INTEGRATION

Let $f(x)$ be a function, the collection of all its premitives is called the indefinite integral of $f(x)$ and is denoted by $\int f(x)\,dx$.

Integration is inverse operation of differentiation.

If $\frac{d}{dx}(\varphi(x)) = f(x)$, $\int f(x)\,dx = \varphi(x) + C$,

where C is the constant of integration or arbitrary constant.

* The process of finding functions whose derivative is given, is called anti-differentiation or integration.

Derivatives	Integrals (Anti derivatives)
(i) $\frac{d}{dx}\left(\frac{x^{n+1}}{n+1}\right) = x^n$;	$\int x^n\,dx = \frac{x^{n+1}}{n+1} + c,\ n \neq -1$
Particularly, note that $\frac{d}{dx}(x) = 1$	$\int dx = x + c$
(ii) $\frac{d}{dx}(\sin x) = \cos x$;	$\int \cos x\,dx = \sin x + C$
(iii) $\frac{d}{dx}(-\cos x) = \sin x$;	$\int \sin x\,dx = -\cos x + C$
(iv) $\frac{d}{dx}(\tan x) = \sec^2 x$;	$\int \sec^2 x\,dx = \tan x + c$
(v) $\frac{d}{dx}(-\cot x) = \operatorname{cosec}^2 x$;	$\int \operatorname{cosec}^2 x\,dx = -\cot x + C$
(vi) $\frac{d}{dx}(\sec x) = \sec x \tan x$;	$\int \sec x \tan x\,dx = \sec x + C$
(vii) $\frac{d}{dx}(-\operatorname{cosec} x) = \operatorname{cosec} x \cot x$;	$\int \operatorname{cosec} x \cot x\,dx = -\operatorname{cosec} x + C$
(viii) $\frac{d}{dx}(\sin^{-1} x) = \frac{1}{\sqrt{1-x^2}}$;	$\int \frac{dx}{\sqrt{1-x^2}} = \sin^{-1} x + C$
(ix) $\frac{d}{dx}(-\cos^{-1} x) = \frac{1}{\sqrt{1-x^2}}$;	$\int \frac{dx}{\sqrt{1-x^2}} = -\cos^{-1} x + C$

(x) $\frac{d}{dx}(\tan^{-1}x) = \frac{1}{1+x^2}$; $\int \frac{dx}{1+x^2} = \tan^{-1}x + C$

(xi) $\frac{d}{dx}(-\cot^{-1}x) = \frac{1}{1+x^2}$; $\int \frac{dx}{1+x^2} = -\cot^{-1}x + C$

(xii) $\frac{d}{dx}(\log_e x) = \frac{1}{x}$; $\int \frac{1}{x}dx = \log_e|x| + C$

* $\frac{d}{dx}(e^x) = e^x$; $\int e^x dx = e^x + C$

* $\frac{d}{dx}\left\{\frac{a^x}{\log_e a}\right\} = a^x, a>0, a\neq 1$; $\int a^x dx = \frac{a^x}{\log_e a} + C$

* $\frac{d}{dx}(\log \sin x) = \cot x$; $\int \cot x\, dx = \log|\sin x| + C$

* $\frac{d}{dx}(-\log \cos x) = \tan x$; $\int \tan x\, dx = -\log|\cos x| + C$

* $\frac{d}{dx}[\log(\sec x + \tan x)] = \sec x$; $\int \sec x\, dx = \log|\sec x + \tan x| + C$

* $\frac{d}{dx}[\log(\operatorname{cosec} x - \cot x)] = \operatorname{cosec} x$; $\int \operatorname{cosec} x\, dx = \log|\operatorname{cosec} x - \cot x| + C$

* $\frac{d}{dx}\sin^{-1}\left(\frac{x}{a}\right) = \frac{1}{x\sqrt{a^2-x^2}}$; $\int \frac{1}{\sqrt{a^2-x^2}}dx = \sin^{-1}\left(\frac{x}{a}\right) + C$

* $\frac{d}{dx}\cos^{-1}\left(\frac{x}{a}\right) = \frac{-1}{\sqrt{a^2-x^2}}$; $\int \frac{-1}{\sqrt{a^2-x^2}}dx = \cos^{-1}\left(\frac{x}{a}\right) + C$

* $\frac{d}{dx}\left(\frac{1}{a}\operatorname{Tan}^{-1}\frac{x}{a}\right) = \frac{1}{a^2+x^2}$; $\int \frac{+1}{a^2+x^2}dx = \frac{1}{a}\operatorname{Tan}^{-1}\left(\frac{x}{a}\right) + C$

