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Superconductivity as a Broken $U(1)_{em}$ Symmetry

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Introduction

The aim of this project was to study superconductivity as a manifestation of spontaneous *U(1)* symmetry breaking. Traditionally, superconductivity has been described by the semi-classical Ginzburg-Landau theory. In 1957, the complete microscopic theory of superconductivity was proposed by Bardeen, Cooper, and Schrieffer. The BCS theory is an effective field theory, which describes electron pairs (Cooper pairs) by a charged scalar field.

Philip Anderson explained the expulsion of magnetic flux from superconductors in terms of spontaneous breaking of gauge symmetry (Anderson-Higgs mechanism). Applying the same idea to elementary particle physics, Peter Higgs was able to explain the origin of mass of elementary particles. Other observed properties of superconductors like flux quantization, Josephsen effect, etc. can also be accounted for elegantly by the Higgs mechanism without referring to field dynamics.

The report is organized as follows: In the first chapter, we start with an informal description of the geometrical foundations of gauge theories. This is followed by a discussion on spontaneous breaking of global symmetry. The most important features of symmetry breaking are explained using the Goldstone theorem, which predicts the presence of massless fields when the symmetry group is continuous. The general group-theoretical nature of this result is stressed. In the second chapter, gauge theory is combined with symmetry breaking which leads to local gauge symmetry breaking. The central feature of this theory is the Higgs mechanism, whereby the gauge fields absorb the Goldstone scalars and become massive. We will explicitly apply these concepts to the theory of interaction of Cooper pairs with electromagnetic field. It will be shown that the Higgs mechanism accounts for the exotic properties of superconductors.

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Chapter 1

Spontaneous Symmetry Breaking

A crucial step was taken by Yang and Mills (1954) when they introduced the concept of a non-abelian gauge theory as a generalization of Maxwell's theory of electromagnetism. The Yang-Mills theory involves a self-interaction among gauge fields, which gives it a certain similarity to Einstein's theory of gravity. At about the same time, the mathematical theory of fiber bundles had reached an advanced stage but was generally unknown to the physics community. The fact that Yang-Mills theories and the affine geometry of principal fiber bundles are one and the same thing was eventually recognized by various authors as early as 1963, but few of the implications were explored. The potential utility of the differential geometric methods of fiber bundles in gauge theories was pointed out to the bulk of the physics community by the paper of Wu and Yang (1975). For example, Wu and Yang showed how the long-standing problem of the Dirac string for magnetic monopoles could be resolved by using overlapping coordinate patches with gauge potentials differing by a gauge transformation. For mathematicians, the necessity of using coordinate patches is a trivial consequence of the fact that non-trivial fiber bundles cannot be described by a single gauge potential defined over the whole coordinate space.

1.1 The Geometry of Gauge Fields

In this section, an elementary presentation of the mathematical tools which are necessary for a geometrical description of gauge fields is given. It will be shown that a gauge potential is a connection on some fiber bundle, and the corresponding gauge field the associated curvature.

The two basic ingredients of a gauge theory are a Lie group G and a manifold (coordinate space) M, with an independent copy $G(x)$ of G assigned to each point x of M.

Definition: A principal fiber bundle consists of a manifold Q(called the total space), a Lie group G, a base manifold M, a right action $\sigma : Q \times G \to Q$ and a smooth projection map $\pi : Q \to M$ such that the following conditions are true:

- σ preserves the fibers of π,

$$\pi(q.g) = \pi(q)$$

  for all $q \in Q$and $g \in G$

- For each $x \in M$ there exists an open neighborhood V of x in $M$and a homeomorphism $\Psi : \pi^{-1}(V) \to V \times G$ of the form,

$$\Psi(q) = (\pi(q), \psi(q))$$

  where $\psi : \pi^{-1}(V) \to G$ satisfies

$$\psi(q.g) = \psi(q)g$$

Q is a principal G-bundle over M and we indicate this diagramatically by writing $G \to Q \to M$.

A principal G bundle is called trivial if it is globally isomorphic to $M \times G$. However, in general principal bundles are only locally trivial.

