# LIMITS

* If $f : \mathbb{R} \to \mathbb{R}$ defined by $f(x) = c$, where $c$ is a constant then

$$\underset{x \to a}{\mathrm{Lt}}\, f(x) = c, \ \forall\, a \in \mathbb{R}.$$

Eg: $\underset{x \to 0}{\mathrm{Lt}}\, 2 = 2$, $\underset{x \to 4}{\mathrm{Lt}}\, (-3) = -3$, $\underset{x \to -1}{\mathrm{Lt}}\, 0 = 0$

* If $\underset{x \to a}{\mathrm{Lt}}\, f(x)$ exists then $\underset{x \to 0}{\mathrm{Lt}}\, f(x) = \underset{x \to 0}{\mathrm{Lt}}\, f(x+a) = \underset{x \to 0}{\mathrm{Lt}}\, f(a-x)$

* If $n$ is a number and $a > 0$, then $\underset{x \to a}{\mathrm{Lt}}\, \dfrac{x^n - a^n}{x - a} = n . a^{n-1}$

* If $m$ & $n$ are real numbers and $a > 0$, then $\underset{x \to a}{\mathrm{Lt}}\, \dfrac{x^m - a^m}{x^n - a^n} = \dfrac{m}{n} . a^{m-n}$

* $\underset{x \to 0}{\mathrm{Lt}}\, \dfrac{\sin x}{x} = 1$, $\underset{x \to 0}{\mathrm{Lt}}\, \dfrac{\sin ax}{x} = a$, $\underset{x \to \infty}{\mathrm{Lt}}\, \dfrac{\sin x}{x} = 0$

* $\underset{x \to 0}{\mathrm{Lt}}\, \dfrac{\tan x}{x} = 1$, $\underset{x \to 0}{\mathrm{Lt}}\, \dfrac{\tan bx}{x} = b$

* $\underset{x \to 0}{\mathrm{Lt}}\, \dfrac{\sin ax}{\sin bx} = \dfrac{a}{b}$, $\underset{x \to 0}{\mathrm{Lt}}\, \dfrac{\tan ax}{\tan bx} = \dfrac{a}{b}$, $\underset{x \to 0}{\mathrm{Lt}}\, \dfrac{\sin ax}{\tan bx} = \dfrac{a}{b}$

* $\underset{x \to 0}{\mathrm{Lt}}\, \dfrac{1 - \cos x}{x} = 0$

* $\underset{x \to 0}{\mathrm{Lt}}\, \dfrac{1 - \cos x}{x^2} = \dfrac{1}{2}$

$$\Rightarrow \underset{x \to 0}{\mathrm{Lt}}\, \frac{2 \sin^2 (x/2)}{x^2} = 2\, \underset{x \to 0}{\mathrm{Lt}} \left( \frac{\sin x/2}{x/2} \right)^2 . \frac{1}{4} = 1/2$$

* $\underset{x \to 0}{\mathrm{Lt}}\, \dfrac{1 - \cos ax}{x^2} = \dfrac{a^2}{2}$

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* $\underset{x \to 0}{\mathrm{Lt}}\, \dfrac{e^x - 1}{x} = 1$

* $\underset{x \to 0}{\mathrm{Lt}}\, \dfrac{a^x - 1}{x} = \log_e a$

* $\underset{x \to 0}{\mathrm{Lt}}\, (1 + x)^{1/x} = e$

* $\underset{x \to \infty}{\mathrm{Lt}}\, \left(1 + \dfrac{1}{x}\right)^x = e$, $\underset{x \to -\infty}{\mathrm{Lt}}\, \left(1 + \dfrac{1}{x}\right)^x = e$

* $\underset{x\to 0}{Lt} \dfrac{(a+x)^n - a^n}{x} = n.a^{n-1}$

* $\underset{h\to 0}{Lt} \dfrac{(a-h)^n - a^n}{-h} = n.a^{n-1}$

* $\underset{h\to 0}{Lt} \dfrac{(a-h)^n - a^n}{h} = -n.a^{n-1}$

* $\underset{x\to 0}{Lt} \dfrac{(a+x)^n - (a-x)^n}{x} = 2.n.a^{n-1}$

$\Rightarrow \underset{x\to 0}{Lt} \left[\dfrac{(a+x)^n - a^n}{x} + \dfrac{(a-x)^n - a^n}{-x}\right]$

Put $a+x=t$ & $a-\cancel{x}=s$

If $x\to 0 \Rightarrow t\to a$, $s\to a$

$\therefore \underset{t\to a}{Lt}\left(\dfrac{t^n - a^n}{t-a}\right) + \underset{s\to a}{Lt}\left(\dfrac{s^n - a^n}{s-a}\right)$

$= na^{n-1} + na^{n-1}$

$= 2na^{n-1}$.

