Predictive Analytics Lecture 3

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Stat 422/722
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The Coin Example from Last Class I

I want to explain the coin example from last class in the context of likelihood. Imagine you flip a coin three times and get heads, heads, tails; thus, $y_1 = 1, y_2 = 1, y_3 = 0$. There is a true probability of heads called θ. We don't know it.

What is the probability of the data? We employ the mass / density function:

$$\mathbb{P}\left(Y_1 = y_1, Y_2 = y_2, Y_3 = y_3; \theta\right) = \prod_{i=1}^{3} \mathbb{P}\left(Y_i = y_i; \theta\right) =$$

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$$= \theta^2 (1-\theta)$$

And now we can calculate the probability of seeing the data assuming θ. Assume $\theta = 0.5$ then,

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i.e. the most likely model for this data is a weighted coin with probability of heads of 2/3.

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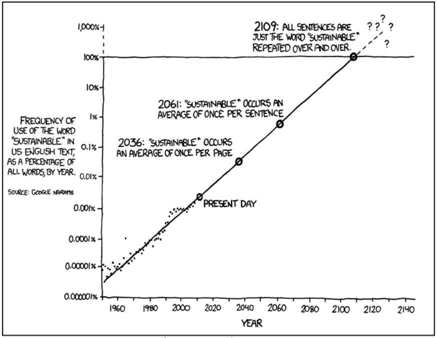

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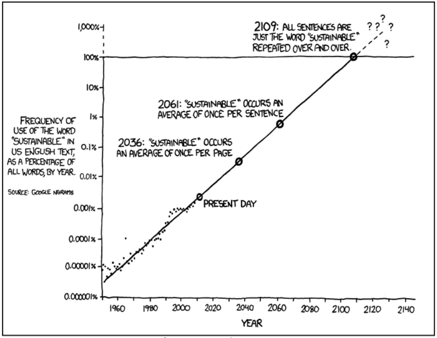

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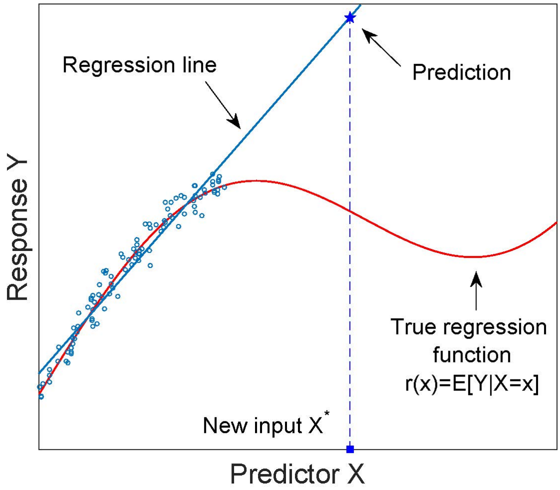
It is important to ask the question for a new observation $\boldsymbol{x}^*$ if it is within the space of $\boldsymbol{x}$'s in the historical data. (Hardly anyone does this when $p > 2$... but you should)!

Reconciliation of These Silly Cartoons



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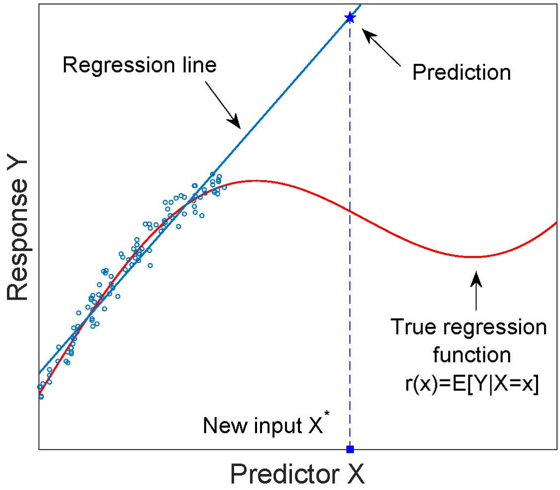

Be aware that extrapolation methods of different algorithms differ considerably! [R Demo]

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What to measure? ✓ Who to measure it on???

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3. Something else?