For each $x \in M$, $\pi^{-1}(x)$ is a closed submanifold of Q, called the fiber over x. Clearly, if u is a point of $\pi^{-1}(x)$, then $\pi^{-1}(x)$ is the set of points u.a, $a \in G$ and is called the fiber through u. Every fiber is diffeomorphic to G. Each fiber can be thought of as a copy of G without a specified identity element (such structures are called torsors, a notion similar to affine spaces). M is the quotient space of Q by the equivalence relation induced by G, $M = Q/G$.

Example of a nontrivial principal bundle: The group $SU(2)$ can be described as a $U(1)$-bundle over S^2. This is an example of the famous Hopf bundle (1.1).

A (local) cross-section of Q defined on an open set $V \in M$ is a smooth map $s : V \to \pi^{-1}(V)$ that satisfies $\pi \circ s = id_V$, i.e., it is a smooth selection of an element from each fiber above V. This is what is known as the choice of gauge $g(x)$ in physics.

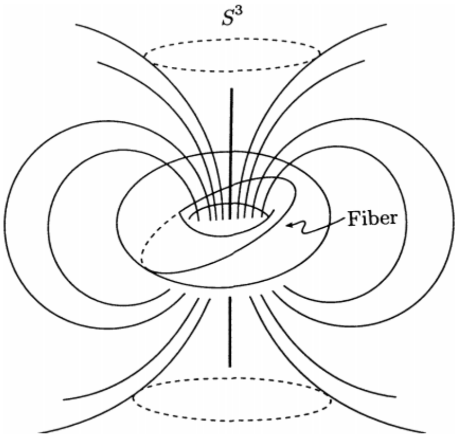

Figure 1.1: Hopf bundle

Suppose M is the four-dimensional Minkowski space with local coordinates x_μ in V. Let ω be a 1-form in V, it can be written in terms of its components (Lie-algebra-valued functions), $A^\mu(x)$ (with respect to section s):

$$\omega = A^\mu(x)dx_\mu$$

Suppose now we transform s into s' (gauge transformation) by the action of some $g(x)$:

$$s'(x) = s(x)g(x)$$

$$\omega' = s'^*\omega = A'^\mu(x)dx_\mu$$

It can be seen that,

$$A'_\mu(x) = g^{-1}A_\mu(x)g + g^{-1}\partial_\mu(x)g$$

Note the similarity in structure with the transformation law for Christoffel symbols,

$$\Gamma'^\kappa_{\lambda\mu} = \Lambda^\kappa_\alpha\Lambda^\beta_\lambda\Lambda^\gamma_\mu\Gamma'^\alpha_{\beta\gamma} + [\textit{inhomogeneous term}]$$

This apparent relation between connection coefficients and gauge potentials can be proved. Thus gauge potentials can be identified with a connection on Q[1]. The covariant derivative with respect to this connection is given by,

[1]More precisely, gauge potentials defined locally on the base manifold are pullbacks of a connection form defined globally on the principal bundle.

$$D_\mu \psi = \partial_\mu \psi - igA_\mu \psi$$

Given an $A_\mu(x)$, is there an intrinsic criterion to determine whether it is gauge-equivalent to zero or not? The answer lies in the curvature tensor of the corresponding connection. The curvature can be identified with the field strength given by,

$$F_{\mu\nu} = \partial_{[\mu} A_{\nu]}$$

The homogeneous Maxwell's equation then follows from the Bianchi identity obeyed by the curvature tensor,

$$\partial_{[\mu} F_{\nu\rho]} = 0$$

Following correspondence can be established between gauge theories and general theory of relativity[2],

$$\text{gauge transformations} \equiv \text{coordinate transformations}$$

$$A_\mu \equiv \Gamma^\kappa_{\mu\nu}$$

$$F_{\mu\nu} \equiv R^\kappa_{\lambda\mu\nu}$$

1.2 Spontaneous Symmetry Breaking[3]

Spontaneous symmetry breaking is a very general phenomenon characterized by Lagrangian having a symmetry (global or local) but the quantum theory, instead of having a unique vacuum state which respects this symmetry, has a family of degenerate vacua that transform into each other under the action of the symmetry group. To get a feel for this concept, let's study the classic example of the Maxican-hat potential.