* $\underset{n\to\infty}{Lt} \dfrac{1+2+3+\cdots+n}{n^2} = 1/2$

* $\underset{n\to\infty}{Lt} \dfrac{1^2+2^2+3^2+\cdots+n^2}{n^3} = 1/3$

* ~~Lt~~ If $K\in\mathbb{N}$, $\underset{n\to\infty}{Lt} \dfrac{1^k+2^k+3^k+\cdots+n^k}{n^{K+1}} = \dfrac{1}{K+1}$

$$\boxed{\begin{array}{l}\text{* If } \underset{x\to a}{Lt} f(x)=1 \text{ and } \underset{x\to a}{Lt} g(x)=\infty \text{ then} \\ \underset{x\to a}{Lt} f(x)^{g(x)} = e^{\underset{x\to a}{Lt} g(x).(f(x)-1)}\end{array}}$$

($1^\infty$ form)

* If $\underset{x\to a}{Lt} f(x) = 1$ and $\underset{x\to a}{Lt} g(x) = \infty$ then

$$\underset{x\to a}{Lt} f(x)^{g(x)} = e^{\underset{x\to a}{Lt}\, g(x).(f(x)-1)}$$

($1^\infty$ form)

Eg: Evaluate $\underset{x\to 0}{Lt} (1+ax)^{b/x}$

This is in $1^\infty$ form so $e^{\underset{x\to 0}{Lt} \frac{b}{x}(\cancel{1}+ax-\cancel{1})}$

$$= e^{\underset{x\to 0}{Lt} \frac{b}{\cancel{x}}(a\cancel{x})}$$

$$= e^{\underset{x\to 0}{Lt}\, ba} = e^{ab}$$

Eg: Evaluate $\underset{x\to \infty}{Lt} \left(1+\frac{1}{ax+b}\right)^{cx+d}$

$\Rightarrow 1^\infty$ form so $e^{\underset{x\to\infty}{Lt} (cx+d)\left(1+\frac{1}{ax+b}-1\right)}$

$$= e^{\underset{x\to\infty}{Lt} (cx+d)\left(\frac{1}{ax+b}\right)}$$

$$= e^{\underset{x\to\infty}{Lt} \left[\frac{(cx+d)/x}{(ax+b)/x}\right]}$$

$$= e^{\underset{x\to\infty}{Lt} \left(\frac{c+d/x}{a+b/x}\right)} =$$

$$= e^{c/a}$$

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Eg: Evaluate $\underset{x\to 0}{Lt} \left(\frac{\sin x}{x}\right)^{\frac{\sin x}{x-\sin x}}$

$\Rightarrow 1^\infty$ form so $e^{\underset{x\to 0}{Lt} \left(\frac{\sin x}{x-\sin x}\right)\left(\frac{\sin x}{x}-1\right)}$

$$= e^{\underset{x\to 0}{Lt} \left(\frac{\sin x}{\cancel{x-\sin x}}\right)\left(\frac{-1}{\cancel{\sin x - x}}\right)}$$

$$= e^{-\underset{x\to 0}{Lt} \frac{\sin x}{x}} = e^{-1}$$

## L-Hospital Rule:

(1) If $f(x)$ and $g(x)$ are two functions such that $\underset{x\to a}{Lt} f(x) = \underset{x\to a}{Lt} g(x) = 0$

then $\underset{x\to a}{Lt} \frac{f(x)}{g(x)} = \underset{x\to a}{Lt} \frac{f'(x)}{g'(x)}$, Provided the latter limits exists.

(2) If $f(x)$ and $g(x)$ are two functions such that $\underset{x\to a}{Lt} f(x) = \underset{x\to a}{Lt} g(x) = \infty$

then $\underset{x\to a}{Lt} \frac{f(x)}{g(x)} = \underset{x\to a}{Lt} \frac{f'(x)}{g'(x)}$, Provided the latter limit exists.