[R demo]

Optimal Design: Split Between Extremes

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General Optimal Design of Linear Models

We seek the best linear approximation of $f(x)$ which is $\beta_0 + \beta_1 x_1 + \ldots + \beta_p x_p$. We pick the $\boldsymbol{x}$'s to give us the best linear approximation. What criteria? JMP gives two ways:

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[R Demo]

General Optimal Design of Linear Models

We seek the best linear approximation of $f(x)$ which is $\beta_0 + \beta_1 x_1 + \ldots + \beta_p x_p$. We pick the $\boldsymbol{x}$'s to give us the best linear approximation. What criteria? JMP gives two ways:

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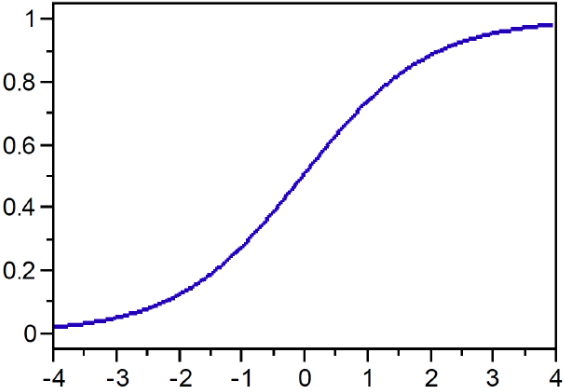
This is (I would say) the most interpretable link function situation we've got.

The Logistic Function (an "S" Shape)



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This does not have a simple, closed form solution. The computer iterates numerically using gradient methods. It usually uses the $\ln(\cdot)$ of above, since it's (1) numerically more stable and (2) the expression is easier to work with. When it "converges" on the values of the parameters that maximize the above, these are shipped to you as $\left\{\hat{\beta}_0, \hat{\beta}_1, \ldots, \hat{\beta}_p\right\}$. This is called "running a logistic regression". The above looks complicated but it is instant on a modern computer for most real-world datasets.

Prediction with Logistic Regression

Recall that predictions in linear regression were easy:

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So on top the computer iterates to find $\left\{\hat{\beta}_0, \hat{\beta}_1, \ldots, \hat{\beta}_p\right\}$, plugs it in and computes the likelihood and on the bottom the computer independently iterates to find $\left\{\hat{\beta}_0\right\}$, plugs it in and computes the likelihood, then together, the LR.

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There is something special about a χ^2 r.v. with one degree of freedom. Note this cool fact from probability theory: $Q \sim \chi^2_1 \Rightarrow \sqrt{Q} \sim \mathcal{N}(0, 1)$ i.e. a "z-score". This is how JMP produces standard errors for logistic regression coefficients.

Telecom Company Churn Example

In marketing lingo, "churn" refers to a customer canceling their service. Studies suggest that it costs 5-10x more to acquire a new customer than to retain an old customer. Thus, predicting churn is of major interest!

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We begin just trying to model y: churn vs. x: tenure (the number of months customer is currently subscribed for). What do you think the relationship will be i.e. what is the sign of $\partial/\partial x[f(x)]$ generally speaking?

Results of Simple Logistic Regression

[JMP demo]

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$$\begin{aligned} Q &= 2\ln(LR) = 2\left(\ell\left(\hat{\theta};x\right) - \ell\left(\hat{\theta}_R;x\right)\right) \\ &= 2\left(-(3595.9341) - -(4075.0729)\right) \\ &= 2\left(479.1389\right) = 958.2778 \end{aligned}$$

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Also note that $LR = 1.22 \times 10^{208}$ but $e^{-3595.9341} = 0$ i.e. it's less than the smallest number a computer can represent (without special handling). Numbers are cool things... and logs are pretty powerful.

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Predict estimated expected probability of churn for someone who has 1 month of tenure

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i.e. a difference in about 0.8% as measured on a probability scale.