$$\mathcal{L} = \partial_\mu \phi \partial^\mu \phi^* - V(\phi, \phi^*) = \partial_\mu \phi \partial^\mu \phi^* - m^2 \phi \phi^* - \lambda(\phi\phi^*)^2$$

Here m^2 is a parameter not necessarily mass. $\mathcal{L}$ is invariant under the global gauge transformation,

$$\phi \to e^{i\Lambda} \phi$$

[2]This similarity in structure between gauge theories and general relativity has led physicists to believe that the geometric viewpoint might provide insights into the unification of gravity with standard model

[3]I couldn't follow the geometric viewpoint of symmetry breaking as rigorously as originally intended, due to lack of time. So I have adopted the more traditional algebraic approach in the following sections.

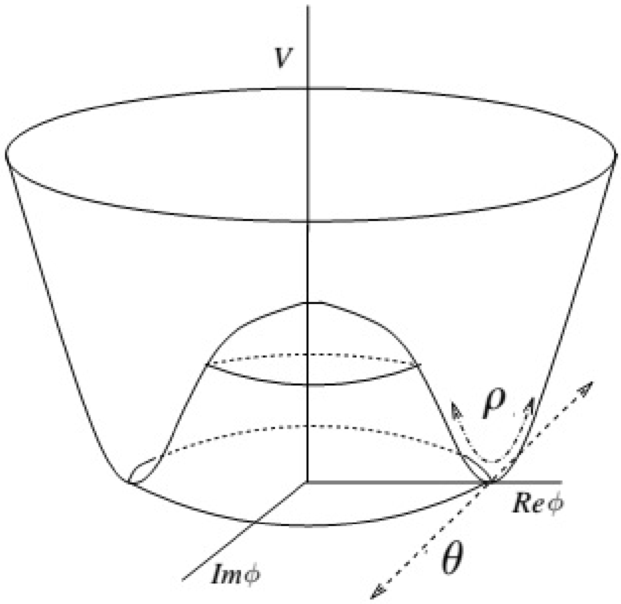

Figure 1.2: Mexican hat potential

where Λ is a constant. The ground state is obtained by minimizing the potential V. We have,

$$\frac{\partial V}{\partial \phi} = m^2\phi^* + 2\lambda\phi^*(\phi^*\phi)$$

When $m^2 > 0$, the minimum occurs at $(\phi^* = \phi = 0)$. For $m^2 < 0$ there is a local maximum at $\phi = 0$ and a minimum at $|\phi|^2 = -\frac{m^2}{2\lambda} = a^2$. The scalar field will choose one of these vacua, and the *U(1)* symmetry is said to be spontaneously broken.

In the quantum theory, where ϕ becomes an operator, this condition can be written in terms of the vacuum expectation value of ϕ,

$$|\langle 0|\phi|0\rangle|^2 = a^2$$

The minima of V lie along the circle $|\phi| = a$, which form a set of degenerate vacua related to each other by rotation. The physical fields, which are excitations above the vacuum, are then realized by performing perturbations about $|\phi| = a$. Working in polar co-ordinates

$$\phi(x) = \rho(x)e^{i\theta(x)}$$

choose the vacuum state

$$\langle 0|\phi|0\rangle = a$$

Thus,

$$\langle 0|\rho|0\rangle = a \qquad \langle 0|\theta|0\rangle = 0$$

Now let's put,

$$\phi(x) = (\rho'(x) + a)e^{i\theta(x)}$$

$$\mathcal{L} = \partial_\mu\rho'\partial^\mu\rho' + (\rho' + a)^2\partial_\mu\theta\partial^\mu\theta - \lambda[(\rho' + a)^2 - a^2]^2 - \lambda a^4$$