* L-Hospital rule can be applied repeatedly.

i.e., $\underset{x\to a}{Lt} \frac{f'(x)}{g'(x)} = \underset{x\to a}{Lt} \frac{f''(x)}{g''(x)} = \underset{x\to a}{Lt} \frac{f'''(x)}{g'''(x)}$

## Evaluation of Limits of form: $\infty-\infty$, $0.\infty$, $0^0$, $\infty^0$

(i) If $\underset{x\to a}{Lt} [f(x)-g(x)]$ is of the form $\infty-\infty$, it can be transformed to $0/0$ form by writing it as $\underset{x\to a}{Lt} \left(\left[\frac{1}{g(x)} - \frac{1}{f(x)}\right] \Big/ \frac{1}{f(x).g(x)}\right)$ and hence can be evaluated.

(ii) If $\underset{x\to a}{Lt} (f(x).g(x))$ is of the form $0.\infty$, it can be transformed to $\frac{0}{0}$ or $\frac{\infty}{\infty}$ form by writing it as $\underset{x\to a}{Lt} \frac{f(x)}{1/g(x)}$ or $\underset{x\to a}{Lt} \frac{g(x)}{1/f(x)}$

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(iii) If $\underset{x\to a}{Lt} f(x)^{g(x)}$ is of the form $0^0$ or $\infty^0$, it can be expressed as

$e^{\underset{x\to a}{Lt} g(x).\ln f(x)}$, So limit in exponent is of $0.\infty$ form as in (ii)

## Monotonic functions:

A function $f:[a,b]\to\mathbb{R}$ is said to be

(i) Monotonically increasing (non-decreasing) on $[a,b]$ if

$$x_1 < x_2 \Rightarrow f(x_1) \le f(x_2),\ \forall x_1, x_2 \in [a,b]$$

(ii) Monotonically decreasing (non-increasing) on $[a,b]$ if

$$x_1 < x_2 \Rightarrow f(x_1) \geq f(x_2),\ \forall x_1, x_2 \in [a,b]$$

(iii) Strictly increasing on $[a,b]$ if

$$x_1 < x_2 \Rightarrow f(x_1) < f(x_2),\ \forall x_1, x_2 \in [a,b]$$

(iv) Strictly decreasing on $[a,b]$ if

$$x_1 < x_2 \Rightarrow f(x_1) > f(x_2),\ \forall x_1, x_2 \in [a,b]$$

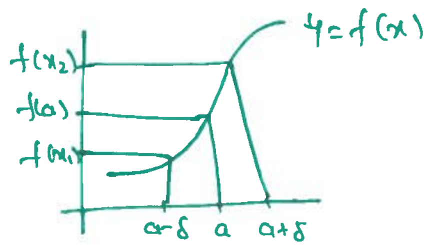
## Locally Increasing functions:

Let $f$ be a function defined in a neighbourhood of a point 'a', then $f$ is said to be increasing at 'a' or locally increasing at 'a' if.

if $\exists$ a, $\delta > 0$ such that

$$x \in (a-\delta, a) \Rightarrow f(x) < f(a)$$

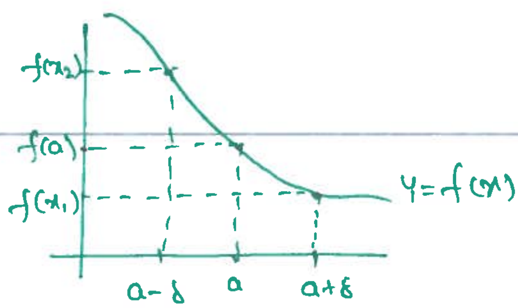
$$x \in (a, a+\delta) \Rightarrow f(x) > f(a)$$




