Numerical Integration:-

Aim:- $I = \int_a^b f(x)\,dx = ?$ (Numerically)

Here $f(x)$ may be explicitely given or given as a tabulated data.

Assumed:- $f(x)$ is integrable on $[a,b]$

$$\boxed{I = \int_a^b f(x)\,dx = \sum_{k=0}^{n} \lambda_k f(x_k)}$$

Recall from 12^{th}

$$= \lambda_0 \underbrace{f(x_0)} + \lambda_1 \underbrace{f(x_1)} + \cdots + \lambda_n \underbrace{f(x_n)}$$

Known $\longrightarrow$ $f(x_i)$ | Unknown $\longrightarrow$ λ_i's ?

Newton - Cotes Integration Rules:-

To find. $I = \int_a^b f(x)\,dx$

$\boxed{\text{The nodes } x_0, x_1, \ldots, x_n \text{ are equispaced.}}$

* we know that $\int_a^b f(x)\,dx$ defines area under the curve $y = f(x)$ above x-axis between the lines $x = a$ & $x = b$.

Integration Rules / Quadrature formula

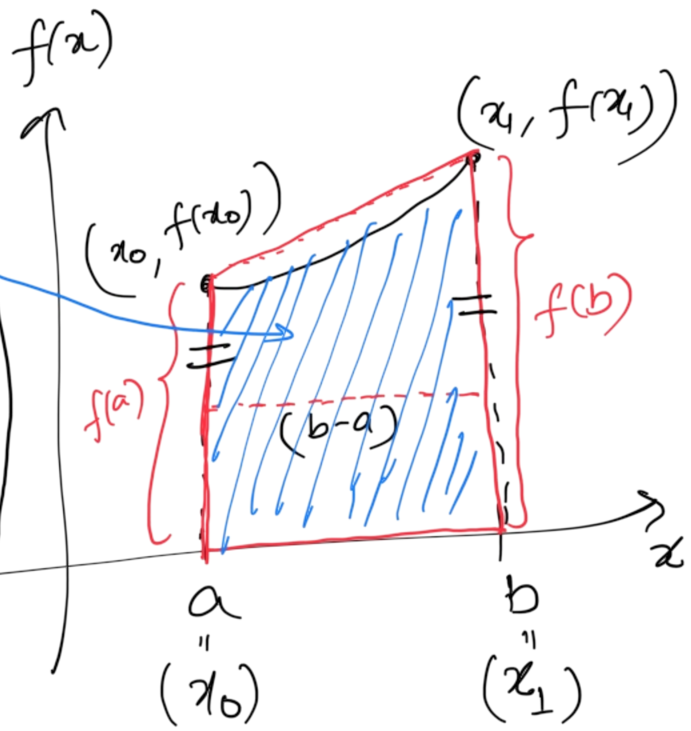
- Newton Cotes
  - Trapezoidal Rule
  - Simpson's $\frac{1}{3}$ Rule
- Gauss Quadrature
  - Gauss Legendre Rule
  - Gauss-Hermite Rule

