Simplex Method Examples

Example 2: (Infinitely many solutions)

Show by simplex method that the following LPP has infinitely many solutions

$$Maximise\ Z = 4x_1 + 10x_2$$

Subject to

$$2x_1 + x_2 \leq 10$$
$$2x_1 + 5x_2 \leq 20$$
$$2x_1 + 3x_2 \leq 18$$
$$x_1, x_2 \geq 0$$

Solution: We write the LPP in the standard form.

$$Maximise\ Z = 4x_1 + 10x_2$$

Subject to

$$2x_1 + x_2 + s_1 = 10$$
$$2x_1 + 5x_2 + s_2 = 20$$
$$2x_1 + 3x_2 + s_3 = 18$$
$$x_1, x_2, s_1, s_2, s_3 \geq 0$$

Let the initial non-basic variables be x_1, x_2. So, the initial basic feasible solution is given by $s_1 = 10, s_2 = 20, s_3 = 18$.

c_j		4	10	0	0	0			
e_i	Basis	x_1	x_2	s_1	s_2	s_3	b	θ	
0	s_1	2	1	1	0	0	10	10	
0	s_2	2	5	0	1	0	20	4	$\rightarrow$
0	s_3	2	3	0	0	1	18	6	
$E_j = \sum_{i=1}^{3} a_{ij}e_i$		0	0	0	0	0			
$c_j - E_j$		4	10	0	0	0			
			$\uparrow$						

$$R_2 \rightarrow \frac{R_2}{5}, R_1 \rightarrow R_1 - R_2, R_3 \rightarrow R_3 - 3R_2$$

c_j		4	10	0	0	0			
e_i	Basis	x_1	x_2	s_1	s_2	s_3	b	θ	
0	s_1	8/5	0	1	-1/5	0	6	15/4	$\rightarrow$
10	x_2	2/5	1	0	1/5	0	4	10	
0	s_3	4/5	0	0	-3/5	1	6	15/2	
$E_j = \sum_{i=1}^{3} a_{ij}e_i$		4	10	0	2	0			
$c_j - E_j$		0	0	0	-2	0			
		$\uparrow$							

All $c_j - E_j \leq 0$, optimal solution is $x_1 = 0, x_2 = 4, Z_{max} = 40$.
If among non-basic original decision variables, $c_j - E_j = 0$ implies the existence of an alternative optima.
We have two original decision variables x_1, x_2, of which x_1 is non-basic with $c_j - E_j = 0$. Therefore, alternative optima exists.

$$R_1 \to R_1(5/8), R_2 \to R_2 - \frac{2}{5}R_1, R_3 \to R_3 - \frac{4}{5}R_1$$

c_j		4	10	0	0	0			
e_i	Basis	x_1	x_2	s_1	s_2	s_3	b	θ	
4	x_1	1	0	5/8	-1/8	0	15/4		
10	x_2	0	1	-1/4	1/4	0	5/2		
0	s_3	0	0	-1/2	-1/2	1	6		
$E_j = \sum_{i=1}^{3} a_{ij}e_i$		4	10	0	2	0			
$c_j - E_j$		0	0	0	-2	0			

All $c_j - E_j \leq 0$, second optimal solution is $x_1 = \frac{15}{4}, x_2 = \frac{5}{2}, Z_{max} = 40$.

Note that $Z_{max} = 40$ in both the optimal solutions.

Let $x_1 = (0,4,6,0,6)\ and\ x_2 = \left(\frac{15}{4}, \frac{5}{2}, 0,0,6\right)$.

Let $x^* = \lambda x_1 + (1-\lambda)x_2, 0 \leq \lambda \leq 1$.

$$\Rightarrow x^* = \lambda(0,4,6,0,6) + (1-\lambda)\left(\frac{15}{4}, \frac{5}{2}, 0,0,6\right)$$

$$= \left(\frac{15}{4} - \frac{15}{4}\lambda, \frac{5}{2} + \frac{3}{2}\lambda, 6\lambda, 0,6\right)$$

$$\Rightarrow Z(x^*) = 4\left(\frac{15}{4} - \frac{15}{4}\lambda\right) + 10\left(\frac{5}{2} + \frac{3}{2}\lambda\right) = 15 - 15\lambda + 25 + 15\lambda = 40$$

$\Rightarrow x^*$ is also an optimal solution.Note that for every λ, there is x^*. Thus, the given LPP has infinitely many optimal solutions.

Example 3: (A case of unbounded optima, no solution)

Solve

$$\begin{gathered} Maximise\ Z = 2x_1 + 3x_2 + 4x_3 + x_4 \\ subject\ to \\ x_1 + 5x_2 + 9x_3 - 6x_4 \geq -2 \\ 3x_1 - x_2 + x_3 + 3x_4 \leq 10 \\ 2x_1 + 3x_2 - 7x_3 + 8x_4 \leq 0 \\ x_1, x_2, x_3, x_4 \geq 0 \end{gathered}$$

Solution: Write the above in the standard form.

$$\begin{gathered} Maximise\ Z = 2x_1 + 3x_2 + 4x_3 + x_4 \\ subject\ to \\ -x_1 - 5x_2 - 9x_3 + 6x_4 + s_1 = 2 \\ 3x_1 - x_2 + x_3 + 3x_4 + s_2 = 10 \\ 2x_1 + 3x_2 - 7x_3 + 8x_4 + s_3 = 0 \\ x_1, x_2, x_3, x_4, s_1, s_2, s_3 \geq 0 \end{gathered}$$

Assume initial non-basic variables are x_1, x_2, x_3, x_4. This implies the initial basic feasible solution is given by $s_1 = 2, s_2 = 10, s_3 = \epsilon \; where \; \epsilon > 0 \; is \; small \; enough$. [Reason: If we start with $s_3 = 0$, solution is degenerate. This degeneracy remains in the simples table till the end. To remove degeneracy in the simplex table we approximate s_3 by a small but positive (>0) number ϵ.]

c_j		2	3	4	1	0	0	0			
e_i	CSV (Basis)	x_1	x_2	x_3	x_4	s_1	s_2	s_3	b	θ	
0	s_1	-1	-5	-9	6	1	0	0	2	$-ve$	
0	s_2	3	-1	1	3	0	1	0	10	10	$\rightarrow$
0	s_3	2	3	-7	8	0	0	1	ϵ	$-ve$	
$E_j = \sum_{i=1}^{3} a_{ij} e_i$		0	0	0	0	0	0	0			
$c_j - E_j$		2	3	4	1	0	0	0			
				$\uparrow$							

$$R_1 \rightarrow R_1 + 9R_2, R_3 \rightarrow R_3 + 7R_2$$

c_j		2	3	4	1	0	0	0			
e_i	CSV (Basis)	x_1	x_2	x_3	x_4	s_1	s_2	s_3	b	θ	
0	s_1	8	-14	0	33	1	9	0	92	$-ve$	
4	x_3	3	-1	1	3	0	1	0	10	$-ve$	
0	s_3	23	-4	0	35	0	7	1	$70 + \epsilon$	$-ve$	
$E_j = \sum_{i=1}^{3} a_{ij} e_i$		12	-4	4	12	0	4	0			
$c_j - E_j$		-10	7	0	-11	0	-4	0			
			$\uparrow$								

Note that in the above table, x_2 is an incoming variable for the next iteration but there is no outgoing variable as all θ column entries are negative. This indicates that an optimal solution exists at the point of infinity. That means this is the case of an unbounded optima. So, the given LPP has no solution.
Note that Z value depends upon x_2. We can take x_2 as large as possible without violating optimality conditions.