## Locally Decreasing functions:

$$x \in (a-\delta, a) \Rightarrow f(x) > f(a)$$

$$x \in (a, a+\delta) \Rightarrow f(x) < f(a)$$




* Let $f$ be a function defined & $f$ be differential at 'a' then

if $f'(a)=0$ & $f'(x)>0$ then $f$ is increasing at a

if $f'(a)=0$ & $f'(x)<0$ then $f$ is decreasing at a

if $f'(a)>0 \Rightarrow f$ is ~~too~~ locally increasing at a

if $f'(a)<0 \Rightarrow f$ is locally decreasing at a eg: For the function $f(x) = x^x$, find the points at which it is increasing and decreasing. ⑥

$$f(x) = x^x$$

$$f'(x) = x(x^{x-1}) + x^x \log x$$

$$= x^x(1 + \log x)$$

If $f'(x) > 0$ then increasing

$$x^x(1 + \log x) > 0$$

$$\log x > -1$$

$$x > e^{-1}$$

$$x > 1/e$$

$\therefore$ $f(x)$ is increasing in $(1/e, \infty)$

If $f'(x) < 0$ then decreasing

$$x^x(1 + \log x) < 0$$

$$\log x < -1$$

$$x < e^{-1}$$

$$x < 1/e$$

$\therefore$ $f(x)$ is decreasing in $(0, 1/e)$

## Greatest and least Values:

Let $f$ be a function defined on a set $A$ and $l \in f(A)$, then $l$ is said to be

(i) the maximum value or greatest value or absolute max or global max of $f$ in $A$ if $f(x) \le l, \forall x \in A$.

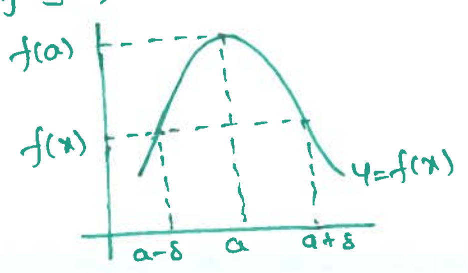
(ii) the minimum value or least value or absolute min or global min of $f$ in $A$ if $f(x) \ge l, \forall x \in A$.

## Local Maximum & Local Minimum:

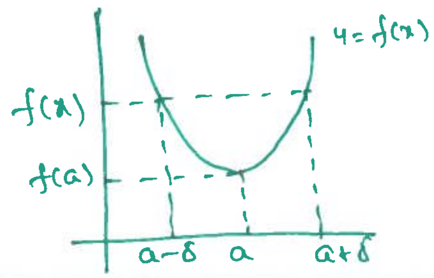
Let $f$ be a func()

L max: if $\exists a, \delta > 0$ such that $f(x) < f(a), \forall x \in (a-\delta, a) \cup (a, a+\delta)$

L min: if $\exists a, \delta > 0$ such that $f(x) > f(a), \forall x \in (a-\delta, a) \cup (a, a+\delta)$




L Max




L-Min

Absolute Max = Max $\{f(a), f(b)$ and all relative max values in $(a,b)\}$ on $[a,b]$

Absolute Min = Min $\{f(a), f(b)$ an all relative ~~max~~ min values in $(a,b)\}$ on $[a,b]$

<u>Stationary</u> <u>Point (SP)</u>: if $f'(a) = 0$, then $f(a)$ is stationary value and $(a, f(a))$ is SP.

<u>Critical</u> <u>point (CP)</u>: If $f'(a)=0$ or $f'(a)$ doesnot exist then $(a, f(a))$ is CP.


→ Find the local maxima and local minima & find the values

(i) $f(x) = x^3 - 6x^2 + 9x + 15$

$f'(x) = 3x^2 - 12x + 9$

$f'(x) = 0$

$3x^2 - 12x + 9 = 0$

$x^2 - 4x + 3 = 0$

$(x-1)(x-3) = 0$

$x = 1, 3$

$f'(x) > 0 \Rightarrow (x-1)(x-3) > 0$

$x < 1$ or $x > 3$

$x \in (-\infty, 1) \cup (3, \infty)$

$f'(x) < 0 \Rightarrow (x-1)(x-3) < 0$

$x \in (1, 3)$

$\therefore$ at $x = 1$, $f'(x)$ changes its sign from +ve to −ve.

$\therefore$ $f(x)$ has local ~~min~~ maxima at $x = 1$, max value $= f(1)$
$= 19$.

at $x = 3$, $f'(x)$ changes sign from −ve to +ve

$\therefore$ $f(x)$ has local minma at $x = 3$, min value $= f(3) = 15$

~~(iii) $f(x) = (x-1)^3(x+1)^2$~~

* Let $f(x)$ be differentiable function and let $f''(a)$ exists
  - → if $f'(a)=0$ and $f''(a)<0$, then $f(x)$ has relative maximum at $x=a$ and has the maximum value at $a$ is $f(a)$
  - → if $f'(a)=0$ and $f''(a)>0$ then $f(x)$ has relative minimum at $x=a$ and the minimum value at $a$ is $f(a)$.
  - → if $f''(a)=0$ then it is not possible to decide whether $f(x)$ has a local max or local min at $x=a$ using 2nd derivative.

