Gauge Kinematics of Deformable Bodies

Ankit Mahajan
Supervisor: Prof. Ravi Banavar

Systems and Control Engineering,
IIT Bombay

SC 624 Paper Review, Spring 2015

The Problem

- A deformable body with zero angular momentum can change its orientation by executing a sequence of deformations.
- This rotation may be naturally expressed in a purely geometric form, in terms of a gauge potential over configuration space.

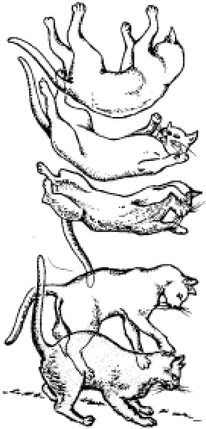


Figure: Falling cat

The Setting

- The space of allowable deformations of the body, forms a principal bundle over the shape space of the body. The fiber of this bundle is the group G of rigid motions (SO(3)), an element of which is the desired orientation.
- Dynamical constraints define a connection on this bundle. For the cat in free-fall with no initial angular momentum this constraint is that the angular momentum remains zero.

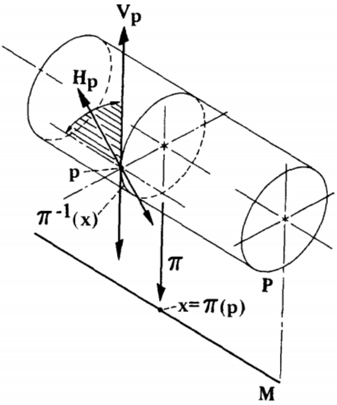
Principle Bundle

X: smooth manifold, G: Lie group, P: total space, $\pi : P \to X$ (projection), $\sigma : P \times G \to P$ (right action of G on P).

- σ preserves the fibers of π,
$$\pi(p.g) = \pi(p)$$
for all p $\in$ P and g $\in$ G
- Local triviality,
$$\Psi(p) = (\pi(p), \psi(p))$$
$$\psi(p.g) = \psi(p)g$$

Connection

A connection assigns to each $p \in P$ a subspace $H_p \in T_pP$ such that for $V_p = \{X \in T_pP | \pi_*(X) = 0\}$ we have $T_pP = H_p \oplus V_p$. We require that $R_{g*}(H_p) = H_{pg}$.



Connection

$$G \xrightarrow{\sigma_p} P \xrightarrow{\pi} X$$

$$\mathcal{G} \rightarrow T_pP \rightarrow T_xX$$

- $(\sigma_g)^*\omega = ad_{g^{-1}} \circ \omega \qquad \forall g \in G$
- $\omega(A^{\#}) = A \qquad \forall A \in \mathcal{G}$

Mechanical Connection

If Q is a Riemannian manifold, there is a unique choice of connection on Q compatible with the metric.

$$< \delta q_1, \delta q_2 >= \int < \delta q_1(y), \delta q_2(y) > dm(y)$$

If $\delta q_2 \in V_q$ then,

$$\delta q_2(y) = \mathbf{A} \times \mathbf{q}(y)$$

$$< \delta q_1, \delta q_2 >=< \mathbf{A}, M(q, \delta q_1) >$$

where,

$$M(q, \delta q_1) = \int q(y) \times \delta q_1(y) > dm(y)$$

motions of the deformable body which lie in the horizontal subspace have zero angular momentum.

Parallel Transport

Definition: Let $\gamma : [0,1] \to M$ be a curve in the base manifold (a base curve). A curve $\gamma_Q : [0,1] \to Q$ is called the horizontal lift of γ if

- $\pi(\gamma_Q) = \gamma$
- All tangent vectors X_Q to γ_Q are horizontal: $X_Q \in H_q Q$

This means that we can (given a connection) uniquely define the parallel transport of a point q in Q along a curve γ in M by moving it along the unique horizontal lift of γ through q. Given a path in the shape space of the deformable body, the unique path horizontally lifted to the configuration space will be its path in real space.