_n \end{bmatrix}$

listing the eigenvalues of A where λ_i is the eigenvalue with eigenvector $\vec{x}_i$

$$AP = A\left[\vec{x}_1, \vec{x}_2 \dots \vec{x}_n\right]$$

$$A\vec{x}_j = \lambda_j \vec{x}_j.$$

Therefore, $A\left[\vec{x}_1, \vec{x}_2 \dots \vec{x}_n\right] = \left[\vec{x}_1, \vec{x}_2 \dots \vec{x}_n\right] \begin{bmatrix} \lambda_1 & & & \\ & \lambda_2 & & \\ & & \ddots & \\ & & & \lambda_n \end{bmatrix}$

$$\Rightarrow AP = PD.$$

$$\cancel{(AP)A^{-1}} \Rightarrow (AP)(P^{-1}) = (PD)P^{-1}$$

$$\Rightarrow A(PP^{-1}) = PDP^{-1}$$

$$\Rightarrow \boxed{A = PDP^{-1}}$$

P is nothing but eigenvector orthonormal basis

Orthonormal matrix

If A is symmetric then there exists a diagonal matrix D ~~exist~~ listing the eigenvalues that is similar to A.

Singular value Decomposition (SVD)

$$A = \begin{bmatrix} 1 & 1 \\ 1 & -1 \\ 1 & 2 \end{bmatrix} \qquad A^T = \begin{bmatrix} 1 & 1 & 1 \\ 1 & -1 & 2 \end{bmatrix}$$

$A^TA \in \mathbb{R}^{2\times 2}$ ← Least Square

Let $A \in \mathbb{R}^{m\times n}$ ~~then~~ with $\text{rank}(A) = r$, $r \in \{0 \dots \min(m,n)\}$ then ther exists orthogonal matrix U and V, $U \in \mathbb{R}^{m\times m}$, $V \in \mathbb{R}^{n\times n}$ and a diagonal matrix $\Sigma \in \mathbb{R}^{m\times n}$ where

$$\underset{m\times n}{A} = \underset{m\times m}{U}\,\underset{m\times n}{\Sigma}\,\underset{n\times n}{V}$$

and $\Sigma = \text{diagonal}(\sigma_1, \sigma_2 \dots \sigma_r, \dots 0)$,

with $\boxed{\sigma_1 \geq \sigma_2 \geq \sigma_3 \dots \geq \sigma_r \geq 0}$

$$\underset{\substack{m\times n \\ m=n}}{\Sigma} = \begin{bmatrix} \sigma_1 & & & \\ & \sigma_2 & & \\ & & \ddots & \\ & & & \sigma_r \end{bmatrix} \qquad \underset{\substack{m\times n \\ m>n}}{\Sigma} = \begin{bmatrix} \sigma_1 & & \\ & \sigma_2 & \\ & & \ddots \; \sigma_r \\ & 0 & \end{bmatrix}$$

$$\underset{\substack{m\times n \\ m<n}}{\Sigma} = \begin{bmatrix} \sigma_1 & & & & \\ & \sigma_2 & & & \\ & & \ddots & & \\ & & & \sigma_r & 0 \end{bmatrix}$$

Given, $A \in \mathbb{R}^{m \times n}$

Let $B = (A^T A)$. $\boxed{A \in \mathbb{R}^{m \times n}}$ $\boxed{A^T \in \mathbb{R}^{n \times m}}$

then $B \in \mathbb{R}^{n \times n}$ and B is <u>symmetric</u> Always symmetric

$B = A^T A$

$= (U \Sigma V)^T (U \Sigma V)$ $\quad [(AB)^T = B^T A^T]$

$= V^T \Sigma^T (U^T U) \Sigma V$

$= V^T \Sigma^T I \Sigma V$

$= V^T \hat{\Sigma} V$ $\quad$ where $\hat{\Sigma} = \Sigma^T \Sigma$

$= \text{diagonal}(\sigma_1^2, \sigma_2^2 \dots \sigma_r^2 \dots 0)$

$\Rightarrow B = V^{-1} \hat{\Sigma} V$ (V is orthogonal)

$\Rightarrow (\sigma_1^2, \sigma_2^2 \dots \sigma_r^2 \dots 0)$ are the eigenvalues of B.

<u>Summary:</u>

For any $A \in \mathbb{R}^{m \times n}$

construct $B = A^T A \in \mathbb{R}^{n \times n}$

then there exists an ortho-normal basis V where

$B = V^{-1} \hat{\Sigma} V$

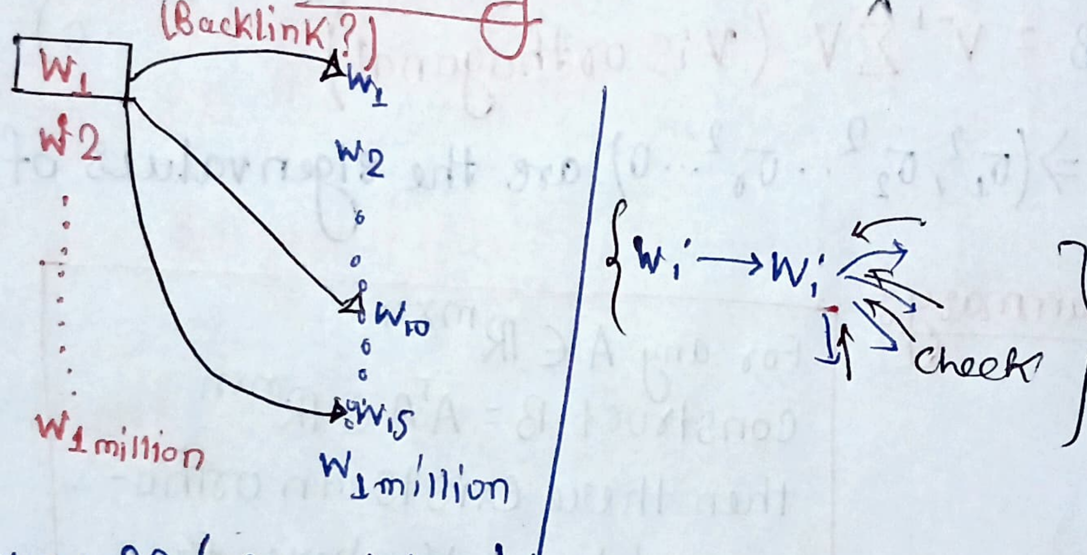
$\Sigma = \text{diagonal}(\sigma_1^2, \sigma_2^2 \dots \sigma_r^2 \dots 0)$

each σ_i^2 is an SVD and also an eigenvalue.

SVD is a versatile tool to address

(a) Data Compression
(b) Image Compression
(c) Noise reduction
(d) signal processing
(e) feature extraction
(f) dimensionality reduction
(g) Constraint optimization

Indexing



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