## MEAN VALUE THEOREMS

### 1. Rolle's Theorem:

If $f:[a,b]\to\mathbb{R}$ is a function such that

(a) $f$ is continuous on $[a,b]$

(b) $f$ is differentiable on $(a,b)$

(c) $f(a)=f(b)$

then there exists atleast one point $c\in(a,b)$ such that $f'(c)=0$

* $f$ is differentiable on $(a,b)$ if it is differentiable at each point of $(a,b)$ i.e., if $f'(c)$ exists, $\forall c\in(a,b)$

* $f$ is differentiable on a point $a$ if

Right Hand derivative of $f(a)$ = Left Hand derivative of $f(a)$

$$Rf'(a) = \lim_{x\to a^+}\frac{f(x)-f(a)}{x-a}$$

$$Lf'(a) = \lim_{x\to a^-}\frac{f(x)-f(a)}{x-a}$$

if $Rf'(a) = Lf'(a)$ then differentiable at $a$

* To check continuity, Lefthand limit & right hand Limit of $f(a)$ should be equal. i.e., $RHL(f(a)) = LHL(f(a)) = f(a)$ then $f(x)$ is continuous at point 'a'.

→ Show that the function

$$f(x) = \begin{cases} x, & \text{if } x < 1 \\ 3-x, & \text{if } 1 \le x \le 3 \\ x^2-4x+3, & \text{if } x > 3 \end{cases}$$

is not differentiable at 1 and 3.

Solution: $$Lf'(1) = \underset{x \to 1^-}{Lt} \frac{f(x)-f(1)}{x-1}$$

$$= \underset{x \to 1^-}{Lt} \frac{x-(3-1)}{x-1}$$

$$= \underset{x \to 1^-}{Lt} \frac{x-2}{x-1}$$ (this is divide by zero form)

$$= \underset{x \to 1^-}{Lt} \left(1 - \frac{1}{x-1}\right) = 1 - \infty = -\infty$$

$$Rf'(1) = \underset{x \to 1^+}{Lt} \frac{f(x)-f(1)}{x-1}$$

$$= \underset{x \to 1^+}{Lt} \frac{(3-x)-(3-1)}{x-1} = \underset{x \to 1^+}{Lt} \frac{1-x}{x-1} = -1$$

$\therefore Lf'(1) \neq Rf'(1)$

$\therefore f(x)$ is not differentiable at 1.

$$Lf'(3) = \underset{x \to 3^-}{Lt} \frac{f(x)-f(3)}{x-3} = \underset{x \to 3^-}{Lt} \frac{3-x-0}{x-3} = -1$$

$$Rf'(3) = \underset{x \to 3^+}{Lt} \frac{f(x)-f(3)}{x-3} = \underset{x \to 3^+}{Lt} \frac{x^2-4x+3-0}{x-3} = \underset{x \to 3^+}{Lt} \frac{(x-1)(x-3)}{x-3}$$

$$= 3-1 = 2$$

$\therefore Lf'(3) \neq Rf'(3)$

$\therefore f(x)$ is not differentiable at 3.

→ Show that $f(x) = \frac{x}{1+|x|}$, $x \in \mathbb{R}$, is differentiable at $x=0$, find $f'(0)$.

$f(0) = 0$.

$$Lf'(0) = \underset{x\to 0^-}{Lt} \frac{f(x)-f(0)}{x-0} = \underset{x\to 0^-}{Lt} \frac{\frac{x}{1-x}-0}{x} = \underset{x\to 0^-}{Lt} \frac{\cancel{x}}{1-x}\cdot\frac{1}{\cancel{x}}$$

$$= \underset{x\to 0^-}{Lt} \frac{1}{1-x} = 1.$$

$$Rf'(0) = \underset{x\to 0^+}{Lt} \frac{f(x)-f(0)}{x-0} = \underset{x\to 0^+}{Lt} \frac{\frac{x}{1+x}-0}{x} = \underset{x\to 0^+}{Lt} \frac{1}{1+x} = 1.$$

$\therefore Lf'(0) = Rf'(0) = 1$

So, $f(x)$ is differentiable at $x=0$.