Trapezoidal Rule :-

$x_0 = a$, $x_1 = b$

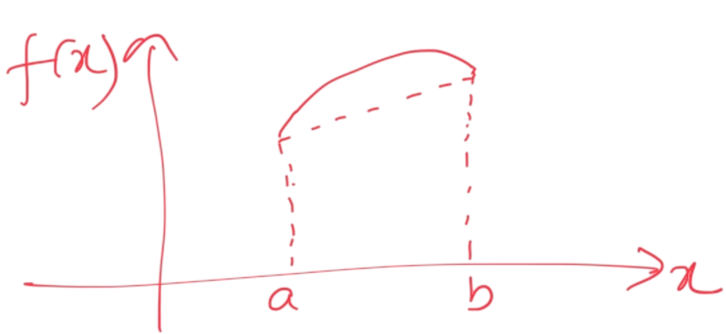
$$I = \int_a^b f(x)\,dx$$

$$= \frac{(b-a)}{2}\left[f(b) + f(a)\right] \quad (\star)$$



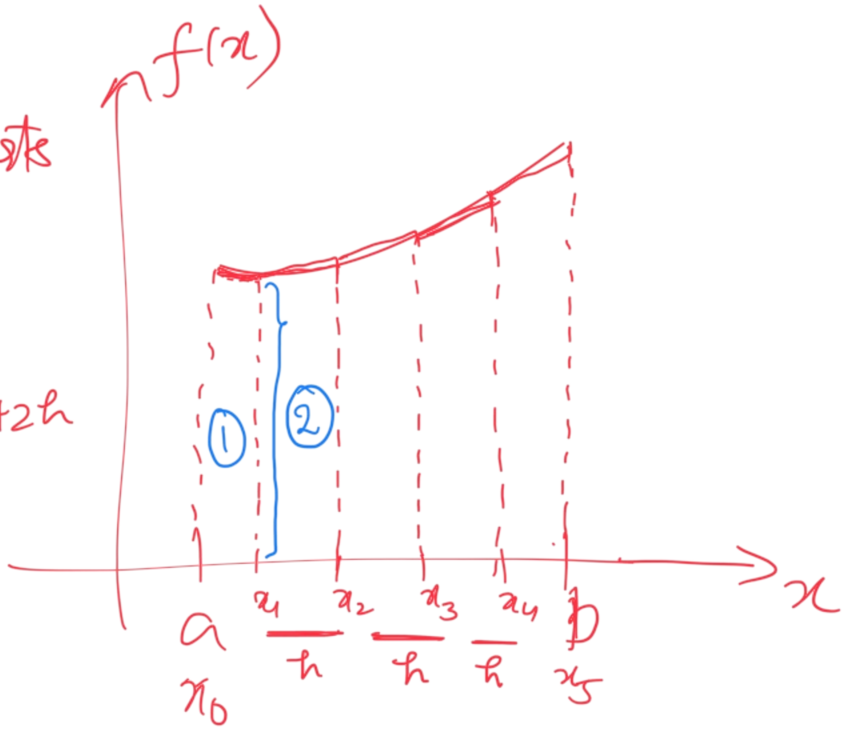
Remark :- Geometrically, R.H.S. of $(\star)$ is the area of trapezium $= \left(f(a) + f(b)\right) \times \frac{(b-a)}{2}$

Think about it :-



Composite trapezoidal Rule :-

devide the interval $[a,b]$ into N equal parts of length h.



$a = x_0,\ x_1 = x_0 + h,\ x_2 = x_0 + 2h$

$\ldots\ \underbrace{x_N = x_0 + Nh = b}$

$$\boxed{h = \frac{b-a}{N}}$$

$$I = \int_a^b f(x)\,dx = \frac{h}{2}\left[f(x_0) + f(x_1)\right] + \frac{h}{2}\left[f(x_1) + f(x_2)\right] + \cdots + \frac{h}{2}\left[f(x_{N-1}) + f(x_N)\right]$$

$$\Rightarrow \boxed{I = \frac{h}{2}\left[f(x_0) + 2\underbrace{\left\{f(x_1) + f(x_2) + \cdots + f(x_{N-1})\right\}} + f(x_N)\right]} \quad ✪✪$$

Remark:- Geometrically R.H.S. of ✪✪ is the sum of areas of N trapeziums with hight h.

Ques:- Find $I = \int_0^1 \frac{1}{1+x}\,dx$ & $N = 8$ $\quad \boxed{h = \frac{1-0}{8}}$

$\boxed{\text{Exact soution:- } I = \ln 2 = 0.693147}$ ✓

x: $x_0 = 0$, $\frac{1}{8}$, $\frac{2}{8}$, $\frac{3}{8}$, $\frac{4}{8}$, $\frac{5}{8}$, $\frac{6}{8}$, $\frac{7}{8}$, $x_8 = 1$

$f(x)$: 1, $\frac{8}{9}$, $\frac{8}{10}$, $\frac{8}{11}$, $\frac{8}{12}$, $\frac{8}{13}$, $\frac{8}{14}$, $\frac{8}{15}$, 0.5

$$f(x) = \frac{1}{1+x} \quad \text{then}$$

$$I = \frac{h}{2}\left[f(x_0) + 2\left\{f(x_1) + f(x_2) + \cdots + f(x_7)\right\} + f(x_8)\right]$$

$$\boxed{I = 0.6941}$$

$$\boxed{\text{Error} = \left|0.693147 - 0.694122\right| = 0.000975}$$

Apply Eq^n ⭐

$x_0 = 0$, $x_1 = 1$, $h = 1$

$$I = \frac{h}{2}\left[f(a) + f(b)\right]$$

$$= \frac{1}{2}\left[1 + 0.5\right] = 0.75$$

Now use $N = 2$:

x_0	x_1	x_2
0	0.5	1

$f(x) = 1$, $\frac{2}{3}$, 0.5

$$I = \frac{h}{2}\left[f(x_0) + 2f(x_1) + f(x_2)\right] = 0.7083.$$

Do it yourself :-

Ques :- Find $I = \int_1^2 \frac{dx}{5+3x}$ with $N=8$

Compare your results with exact solution.