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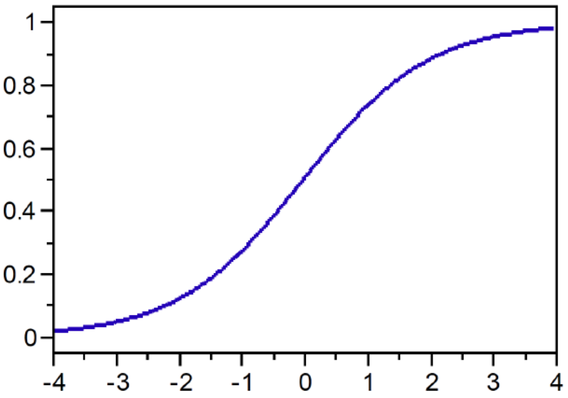
i.e. a difference in about 0.2% as measured on a probability scale (i.e. a 4x difference from before). But isn't the model supposedly to be linear??

The Logistic Function



A move of one unit in x when $x \approx 0$ is a much bigger move than one unit in x when $x \approx 3$

Parameter Standard Error

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are expectedly about the same

$$\begin{aligned} s_{\hat{\beta}_1} &= \frac{|-0.0387682|}{\sqrt{761.00}} = 0.0014 \quad \text{(via the Wald test)} \\ s_{\hat{\beta}_1} &= \frac{|-0.0387682|}{\sqrt{958.28}} = 0.0013 \quad \text{(via the LR test)} \end{aligned}$$

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Now let's use all variables. Questions:

- Which variable(s) should we leave out?

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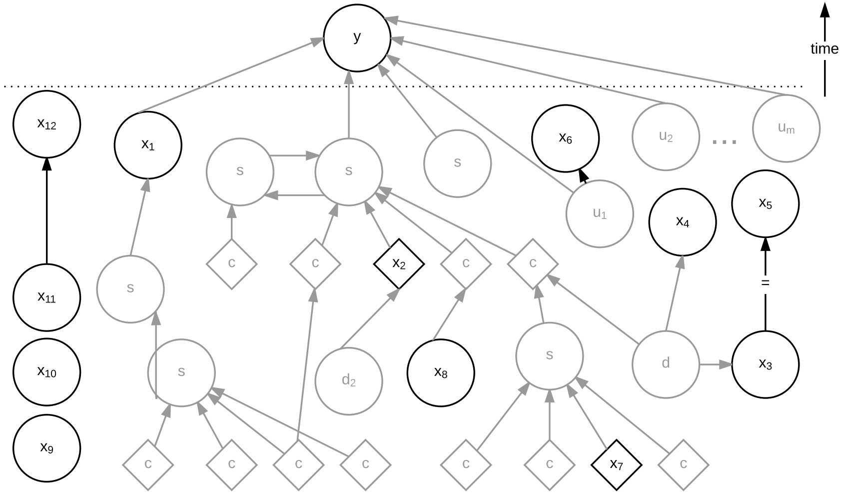
- Which variable(s) should we leave out? `customerID` — no causal mechanism.
- Why are we getting biased estimates and zeroes? Perfect multicollinearity. Solution? Kill columns until you don't get the error anymore. Turns out 6 have to be removed.

Realistic Predictors Illustration (updated)



Multivariate Logistic Regr. Interp. II

Now let's use all variables. Questions:

- Which variable(s) should we leave out? `customerID` — no causal mechanism.
- Why are we getting biased estimates and zeroes? Perfect multicollinearity.
- Are all these variables "significant"?

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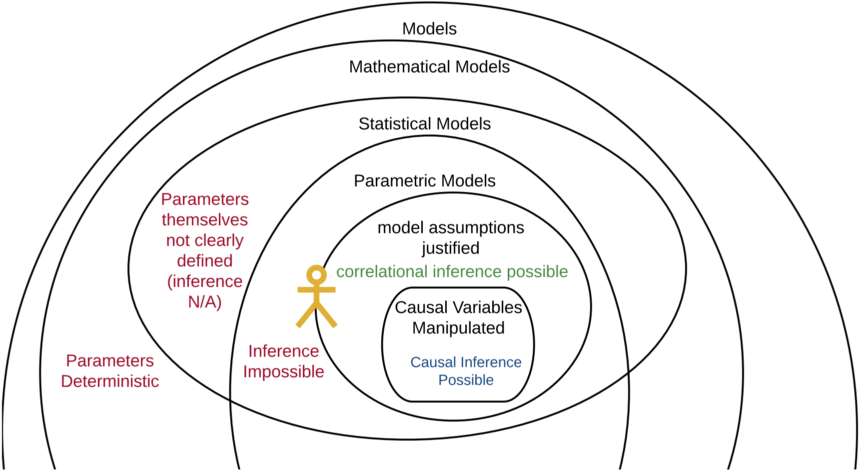
- Which of the "driving" variables are *not* able to be manipulated? Senior citizen, total charges (?)
- Should you (the manager) try to incentivize electronic checks? Paperless billing? Dropping multiple phone lines? You can try these things if you have nothing to lose, but remember, they are not guaranteed to be causal! And they may backfire!! (Can you think of