$\therefore f'(0) = 1$

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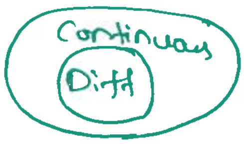
* If $f(x)$ is differentiable at a point 'a', then it is continuous at 'a', but the converse may not be true

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→ Examine the applicability of Rolle's theorem to $f(x) = 1-(x-1)^{2/3}$ on $[0,2]$

Sol$^n$: $f(0) = 1-(0-1)^{2/3} = 0$

$f(2) = 1-(2-1)^{2/3} = 0$

$\therefore f(0) = f(2)$

$$f'(x) = 0 - \frac{2}{3}(x-1)^{2/3-1} = -\frac{2}{3}(x-1)^{-1/3} = -\frac{2}{3}\cdot\frac{1}{(x-1)^{1/3}}$$

clearly $f'(x)$ doesnot exist at $x=1 \in (0,2)$

So $f(x)$ is not differentiable at $x=1 \in (0,2)$

$\therefore$ Rolle's theorem cannot be applied.

→ Verify Rolle's theorem for $f(x) = x(x+3)e^{-x/2}$, on $[-3,0]$

$$f(-3) = -3(-3+3)e^{-3/2} = 0$$

$$f(0) = 0(0+3)e^{0/2} = 0$$

$$\therefore f(-3) = f(0) = 0 \checkmark$$

$$f'(x) = \left(e^{-x/2}(x^2+3x)\right)'$$

$$= -\frac{1}{2}e^{-x/2}(x^2+3x) + e^{-x/2}(2x+3)$$

$$= e^{-x/2}\left(-\frac{x^2}{2} - \frac{3x}{2} + 2x + 3\right)$$

$$= \frac{1}{2}e^{-x/2}(x - x^2 + 6)$$

Clearly $f'(x)$ exists $\forall\, x \in [-3,0]$

$\therefore$ $f$ is continuous on $[-3,0]$ and differentiable in $(-3,0)$

Hence by ~~Roo~~ Rolle's theorem, $\exists\, c \in (-3,0)$ such that $f'(c) = 0$

$$\frac{1}{2}e^{-x/2}(x - x^2 + 6) = 0$$

$$\frac{1}{2}e^{-c/2}(c - c^2 + 6) = 0$$

$$e^{-c/2} = 0, \qquad c - c^2 + 6 = 0$$

$\times$

$$c^2 - c - 6 = 0$$

$$(c-3)(c+2) = 0$$

$$c = +3, -2$$

$$c = -2 \in (-3,0)$$

2. LAGRANGE's MEAN VALUE THEOREM (LMVT):

If $f:[a,b]\to\mathbb{R}$ is a function such that

(i) $f$ is continuous on $[a,b]$

(ii) $f$ is differentiable on $(a,b)$

then $\exists$ a point $c\in(a,b)$ such that $f'(c)=\dfrac{f(b)-f(a)}{b-a}$

Some times, the LMVT is referred to as the "Mean Value Theorem".

3. CAUCHY's MEAN VALUE THEOREM:

If $f:[a,b]\to\mathbb{R}$, $g:[a,b]\to\mathbb{R}$ are two distinct functions such th

(i) $f$ and $g$ are continuous on $[a,b]$

(ii) $f$ and $g$ are differentiable on $(a,b)$ and

(iii) $g'(x)\neq 0$ for any $x\in(a,b)$

then there exists a point $c\in(a,b)$ such that

$$\frac{f'(c)}{g'(c)}=\frac{f(b)-f(a)}{g(b)-g(a)}$$

This theorem is also known as "Generalized Mean Value Theorem".