We see that there is a term in ρ'^2, so ρ' has a mass given by, $m_{\rho'}^2 = 4\lambda a^2$. but there is no term in θ^2, so θ is a massless field. As a result of spontaneous symmetry breaking, what would otherwise be two massive fields (the real parts of ϕ), become one massive and one massless field. It clearly costs energy to displace ρ' against the restoring forces of the potential, but there are no restoring forces corresponding to displacements along the circular valley $|\phi| = a$, in view of the vacuum degeneracy. Hence for the angular excitations θ, of wavelength λ, we have $\omega \to 0$ as $\lambda \to \infty$, and the relativistic particles are massless. The θ particle is known as a Goldstone boson. This phenomenon is general: spontaneous breaking of a (continuous) symmetry entails the existence of a massless particle, the Goldstone particle. This statement, known as the Goldstone's theorem, will be proved in the next section.

1.2.1 Goldstone's Theorem: SSB of Global Symmetries

If $\mathcal{L}$ is invariant under a group of transformations then according to Noether's theorem, the currents

$$j_\mu^a(x) = \frac{\partial\mathcal{L}}{\partial(\partial^\mu\phi)}\frac{\delta\phi(x)}{\delta\alpha^a}$$

have zero divergence, and the corresponding charges

$$Q^a = \int d^3x j_0^a(x)$$

are conserved. These charges have the commutation relations of the symmetry group

$$[Q^a, Q^b] = C^{abc}Q^c$$

The unitary operator corresponding to a group transformation is

$$U = e^{iQ^a\alpha^a}$$

In case of SSB, $U\,|0\rangle = |0'\rangle \neq |0\rangle$ or $Q^a\,|0\rangle \neq 0$. Since $\phi(x)$ is not a singlet under the group, there must exist an operator $\psi'(x)$ such that, for some a,

$$[Q^a, \phi'(x)] = \phi(x)$$

and, since $\langle 0|\phi(x)|0\rangle \neq 0$,

$$\langle 0|Q^a\phi'(x) - \phi'(x)Q^a|0\rangle \neq 0$$

$$\sum_n \int d^3y[\langle 0|\, j_0^a(y)\,|n\rangle\, \langle n|\, \phi'(x)\,|0\rangle - \langle 0|\, \phi'(x)\,|n\rangle\, \langle n|\, j_0^a(y)\,|0\rangle]|_{x^0=y^0} \neq 0$$

Now translation invariance implies that

$$j_0^a(y) = e^{-ipy} j_0^a(0) e^{ipy}$$

Thus,

$$(2\pi)^3 \sum_n \delta^3(\mathbf{p}_n)[\langle 0|\, j_0^a(y)\,|n\rangle\, \langle n|\, \phi'(x)\,|0\rangle\, e^{iM_n y_0} - \langle 0|\, \phi'(x)\,|n\rangle\, \langle n|\, j_0^a(y)\,|0\rangle\, e^{-iM_n y_0}]|_{x^0=y^0} \neq 0$$

where, $M_n = p_{n0}$, is the mass of the state n. To show that the above expression is independent of y_0, we start from the fact that $j_\mu^a(y)$ has zero divergence:

$$\partial^\mu j_\mu^a(y) = 0$$

$$\begin{aligned}
\frac{\partial}{\partial y_0}\langle 0|[Q^a, \phi'(x)]|0\rangle &= \frac{\partial}{\partial y_0}\int d^3y\, \langle 0|[j_0^a(y), \phi'(x)]|0\rangle \\
&= -\int d\mathbf{S}.\, \langle 0|[\mathbf{j}^a(y), \phi'(x)]|0\rangle \\
&= 0
\end{aligned}$$

Thus the expression derived above is independent of y_0. We conclude that $M_n = 0$. Therefore we have proven Goldstone's theorem: associated with each broken symmetry there is a massless mode in the theory. These modes are known as Nambu-Goldstone modes.

Chapter 2

Higgs Mechanism and Superconductivity

In this chapter, the phenomenon of superconductivity is explained as a consequence of the Higgs mechanism. The photon acquires mass in the sense that the electromagnetic field becomes a massive field.