Remember Where you At!!



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We now cover evaluating classification models in general (not only in the context of logistic regression models specifically).

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		$\hat{y}$ (decision) 1	$\hat{y}$ (decision) 0
y (truth)	1	true positive (TP)	false negative (FN)
	0	false positive (FP)	true negative (TN)

Why do "correlations rock" here??

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Why do "correlations rock" here?? We are purely evaluating predictive performance... no inferential claims!

Confusion Matrix for Churn Model

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JMP gives us the matrix [JMP], but they don't annotate it well. Here are some numbers I like to see:

		$\hat{y}$ 1	$\hat{y}$ 0	Totals	Model Errors
y	1	$TP = 1012$	$FN = 857$	$P = 1869$	$FNR = 45.9\%$
	0	$FP = 531$	$TN = 4632$	$N = 5163$	$FPR = 10.2\%$
	Totals	$\hat{P} = 1543$	$\hat{N} = 5489$	$n = 7032$	
	Use errors	$FDR = 34.3\%$	$FOR = 15.6\%$		$\boxed{ME = 19.7\%}$

There are other metrics commonly reported

- Sensitivity = Recall = $\frac{TP}{TP+FN} = \frac{TP}{P}$, the proportion of positives successfully recovered (large value = good model), 54.9% above
- Specificity = $\frac{TN}{TN+FP} = \frac{TN}{N}$, the proportion of negatives successfully recovered (large value = good model), 89.8% above

Misclassification Error

Already... what is one broad, general metric to evaluate the model?

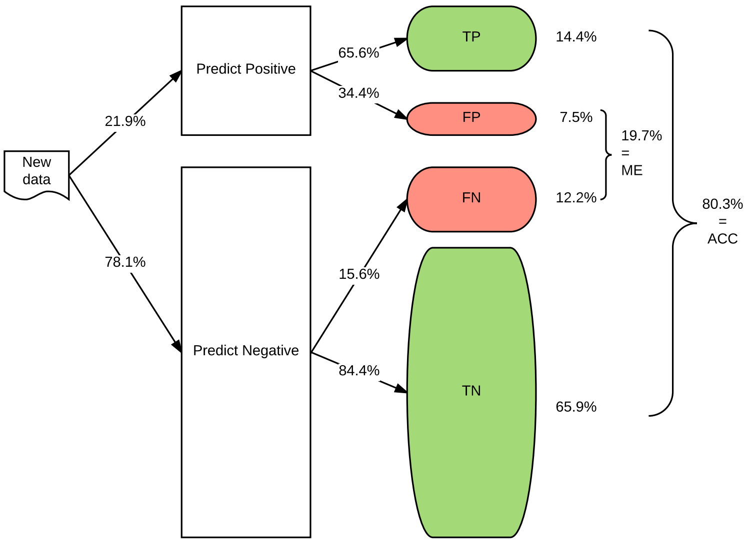
Misclassification Error

Already... what is one broad, general metric to evaluate the model? Misclassification error cost function (or Accuracy):

$$
\begin{aligned}
ME &:= \frac{1}{n}\sum_{i=1}^{n} \mathbb{1}_{y_i \neq \hat{y}_i} \\
ACC &:= 1 - ME = \frac{1}{n}\sum_{i=1}^{n} \mathbb{1}_{y_i = \hat{y}_i}
\end{aligned}
$$

This essentially treats both types of errors (the FN's and the FP's) equally (more on this later).