→ verify LMVT for $f(x)=lx^2+mx+n$, $(l\neq 0)$ in $[a,b]$

Sol$^n$: $f(x)=lx^2+mx+n$, $x\in[a,b]$

Since $f(x)$ is quadratic eq. it is continuous on $[a,b]$ and differentiable on $(a,b)$

$\therefore\ \exists\ c\in(a,b)$ such that $f'(c)=\dfrac{f(b)-f(a)}{b-a}$

$\therefore f'(x) = 2lx+m$

$f'(c) = 2lc+m$

$$2lc+m = \frac{lb^2+mb+\cancel{n}-la^2-mb-\cancel{n}}{b-a}$$

$$= \frac{l(b^2-a^2)+m(b-a)}{b-a}$$

$$2lc+m = \frac{\cancel{(b-a)}[l(b+a)+m]}{\cancel{b-a}}$$

$$2\cancel{l}c = \cancel{l}(b+a)$$

$$c = \frac{b+a}{2}, \in (a,b)$$

→ Verify the LMVT for the function $f(x) = \sin x - \sin 2x$ on $[0,\pi]$

$\underline{\underline{Sol^n}}$: Since ~~sine~~ Sine function is continuous & differentiable on $\mathbb{R}$, $f$ is continuous on $[0,\pi]$ and differentiable on $(0,\pi)$

$$f'(x) = \cos x - 2\cos 2x$$

$\therefore \exists\, c \in (0,\pi)$ such that $f'(c) = \dfrac{f(b)-f(a)}{b-a}$

$$\cos c - 2\cos 2c = \frac{(\sin\pi - \sin 2\pi)-(\sin 0 - \sin 0)}{\pi - 0}$$

$$= 0$$

$$\cos c - 2\cos 2c = 0$$

$$\cos c = 2\cos 2c$$

$$\cos c = 4\cos^2 c - 2$$

$$4\cos^2 c - \cos c - 2 = 0$$

$$\cos c = \frac{+1 \pm \sqrt{1-(-32)}}{8} = \frac{1\pm\sqrt{33}}{8}$$

$$c = \cos^{-1}\left(\frac{1\pm\sqrt{33}}{8}\right) \in (0,\pi)$$

# CONTINUITY



## Continuity in an interval:

* A function $f$ is defined on $(a,b)$ is said to be continuous on $(a,b)$ iff it is continuous at every point of $(a,b)$ i.e.,

$$\text{if} \quad \underset{x \to c}{Lt}\, f(x) = f(c), \ \forall\, c \in (a,b)$$

* A function $f$ is defined on $[a,b]$ is said to be continuous on $[a,b]$ if

(i) $f$ is continuous on $(a,b)$

(ii) $f$ is right continuous at $a$

$$\text{i.e.,} \quad \underset{x \to a^+}{Lt}\, f(x) = f(a)$$

(iii) $f$ is left continuous at $b$

$$\text{i.e.,} \quad \underset{x \to b^-}{Lt}\, f(x) = f(b)$$

## Continuity ~~of~~ at a point:

Let $f$ be a function defined at a point '$a$' then $f$ is continuous at '$a$' if

(i) Left continuous at $a$ iff $\underset{x \to a^-}{Lt}\, f(x) = f(a)$

(ii) Right continuous at $a$ iff $\underset{x \to a^+}{Lt}\, f(x) = f(a)$

→ Show that $f(x) = \begin{cases} x^2, & \text{if } 0 \le x \le 1 \\ x, & \text{if } 1 \le x \le 2 \end{cases}$ is continuous on $[0,2]$

Sol$^n$: Case 1: check continuity at $x=0$

$f(0)=0$, $\lim_{x\to 0} f(x) = \lim_{x\to 0^+} f(x) = \lim_{x\to 0^-} f(x) = 0 = f(0)$

$\therefore$ Continuous at $x=0$

Case 2: check continuity at $x=1$

$f(1)=1$, $\lim_{x\to 1^-} f(x) = \lim_{x\to 1^-} x^2 = 1$

$\lim_{x\to 1^+} f(x) = \lim_{x\to 1^+} x = 1$

$\Rightarrow$ continuous at $x=1$

Case 3: Check continuity in $x\in(0,1)$

Let $a\in(0,1)$, then $f(a)=a^2$

$\lim_{x\to a} f(x) = \lim_{x\to a} x^2 = a^2 = f(a)$

$\therefore$ continuous on $(0,1)$

Case 4: check continuity at $x=2$

$f(2)=2$, $\lim_{x\to 2^-} f(x) = \lim_{x\to 2^-} x = 2 = f(2)$

$\therefore$ continuous at 2.

Case 5: Check continuity in the interval $(1,2)$

let $b\in(1,2)$, $f(b)=b$

$\lim_{x\to b} f(x) = \lim_{x\to b} x = b = f(b)$

continuous on $(1,2)$

$\therefore$ $f(x)$ is continuous in $[0,2]$