* $\frac{d}{dx}\left(\frac{1}{a}\cot^{-1}\frac{x}{a}\right) = \frac{-1}{a^2+x^2}$; $\int \frac{-1}{a^2+x^2}dx = \frac{1}{a}\cot^{-1}\left(\frac{x}{a}\right) + C$

* $\frac{d}{dx}\left(\frac{1}{a}\sec^{-1}\frac{x}{a}\right) = \frac{1}{x\sqrt{x^2-a^2}}$; $\int \frac{1}{x\sqrt{x^2-a^2}}dx = \frac{1}{a}\sec^{-1}\left(\frac{x}{a}\right) + C$

* $\frac{d}{dx}\left(\frac{1}{a}\operatorname{cosec}^{-1}\frac{x}{a}\right) = \frac{-1}{x\sqrt{x^2-a^2}}$; $\int \frac{-1}{x\sqrt{x^2-a^2}}dx = \frac{1}{a}\operatorname{cosec}^{-1}\left(\frac{x}{a}\right) + C$

* Comparision between Differentiation and Integration.

(i) Both differentiation and integration are linear operators on functions as:

$$\frac{d}{dx}\left(af(x) \pm bg(x)\right) = a\frac{d}{dx}f(x) \pm b\frac{d}{dx}g(x)$$

$$\int (af(x) \pm bg(x))\,dx = a\int f(x)\,dx \pm b\int g(x)\,dx$$

(ii) All functions are not differentiable, Similarly there are some functions which are not integrable.

(iii) Integral of a function is always discussed in an interval but derivative of a function can be discussed in an interval as well as ~~an~~ at a point.

$$\therefore \int_a^a f(x)\,dx = 0.$$

→ $\int \cos 2x\,dx$

Sol^n: Look for a function whose derivative is $\cos 2x$.

$$\frac{d}{dx}\sin 2x = 2\cos 2x$$

$$\therefore \cos 2x = \frac{1}{2}\frac{d}{dx}\sin 2x = \frac{d}{dx}\left(\frac{1}{2}\sin 2x\right)$$

$$\therefore \int \cos 2x\,dx = \frac{1}{2}\sin 2x$$

→ $\int \frac{x^3-1}{x^2}\,dx$

Sol^n: $\int \frac{x^3-1}{x^2}\,dx = \int x\,dx - \int x^{-2}\,dx = \frac{x^2}{2} + C_1 - \frac{x^{-1}}{-1} - C_2 = \frac{x^2}{2} + \frac{1}{x} + C_1 - C_2$

$$= \frac{x^2}{2} + \frac{1}{x} + C \text{, where } C = C_1 - C_2 \text{ is a Constant.}$$

→ $\int \operatorname{cosec} x\,(\operatorname{cosec} x + \cot x)\,dx$

Sol^n:
$$= \int (\operatorname{cosec}^2 x + \operatorname{cosec} x \cot x)\,dx$$
$$= -\cot x - \operatorname{cosec} x + c$$

Methods of Integration:

(1) Integration by Substitution.

$\int f(x)\,dx$ can be transformed into another form by changing the independent variable x to 't' by substituting $x = g(t)$

Consider $I = \int f(x)\,dx$

Put $x = g(t)$ So that $\frac{dx}{dt} = g'(t)$

$\Rightarrow dx = g'(t)\,dt$

$$\therefore I = \int f(x)\,dx = \int f(g(t))\,g'(t)\,dt$$

Eg: $\int \tan x\,dx$

Sol^n: $\int \tan x\,dx = \int \frac{\sin x}{\cos x}\,dx$

Put $\cos x = t \Rightarrow \sin x\,dx = -dt$

$$\therefore \int \tan x\,dx = \int -\frac{dt}{t}$$
$$= -\log|t| + c$$
$$= -\log|\cos x| + c$$

$$\therefore \int \tan x\,dx = \log|\sec x| + c$$

Eg: $\int \sin^3 x \cos^2 x\,dx$

Sol^n: $\int \sin^3 x \cos^2 x\,dx = \int \sin^2 x \cos^2 x\,(\sin x)\,dx$
$$= \int (1-\cos^2 x)\cos^2 x\,(\sin x)\,dx$$

Put $\cos x = t \Rightarrow dt = -\sin x\,dx$

$$\therefore \int \sin^2 x \cos^2 x\,(\sin x)\,dx = -\int (1-t^2)\,t^2\,dt$$

$$= -\int (t^2 - t^4)\,dt = -\left(\frac{t^3}{3} - \frac{t^5}{5}\right) + c$$

$$= -\frac{1}{3}\cos^3 x + \frac{1}{5}\cos^5 x + c$$

Eg: $\int \frac{1}{1+\tan x}\,dx$

Sol^n: $= \int \frac{\cos x}{\cos x + \sin x}\,dx$

$$= \frac{1}{2}\int \frac{(\cos x + \sin x + \cos x - \sin x)}{\cos x + \sin x}\,dx$$

$$= \frac{1}{2}\int dx + \frac{1}{2}\int \frac{\cos x - \sin x}{\cos x + \sin x}\,dx$$

$$= \frac{1}{2}x + \frac{c_1}{2} + \frac{1}{2}\int \frac{\cos x - \sin x}{\cos x + \sin x}\,dx$$