Production Classifier Flowchart



There's a Ton of Metrics...

From wikipedia...

		Predicted condition			
	Total population	Predicted Condition positive	Predicted Condition negative	Prevalence $= \frac{\Sigma \text{ Condition positive}}{\Sigma \text{ Total population}}$	
True condition	condition positive	**True positive**	**False Negative** (Type II error)	True positive rate (TPR), Sensitivity, Recall, probability of detection $= \frac{\Sigma \text{ True positive}}{\Sigma \text{ Condition positive}}$	False negative rate (FNR), Miss rate $= \frac{\Sigma \text{ False negative}}{\Sigma \text{ Condition positive}}$
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Others (from above) commonly used:

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Recall the classification rule $\hat{y} = \mathbb{1}_{\hat{p} \geq 0.5}$. Using 0.5 is a principled default but we can use any rule $p_0 \in (0, 1)$:

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So now it's simple, vary p_0 and pick the best model according to your cost / error / loss function (the ME at the moment). Let's just do every p_0!

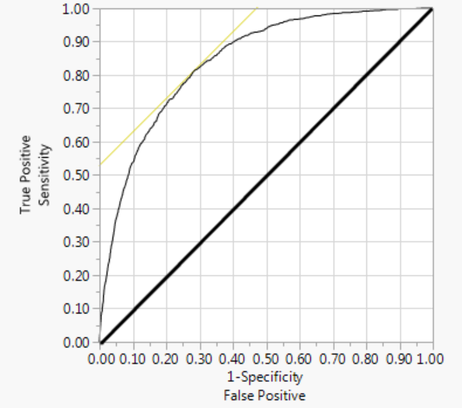
Receiver-Operator Characteristic Table

ROC Table

Prob	1-Specificity	Sensitivity	Sens-(1-Spec)	True Pos	True Neg	False Pos	False Neg
.	0.0000	0.0000	0.0000	0	5163	0	1869
0.8117	0.0000	0.0005	0.0005	1	5163	0	1868
0.8104	0.0000	0.0011	0.0011	2	5163	0	1867
0.8093	0.0000	0.0016	0.0016	3	5163	0	1866
0.8092	0.0000	0.0021	0.0021	4	5163	0	1865
0.8090	0.0000	0.0027	0.0027	5	5163	0	1864
0.8085	0.0000	0.0032	0.0032	6	5163	0	1863
0.8083	0.0000	0.0037	0.0037	7	5163	0	1862
0.8082	0.0000	0.0043	0.0043	8	5163	0	1861
0.8079	0.0000	0.0048	0.0048	9	5163	0	1860
0.8079	0.0000	0.0054	0.0054	10	5163	0	1859
0.8077	0.0000	0.0059	0.0059	11	5163	0	1858
0.8076	0.0002	0.0059	0.0057	11	5162	1	1858
0.8072	0.0002	0.0064	0.0062	12	5162	1	1857
0.8065	0.0002	0.0070	0.0068	13	5162	1	1856
0.8064	0.0002	0.0075	0.0073	14	5162	1	1855
0.8061	0.0002	0.0080	0.0078	15	5162	1	1854

Here, `Prob` is what we denoted p_0. "Best model" is not defined here by highest ACC (lowest ME), it's determined by highest **specificity** + **sensitivity** or equivalently, the highest **sensitivity - (1 - specificity)**.JMP indicates that row with a $\star$. This is an arbitrary metric, but is a good default.

Receiver-Operator Characteristic Curve



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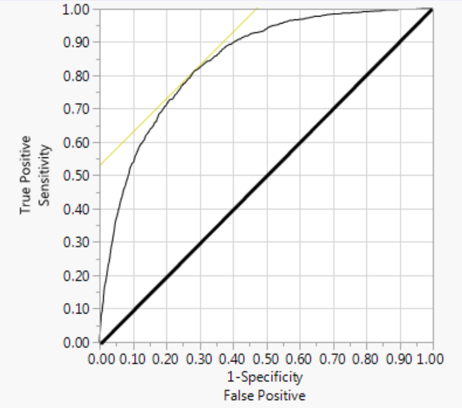