2.1 The problem of the gauge field masses

Consider the following gauge invariant Lagrangian,

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} - \frac{1}{2}D_\mu\phi D^\mu\phi^* - \bar{\Psi}\gamma^\mu D_\mu\Psi - V(\phi,\phi^*)$$

It contains no mass terms for the gauge fields. A term of the form $\mathcal{L}_m(A) = \frac{1}{2}M_{rs}A^r_\mu A^{\mu s}$ would destroy the gauge invariance. This is a serious defect, because if the above is to be applied to the weak interactions the gauge fields must be identified with observable vector fields, and it is known experimentally that such vector fields must be massive (because massless fields should be easily observed because of their long range and yet the only massless vector field which has been observed is the photon).

The resolution of this problem lies in SSB. Thus while the ordinary gauge theory can be used for gravitational, electromagnetic (and possibly strong coloured) interactions, its use for weak (and possibly strong flavoured) interactions depends on the introduction of SSB. In the following section we show how SSB solves the gauge field mass problem in a simple and elegant way (Higgs mechanism).

2.2 Higgs mechanism

In this section, we will analyze the consequences of SSB in case of local gauge symmetry. Consider a *U(1)* gauge field coupled to a self-interacting charged complex scalar field ϕ with Lagrangian,

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} - \frac{1}{2}D_\mu\phi D^\mu\phi^* - \frac{\lambda}{4}(\phi^*\phi - \frac{\nu^2}{2})^2$$

This $\mathcal{L}$ is invariant under the gauge transformations,

$$\phi \to e^{i\Lambda(x)}\phi \qquad A_\mu \to A_\mu + \partial_\mu\Lambda(x)$$

The minimum of the potential is defined by the equation $|\phi| = \frac{\nu}{\sqrt{2}}$. Thus, there is a continuum of different vacua labelled by the phase of the scalar field. None of these vacua is invariant under the gauge symmetry,

$$\langle 0|\phi|0\rangle = \frac{\nu}{\sqrt{2}}e^{i\theta_0} \to \frac{\nu}{\sqrt{2}}e^{i\theta_0 + i\Lambda(x)}$$

and therefore the symmetry is spontaneously broken. As before, let's study the theory around one of these vacua, $\langle 0|\phi|0\rangle = \frac{\nu}{\sqrt{2}}$, by writing the field ϕ in terms of the excitations around this particular vacuum,

$$\phi(x) = \frac{1}{\sqrt{2}}(\nu + \sigma(x))e^{i\theta(x)}$$

As $\mathcal{L}$ is gauge invariant, we are at liberty to perform a gauge transformation with parameter $\Lambda(x) = -\theta(x)$ in order to get rid of the phase factor. Thus by substitution,

$$\begin{aligned}\mathcal{L} = &-\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \frac{1}{2}e^2\nu^2 A_\mu A^\nu + \frac{1}{2}\partial_\mu\sigma\partial^\mu\sigma - \frac{1}{4}\lambda\nu^2\sigma^2 - \frac{1}{\sqrt{2}}\lambda\nu\sigma^3 - \frac{\lambda}{4}\sigma^4 \\ &+ \frac{1}{\sqrt{2}}e^2\nu A_\mu A^\mu\sigma + e^2 A_\mu A^\mu\sigma^2\end{aligned}$$

There is a real scalar field $\sigma(x)$ with mass squared $\frac{1}{2}\lambda\nu^2$. What makes the construction interesting is that the gauge field A_μ has acquired a mass $m_\gamma = e\nu$. What is really remarkable about this way of giving a mass to the photon is that at no point have we given up gauge invariance. The symmetry is only hidden. Therefore in quantizing the theory we can still enjoy all the advantages of having a gauge theory, while at the same time we have managed to generate a mass for the gauge field.