Let $I = \int \frac{\cos x - \sin x}{\cos x + \sin x}\,dx$

Put $\cos x + \sin x = t \Rightarrow (\cos x - \sin x)dx = dt$

$$\therefore I = \int \frac{dt}{t} = \log|t| + c_2$$

$$= \log|\cos x + \sin x| + c_2$$

$$\therefore \int \frac{1}{1+\tan x}\,dx = \frac{x}{2} + \frac{c_1}{2} + \frac{1}{2}\log|\cos x + \sin x| + \frac{c_2}{2}$$

$$= \frac{x}{2} + \frac{1}{2}\log|\cos x + \sin x| + c$$

* Integrals of some particular functions:

(i) $\int \frac{dx}{x^2 - a^2} = \frac{1}{2a}\log\left|\frac{x-a}{x+a}\right| + c$

(ii) $\int \frac{dx}{a^2 - x^2} = \frac{1}{2a}\log\left|\frac{a+x}{a-x}\right| + c$

(iii) $\int \frac{dx}{x^2 + a^2} = \frac{1}{a}\tan^{-1}\frac{x}{a} + c$

(iv) $\int \frac{dx}{\sqrt{x^2 - a^2}} = \log\left|x + \sqrt{x^2 - a^2}\right| + c$

(v) $\int \frac{dx}{\sqrt{x^2 + a^2}} = \log\left|x + \sqrt{x^2 + a^2}\right| + c$

(2) Integration by Parts:

This method is used to integrate the product of two functions.

If $f(x)$ and $g(x)$ be two integrable functions then,

$$\int f(x).g(x)\,dx = f(x)\int g(x)\,dx - \int\left[\int g(x)\,dx\right] f'(x)\,dx$$

$$\boxed{\text{i.e., } \int f(x).g(x)\,dx = f(x)\int g(x)dx - \int\left(f'(x).\int g(x)dx\right)dx}$$

Eg: $\int x\cos x\,dx$

Sol^n: Put $f(x) = x$, $g(x) = \cos x$ and integrate by parts

$$\therefore \int x\cos x\,dx = x\int \cos x\,dx - \int\left(\frac{d}{dx}(x)\int \cos x\,dx\right)dx$$

$$= x\sin x - \int \sin x\,dx$$

$$= x\sin x + \cos x + C$$

If we put $f(x) = \cos x$ and $g(x) = x$ then

$$\int x\cos x\,dx = \cos x\int x\,dx - \int\left(\frac{d}{dx}(\cos x)\int x\,dx\right)dx$$

$$= (\cos x)\frac{x^2}{2} + \int \sin x.\frac{x^2}{2}\,dx$$

Thus, it shows that integral $\int x\cos x\,dx$ is reduced to the comparitively more complicated integral having more power of x.

$\therefore$ The proper choice of the first and 2nd function is significant.

* Integration by Parts is not applicable to product of functions in all cases. For example, the method does not work for $\int\sqrt{x}\sin x\,dx$ because there does not exist any function whose derivative is $\sqrt{x}\sin x$.

Eg: $\int x e^x dx$

Take first fun(x) as x and second func() as e^x.

$$\therefore \int x e^x dx = x e^x - \int 1.e^x dx = xe^x - e^x + C$$

3, Integration by Partial ~~functions~~ fractions :-

Recall that a rational function is defined as the ratio of two polynomials in the form $\frac{P(x)}{Q(x)}$, where $P(x)$ and $Q(x)$ are polynomials in x and $Q(x) \neq 0$.

It is always possible to write the integrand as a sum of simpler rational functions by a method called partial fraction decomp-osition.