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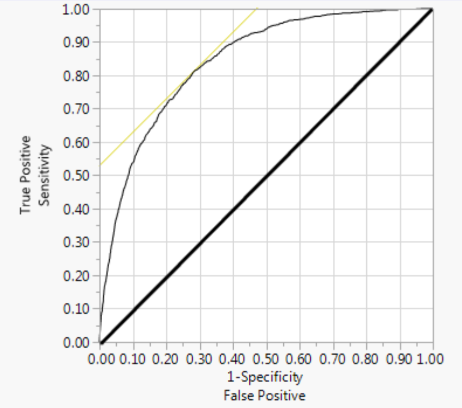

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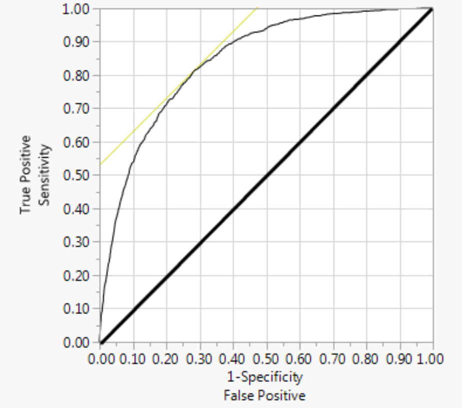

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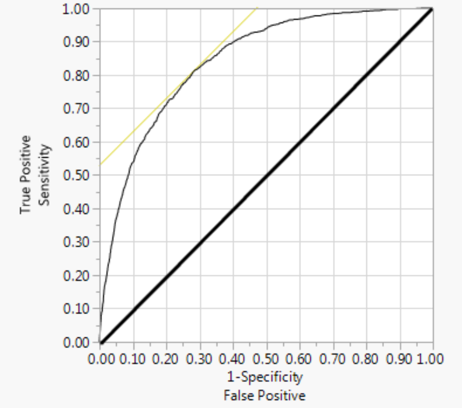

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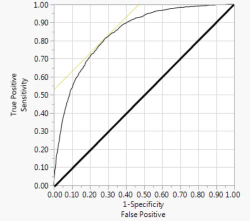
Area Under the Curve (AUC) Metric



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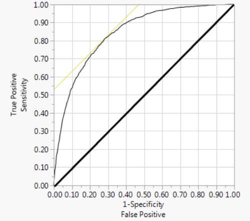

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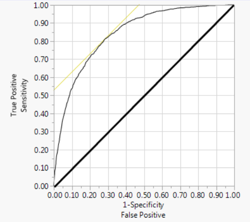
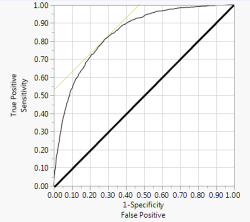


If you built a model by chance the "area under the curve" (or to the right of the curve) on the graph would be ... 0.5 since the graph is a unit square. Under the ROC curve itself (or to its right) is an area ... greater than 0.5. Here, it's 0.844.

Area Under the Curve (AUC) Metric



If you built a model by chance the "area under the curve" (or to the right of the curve) on the graph would be ... 0.5 since the graph is a unit square. Under the ROC curve itself (or to its right) is an area ... greater than 0.5. Here, it's 0.844. This metric is called AUC and is widely used as a metric to assess performance of all possible classifiers in this set of models together, it is a composite metric unlike *ME* or anything derived from an individual confusion table.

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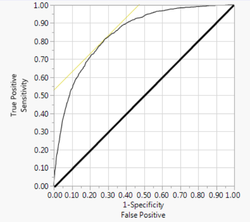

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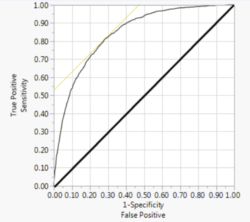

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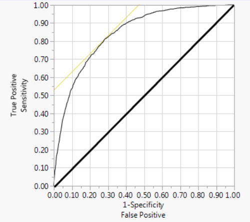

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AUC is nice to evaluate overall performance of all possible models... but at the end of the day... you ship **ONE** model! So we still need a means of evaluating our one model from one confusion table.