Since the vacuum chosen by the scalar field breaks the single generator of *U(1)*, we would have expected from Goldstone's theorem to have one massless particle. Were we dealing with a global *U(1)* theory, the Goldstone boson would correspond to excitations of the scalar field along the valley of the potential associated with the phase $\theta(x)$. In writing the Lagrangian we managed to get rid of $\theta(x)$ using a gauge transformation. With this we shifted the Goldstone mode into the gauge field A_μ. In fact, by identifying the gauge parameter with the Goldstone excitation we have completely fixed the gauge and the Lagrangian does not have any residual gauge symmetry. A massive vector field has three polarizations: two transverse ones with helicities $\lambda = \pm 1$ plus a longitudinal one. In gauging away the massless Goldstone boson $\theta(x)$ we have transformed it into the longitudinal polarization of the massive vector field. In literature, this is usually expressed by saying that the Goldstone mode is 'eaten up' by the longitudinal component of the gauge field. This phenomenon is called the Higgs phenomenon. In this Abelian model it can be summarized by saying that spontaneous breaking of a gauge symmetry results, not in the presence of a massless Goldstone boson, but in the disappearance of that field altogether and the appearance instead of a massive, rather than a massless, gauge field.

2.3 Superconductivity

To begin, we consider a static situation, so $\partial_0\phi = 0$, etc., and $\mathcal{L}$ takes the form

$$\mathcal{L} = -\frac{1}{2}(\boldsymbol{\nabla} \times \mathbf{A})^2 - |(\boldsymbol{\nabla} - ie\mathbf{A})\phi|^2 - m^2|\phi|^2 - \lambda|\phi|^4$$

where $m^2 = a(T - T_c)$ near the critical temperature $T = T_c$, ϕ is the macroscopic many-particle wave function, and its use is justified by the Bardeen-Cooper-Schrieffer (BCS) theory, according to which, under certain conditions, there is an attractive force between electrons, and the field quanta are electron pairs, which are bosons. At low temperatures, these fall into the same quantum state (Bose-Einstein condensation), and because of this, a many-particle wave function ϕ may be used to describe the macroscopic system. For $T > T_c$, $m^2 > 0$ and the minimum free energy is at $|\phi| = 0$. But when $T < T_c$, $m^2 < 0$ and the minimum free energy is at,

$$|\phi|^2 = -\frac{m^2}{2\lambda} > 0$$

which is an example of spontaneous symmetry breaking. $\mathcal{L}$ is invariant under the usual phase transformation,

$$\phi \rightarrow e^{i\Lambda(x)}\phi \qquad \mathbf{A} \rightarrow \mathbf{A} + \frac{1}{e}\boldsymbol{\nabla}\Lambda(x)$$

and the associated conserved current is

$$\mathbf{j} = -i(\phi^* \boldsymbol{\nabla}\phi - \phi\boldsymbol{\nabla}\phi^*) - 2e|\phi|^2\mathbf{A}$$

When $T < T_c$, and ϕ varies only very slightly over the sample, the second of these terms dominates over the first, and

$$\mathbf{j} = \frac{em^2}{\lambda}\mathbf{A} = \kappa^2\mathbf{A}$$

This is the London equation. The electric field $E = -\partial\mathbf{A}/\partial t = 0$, and Ohm's law defines resistance by $\mathbf{E} = R\mathbf{j}$, so $R = 0$ and we have superconductivity.

The Meissner effect (expulsion of magnetic flux) is easily derived. Ampere's equation is

$$\boldsymbol{\nabla} \times \mathbf{B} = \mathbf{j}$$

Thus,

$$\boldsymbol{\nabla}^2\mathbf{B} = k^2\mathbf{B}$$

Confining ourselves, for simplicity, to one spatial dimension, has the solution,

$$B_x = B_0 e^{(-kx)}$$

so that magnetic field only penetrates the specimen to a characteristic depth $1/k$. Because the energetic cost of supporting massive fields over an extended volume is prohibitive, a superconducting material finds ways to expel magnetic fields.

It also follows that,

$$\Box A_\mu = -k^2 A_\mu$$

which means the 'photons' have a mass k , which is the characteristic feature of the Higgs phenomenon.

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