Eg: $\int \frac{dx}{(x+1)(x+2)}$

$$\frac{1}{(x+1)(x+2)} = \frac{A}{x+1} + \frac{B}{x+2}$$

where A and B are real numbers that are tobe determined suitably

$\therefore$ $A(x+2) + B(x+1) = 1 \Rightarrow (A+B)x + (2A+B) = 1$

Equating the coefficients of x and the constant term,

$$A + B = 0$$
$$2A + B = 1$$

Solving these equations, we get $A = 1, B = -1$

$$\therefore \frac{1}{(x+1)(x+2)} = \frac{1}{x+1} - \frac{1}{x+2}$$

$$\int \frac{1}{(x+1)(x+2)} dx = \int \frac{1}{x+1} dx - \int \frac{1}{x+2} dx$$

$$= \log|x+1| - \log|x+2| + c$$

$$= \log\left|\frac{x+1}{x+2}\right| + c$$

Eg: Find $\int \frac{x^2}{(x^2+1)(x^2+4)} dx$

Sol^n: Put $x^2 = y$

$$\therefore \frac{x^2}{(x^2+1)(x^2+4)} = \frac{y}{(y+1)(y+4)} = \frac{A}{(y+1)} + \frac{B}{(y+4)}$$

$$\therefore y = (y+4)A + B(y+1)$$

$$y = y(A+B) + (4A+B)$$

Comparing the coefficients of y and constant terms

$$A+B=1$$
$$4A+B=0$$
$$A = \frac{-1}{3}, \quad B = \frac{4}{3}$$

$$\therefore \frac{x^2}{(x^2+1)(x^2+4)} = -\frac{1}{3(x^2+1)} + \frac{4}{3(x^2+4)}$$

$$\therefore \int \frac{x^2}{(x^2+1)(x^2+4)}\,dx = -\frac{1}{3}\int \frac{1}{x^2+1}\,dx + \frac{4}{3}\int \frac{1}{x^2+4}\,dx$$

$$= -\frac{1}{3}\,\mathrm{Tan}^{-1}x + \frac{4}{3}\cdot\frac{1}{2}\,\mathrm{Tan}^{-1}\left(\frac{x}{2}\right) + c$$

$$= -\frac{1}{3}\,\mathrm{Tan}^{-1}x + \frac{2}{3}\,\mathrm{Tan}^{-1}\left(\frac{x}{2}\right) + c$$

Definite Integrals:

Eg: Find $\displaystyle\int_4^9 \frac{\sqrt{x}}{(30-x^{3/2})^2}\,dx$

Sol^n: Put $30 - x^{3/2} = t \Rightarrow -\frac{3}{2}\sqrt{x}\,dx = dt$ or $\sqrt{x}\,dx = -\frac{2}{3}dt$

$$\therefore \int_4^9 \frac{\sqrt{x}}{(30-x^{3/2})^2}\,dx = -\frac{2}{3}\int_4^9 \frac{dt}{t^2} = \frac{2}{3}\left[\frac{1}{t}\right]_4^9$$

$$= \frac{2}{3}\left[\frac{1}{(30-x^{3/2})}\right]_4^9$$

$$= \frac{2}{3}\left[\frac{1}{30-27} - \frac{1}{30-8}\right] = \frac{19}{99}$$

Properties of definite integrals:

P_0 : $\displaystyle\int_a^b f(x)\,dx = \int_a^b f(t)\,dt$

P_1 : $\displaystyle\int_a^b f(x)\,dx = -\int_b^a f(x)\,dx$. In particular, $\displaystyle\int_a^a f(x)\,dx = 0$

P_2 : $\displaystyle\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^a f(x)\,dx$

P_3: $\int_a^b f(x)\,dx = \int_a^b f(a+b-x)\,dx$

P_4: $\int_0^a f(x)\,dx = \int_0^a f(a-x)\,dx$

P_5: $\int_0^{2a} f(x)\,dx = \int_0^a f(x)\,dx + \int_0^a f(2a-x)\,dx$

P_6: $\int_0^{2a} f(x)\,dx = 2\int_0^a f(x)\,dx$, if $f(2a-x) = f(x)$ and
0, if $f(2a-x) = -f(x)$

P_7: (i) $\int_{-a}^{a} f(x)\,dx = 2\int_0^a f(x)\,dx$, if f is an even function
i.e., if $f(-x) = f(x)$

(ii) $\int_{-a}^{a} f(x)\,dx = 0$, if f is an odd function, i.e., if $f(-x) = -f(x)$

Eg: Evaluate $\int_{-\pi/4}^{\pi/4} \sin^2 x\,dx$

Soln: Observe that $\sin^2 x$ is an even func.

$\therefore$ by P_7 (i), we get

$$\int_{-\pi/4}^{\pi/4} \sin^2 x\,dx = 2\int_0^{\pi/4} \sin^2 x\,dx$$

$$= 2\int_0^{\pi/4} \left(\frac{1-\cos 2x}{2}\right) dx = \int_0^{\pi/4} (1-\cos 2x)\,dx$$

$$= \left[x - \frac{1}{2}\sin 2x\right]_0^{\pi/4}$$

$$= \frac{\pi}{4} - \frac{1}{2}\sin\frac{\pi}{2} - 0$$

$$= \frac{\pi}{4} - \frac{1}{2}$$