21st Feb

Linear Diophantine Eqn

$ax + by = c$

a, b, c are given

x & y are unknown.

→ The Degenerate case:

if ($a = 0$) and $b = 0$)

then $c = 0 \Rightarrow$ infinite soln

$c \neq 0 \Rightarrow$ No soln.

→ Finding any Soln :

* 1 soln, with diophantine Eqn and 2 unknowns, we can use extended Euclid algo.

* Assume a and b are non-negative.

using ex-gcd

$(a)x_g + (b)y_g = g$

$g \to$ gcd of a and b.

* Now instead of 1 eqn we have 2 eqns.

$ax + by = c$ — ① (given)

$a\ x_g + b\ y_g = g$ — ② (from Ex-gcd)

Fact: a linear combination of 2 #'s, is divisible by their common divisor.

Let divide every term by g in eqn 2.

$$\frac{a \cdot x_g}{g} + \frac{b}{g} \cdot y_g = 1$$

Let's multiply c both sides.

$$\frac{a}{g} \cdot x_g \cdot c + \frac{b}{g} \cdot y_g \cdot c = c.$$

rewrite i.e.

$$a\left(x_g \cdot \frac{c}{g}\right) + b\left(\frac{y_g \cdot c}{g}\right) = c$$

$$a\ x \quad + b\ y \quad = c$$

$\Rightarrow x = x_g \cdot c/g$ ✓

$\Rightarrow y = y_g \cdot c/g$ ✓

Ex: Solve:

$\Rightarrow$ ① $10x + 15y = 55$

$\boxed{11x_1 = x \quad \Rightarrow x = -11 \\ 11y_1 = y \quad y_1 = 11}$

Soln. $c \% (\gcd(a, b))$

$= 5 \% (5) = 0$ ✓

One soln exists.

$10x_1 + 15y_1 = 5$ $\Rightarrow \boxed{x_1 = -1, y_1 = 1}$

$x_1 = -1 \quad y_1 = 1 - \cancel{\frac{x}{15}}\ 0$

$15x_2 + 10y_2 = 5 \quad x_2 = 1 \quad y_2 = 0 - \frac{15}{10} \cdot 1 \times 1 = -1$

$10x_3 + 5y_3 = 5 \quad x_3 = 0 \quad y_3 = 1 - 5 \cdot 0 = 1$

$5x_4 + 0 \cdot y_4 = 5 \quad x_4 = 1, y_4 = 0$

Overview

Given:

$ax + by = c$ — (1)

you said, i know another eqn with a and b as parameters i.e euclid eqn.

$ax_1 + by_1 = g$ — (2)

Now you can compare both eqns and find x_1 and y_1.

For Euclid

if there are 2 numbers a and b and their gcd is g then

$ax + by = g$ — (1)

and

gcd (b, a%b) = g, so

$bx_1 + (a\%b)\, y_1 = g$ — (2)

compare coefficients of a and b to find the soln of x and y.

As we can recursively continue this process until a becomes zero

Special case :

If $a < 0$ or $b < 0$.

earlier: $10x + 15y = 5$ → $x = -1, y = 1$

Now $-10x + -15y = 5$ → $x = 1, y = -1$

∃ gcd( ) in both cases $= 5$

if $a < 0 \Rightarrow$ change sign of derived x.

$b < 0 \Rightarrow$ change sign of derived y.

Code Time

Find the values of x_1, y_1 from Euclid Gcd.

$$x = x_1 * \frac{c}{g}, \quad y = x_2 * \frac{c}{g};$$

if $(a<0)$ $x = -x;$

if $(b<0)$ $y = -y$

Generating All Soln's.

$$ax + by = c$$

We find one solution, (x_0, y_0)

$\rightarrow$ $$ax_0 + by_0 = c$$

$$ax_0 + by_0 + \frac{ab}{g} - \frac{ab}{g} = c$$

$$a(x_0 + b/g) + b(y_0 - b/g) = c$$

$$\uparrow \qquad\qquad \uparrow$$

New Soln's

$$x = x_0 + b/g * k$$

$$y = y_0 - b/g * k$$

We can repeat it any number of times. Thus (k)

Finding number of soln's and soln's in an interval:

$x \in [min_x, max_x]$

$y \in [min_y, max_y]$

* Find a soln from DPE'qn.

Suppose u get (x_0, y_0) as one soln.

$x = x_0 - k * b/g$ — 1

$y = y_0 - k * a/g$

* Consider Xmin is Left limit.

using eqn 1 we can find $x \geq x_{min}$

Similarly $x \leq x_{max} \Rightarrow [l_1, r_1]$

→ Sim - we can find values of y for ymin and ymax

→ From we can compute corresponding values of x $[l_2, r_2]$

have 2 ranges now:

$[l_1, r_1]$ $[l_2, r_2]$

ans is $[l_1, r_1] \cap [l_2, r_2]$

and we can find corresponding values of y.

Find sol^n s.t $x+y$ is min:

$$x' = x + \frac{b}{g} \cdot k$$

$$y' = y - k \cdot a/g$$

$\Rightarrow$

$$x' + y' = x + y + \frac{k(b-a)}{g}$$

depending on the values

Of a and b, decide k.

Overview

just find one (integer) soln to $ax+by=c$, using $ax_1+by_1 = \gcd(a,b)$. Then use your tricks to find all soln within limit.

$ax+by=c$, if a and b are coprime, then the max value of c, which will not have any soln for non-negative values of x and y is $c = a.b - a - b$.

∴ Ex: $4x+7y=c$

then $c_1 = 7*4 - 7 - 4 = 17$ is the last value which does not have any soln for non-negative values of x and y.